Advanced Reinforced Concrete

University of Basrah – college of Engineering

Department of civil Engineering

Master of Science Course

STRUCTURES

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Syllabus:

1. Introduction
2. Deflection of reinforced concrete beams and slabs
3. Cracks in reinforced concrete members
4. Inelastic analysis of reinforced concrete members
5. Strut and tie method
6. Design of deep beams
7. Design of corbels
8. Design of slender columns

References:

1) ACI MNL-17(21) “ACI REINFORCED CONCRETE DESIGN HANDBOOK” 2019

2) Nilsson A. H., Darwin D. , and Dolan C. W. “ Design of concrete structures” 14th and 15th editions.

3) McCormac J. C. and Brown R. H. “Design of Reinforced Concrete” Ninth edition.

4) Hassoun M. Nadim and Al-Manaseer Akthem “Structural Concrete Theory and Design” 7th Edition, 2020.

Introduction:

STRUCTURAL CONCRETE

The design of different structures is achieved by performing two main steps:

(1) determining the different forces acting on the structure using proper methods of structural analysis, and

(2) proportioning all structural members economically, considering:

i. the safety,
ii. stability,
iii. serviceability, and
iv. functionality of the structure.

- Structural concrete is one of the materials commonly used to design all types of buildings.
- Its two component materials, concrete and steel, work together to form structural members that can resist many types of loadings.
- The key to its performance lies in strengths that are complementary:
- Concrete resists compression and
- steel reinforcement resists tension forces.

Design Codes:

- The design engineer is usually guided by specifications called the “codes of practice”.
- Engineering specifications are set up by various organizations to represent the minimum requirements necessary for the safety of the public.
- Most codes specify design loads, allowable stresses, material quality, construction types, and other requirements for construction.
- The most significant standard for structural concrete design is the Building Code Requirements for Structural Concrete, ACI 318, or the ACI Code.
- This code is used primarily for the design of buildings.
- Design requirements for various types of reinforced concrete members are presented in the code along with a “commentary” on those requirements.
- The commentary provides explanations, suggestions, and additional information concerning the design requirements.

Other codes of practice and material specifications include:

1) The International building Code (IBC),
2) The American Society of Civil Engineers standard ASCE 7,
3) The American Association of State Highway and Transportation Officials (AASHTO) specifications,
4) Specifications issued by the American Society for Testing and Materials (ASTM),
5) The American Railway Engineering Association (AREA).

LOADS

- Structural members must be designed to support specific loads.
- Loads are those forces for which a given structure should be proportioned.
- In general, loads may be classified as dead or live.

A. Dead loads include the weight of the structure (its self-weight) and any permanent material placed on the

structure, such as tiles, roofing materials, and walls.

Dead loads can be determined with a high degree of accuracy from the dimensions of the elements and the unit weight of materials.

B. Live loads are all other loads that are not dead loads.

They may be steady or unsteady or movable or moving;

- they may be applied slowly, suddenly, vertically, or laterally, and their magnitudes may fluctuate with time. These loads are given in References 6 and 7.

Table 1.2 Density and Specific Gravity of Various Materials

Material	Density lb/ft³	Density kg/m³	Specific Gravity
Building materials			
Bricks	120	1924	1.8–2.0
Cement, portland, loose	90	1443	—
Cement, portland, set	183	2933	2.7–3.2
Earth, dry, packed	95	1523	—
Sand or gravel, dry, packed	100–120	1600–1924	—
Sand or gravel, wet	118–120	1892–1924	—
Liquids			
Oils	58	930	0.9–0.94
Water (at 4°C)	62.4	1000	1.0
Ice	56	898	0.88–0.92
Metals and minerals			
Aluminum	165	2645	2.55–2.75
Copper	556	8913	9.0
Iron	450	7214	7.2
Lead	710	11, 380	11.38
Steel, rolled	490	7855	7.85
Limestone or marble	165	2645	2.5–2.8
Sandstone	147	2356	2.2–2.5
Shale or slate	175	2805	2.7–2.9
Normal-weight concrete			
Plain	145	2324	2.2–2.4
Reinforced or prestressed	150	2405	2.3–2.5

In general, live loads include the following:

1) Occupancy loads caused by the weight of the people, furniture, and goods
2) Forces resulting from wind action and temperature changes
3) The weight of snow if accumulation is probable
4) The pressure of liquids or earth on retaining structures
5) The weight of traffic on a bridge
6) Dynamic forces resulting from moving loads (impact), earthquakes, or blast loading

- The ACI 318–19 Code does not specify loads on structures.
- Occupancy loads on different types of buildings are prescribed by IBC-2018 and the American National Standards Institute (ANSI).
- AASHTO and AREA specifications prescribe vehicle loadings on highway and railway bridges, respectively.

Table 1.1 Typical Uniformly Distributed Design Loads

Occupancy	Contents	Design Live Load lb/ft^2	Design Live Load kN/m^2
Assembly hall	Fixed seats	60	2.9
	Movable seats	100	4.8
Hospital	Operating rooms	60	2.9
	Private rooms	40	1.9
Hotel	Guest rooms	40	1.9
	Public rooms	100	4.8
	Balconies	100	4.8
Housing	Private houses and apartments	40	1.9
	Public rooms	100	4.8
Institution	Classrooms	40	1.9
	Corridors	100	4.8
Library	Reading rooms	60	2.9
	Stack rooms	150	7.2
Office building	Offices	50	2.4
	Lobbies	100	4.8
Stairs (or balconies)		100	4.8
Storage warehouses	Light	100	4.8
	Heavy	250	12.0
Yards and terraces		100	4.8

Design of reinforced concrete:

The first step in the design of a building is the general planning carried out by the architect to determine the layout of

each floor of the building to meet the owner's requirements.

Once the architectural plans are approved, the structural engineer then determines the most adequate structural system to ensure the safety and stability of the building.

Different structural options must be considered to determine the most economical solution based on the materials available and the soil condition.

Steps of structural design:

1) Idealizing the building into a structural model of load-bearing frames and elements.
2) Estimating the different types of loads acting on the building.
3) Performing the structural analysis using computer or manual calculations to determine the maximum moments, shear, torsional forces, axial loads, and other forces.
4) Proportioning the different structural elements and calculating the reinforcement needed.
5) Producing structural drawings and specifications with enough details to enable the contractor to construct the building properly.

Advanced Reinforced Concrete

Serviceability

Serviceability:

- In working and strength design reinforced concrete beams, methods have been developed to ensure that beams will have a proper safety margin against failure in flexure or shear, or due to inadequate bond and anchorage of the reinforcement.
- It is also important that member performance in normal service be satisfactory, when loads are those actually expected to act, that is, when load factors are 1.0.

- This is not guaranteed simply by providing adequate strength.
- Service load deflections under full load may be excessively large, or long-term deflections due to sustained loads may cause damage.
- Tension cracks in beams may be wide enough to be visually disturbing, and in some cases may reduce the durability of the structure.
- These and other questions, such as vibration or fatigue, require consideration.
- Serviceability studies are carried out based on elastic theory, with stresses in both concrete and steel assumed to be proportional to strain.
- The concrete on the tension side of the neutral axis may be assumed uncracked, partially cracked, or fully cracked, depending on the loads and material strengths.

- In early reinforced concrete designs, questions of serviceability were dealt with indirectly, by limiting the stresses in concrete and steel at service loads to the rather conservative values that had resulted in satisfactory performance.
- In contrast, with current design methods that permit more slender members through more accurate assessment of capacity, and with higher-strength materials further contributing to the trend toward smaller member sizes, such indirect methods no longer work.
- The current approach is to investigate service load cracking and deflections specifically, after proportioning members based on strength requirements.

CRACKING IN FLEXURAL MEMBERS:

- All reinforced concrete beams crack, generally starting at loads well below service level, and possibly even prior to loading due to restrained shrinkage.
- Flexural cracking due to loads is not only inevitable but actually necessary for the reinforcement to be used effectively.
- Prior to the formation of flexural cracks, the steel stress is no more than n times the stress in the adjacent concrete, where n is the modular ratio (ES/Ec).
- For materials common in current practice, n is approximately 8.
- Thus, when the concrete is close to its modulus of rupture of about 3.5 MPa, the steel stress will be only 8 × 3.5 = 28 MPa, far too low to be very effective as reinforcement.
- At normal service loads, steel stresses 8 or 9 times that value can be expected.

- In a well-designed beam, flexural cracks are fine, so-called hairline cracks, almost invisible to the casual observer, and they permit little if any corrosion of the reinforcement.
- As loads are gradually increased above the cracking load, both the number and the width of cracks increase, and at service load level a maximum width of crack of about 0.4 mm is typical.
- If loads are further increased, crack widths increase further, although the number of cracks is more or less stable.

- Cracking of concrete is a random process, highly variable and influenced by many factors.
- Because of the complexity of the problem, present methods for predicting crack widths are based primarily on test observations.
- Most equations that have been developed predict the *probable maximum crack width*, which usually means that about 90 percent of the crack widths in the member are below the calculated value.
- However, isolated cracks exceeding twice the computed width can sometimes occur.

Variables Affecting Width of Cracks:

1. The bond between steel and concrete:

- If good resistance to slip is provided along beam span crack widths will be smaller
- In general, beams with smooth round bars will display a relatively small number of rather wide cracks in service, while beams with good slip resistance ensured by proper surface deformations on the bars will show a larger number of very fine, almost invisible cracks.

2. The stress in the reinforcement:

- Crack width is proportional to f_s^n, where f_s is the steel stress and n is an exponent that varies in the range from about 1.0 to 1.4.
- For steel stresses in the range of practical interest, say from 140 to 250 MPa, n may be taken equal to 1.0.
- The steel stress is easily computed based on elastic cracked-section analysis.
- Alternatively, f_s may be taken equal to $\frac{2}{3} f_y$.

3. Concrete cover:

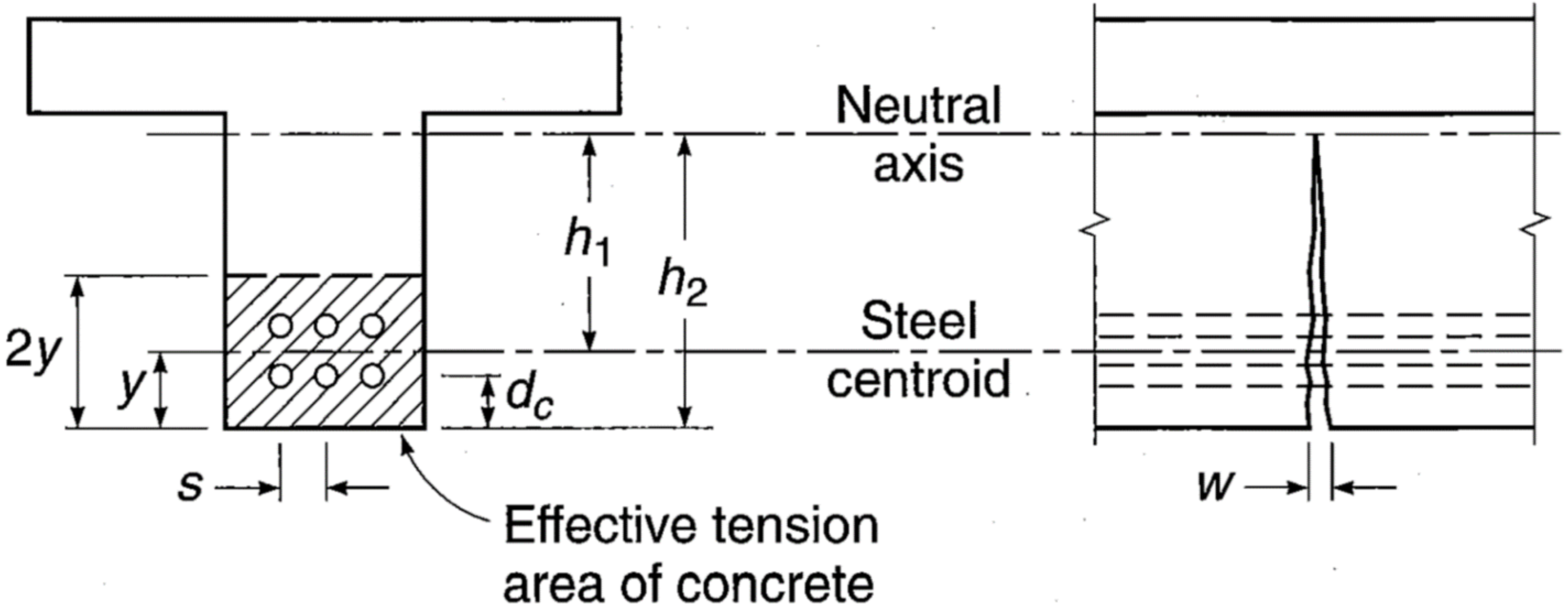
- Increasing the concrete cover distance d_c, measured from the center of the bar to the face of the concrete increases the spacing of cracks and also increases crack width.

4. Distribution of tensile reinforcement:

- Generally, to control cracking, it is better to use a larger number of smaller-diameter bars to provide the required *As* than to use the minimum number of larger bars.
- For deep flexural members, this includes additional reinforcement on the sides of the web to prevent excessive surface crack widths above or below the level of the main flexural reinforcement.
- For reasons of durability and appearance, many fine cracks are preferable to a few wide cracks.
- Detailing practices limiting bar spacing will usually lead to adequate crack control where Grade 420 reinforcement is used.

Equations for Crack Width:

- A number of expressions for maximum crack width have been developed based on the statistical analysis of experimental data.
- Two expressions that have figured prominently in the development of the crack control provisions in the ACI Code are those developed by Gergely and Lutz and Frosch for the maximum crack width at the tension face of a beam.



$$w = 1.1\beta f_s \sqrt[3]{d_c A} \times 10^{-5} \tag{6.1}$$

and

$$w = 2\,\frac{f_s}{E_s}\,\beta\sqrt{d_c^2 + \left(\frac{s}{2}\right)^2} \tag{6.2}$$

where w = maximum width of crack, mm
f_s = steel stress at load for which crack width is to be determined, MPa
E_s = modulus of elasticity of steel, MPa

The geometric parameters are shown in Fig. 6.1 and are as follows:

d_c = thickness of concrete cover measured from tension face to center of bar closest to that face, mm
β = ratio of distances from tension face and from steel centroid to neutral axis, equal to h_2/h_1
A = concrete area surrounding one bar, equal to total effective tension area of concrete surrounding reinforcement and having same centroid, divided by number of bars, mm^2
s = maximum bar spacing, mm

- Equations (6.1) and (6.2), which apply only to beams in which deformed bars are used, include all the factors just named as having an important influence on the width of cracks:

i. steel stress,

ii. concrete cover, and

iii. the distribution of the reinforcement in the concrete tensile zone.

- In addition, the factor β is added to account for the increase in crack width with distance from the neutral axis.

Example 6.3

Assuming $\beta_h = 1.20$ and $f_y = 60$ ksi, calculate the estimated width of flexural cracks that will occur in the beam of Figure 6.13. If the beam is to be exposed to moist air, is this width satisfactory as compared to the values given in Table 6.3 of this chapter? Should the cracks be too wide, revise the design of the reinforcing structure and recompute the crack width.

SOLUTION

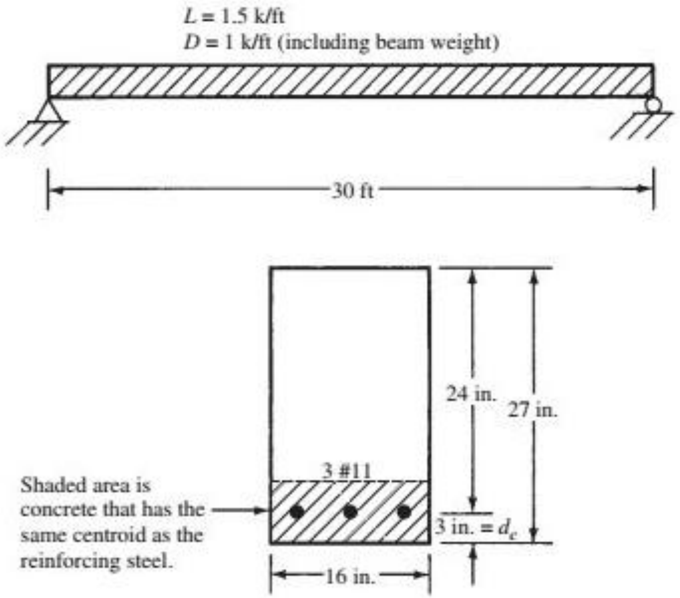
Substituting into the Gergely–Lutz Equation

$$d_c = 3 \text{ in.}$$

$$A = \frac{(6 \text{ in.})(16 \text{ in.})}{3} = 32 \text{ in.}^2$$

$$w = (0.076)(1.20)(0.6 \times 60 \text{ ksi})\sqrt[3]{(3 \text{ in.})(32 \text{ in.}^2)}$$

$$= \frac{15.03 \text{ in.}}{1000} = 0.015 \text{ in.} > 0.012 \text{ in.} \quad \underline{\underline{\text{No good}}}$$



Cyclic and Sustained Load Effects:

- Both cyclic and sustained loading account for increasing crack width.
- While there is a large amount of scatter in test data, results of fatigue tests and sustained loading tests indicate that a doubling of crack width can be expected with time.
- Under most conditions, the spacing of cracks does not change with time at constant levels of sustained stress or cyclic stress range.

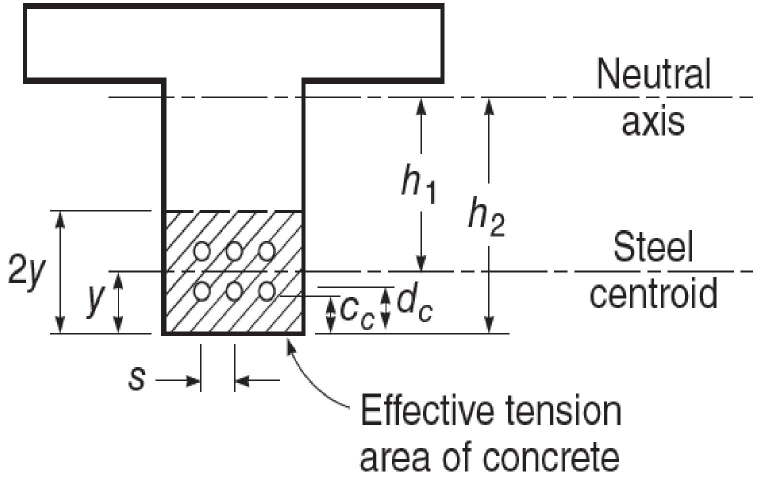
ACI CODE PROVISIONS FOR CRACK CONTROL:

- Crack width is controlled in accordance with ACI Code 24.3.2 by establishing a maximum center-to-center spacing *s* for the reinforcement closest to the surface of a tension member as a function of:

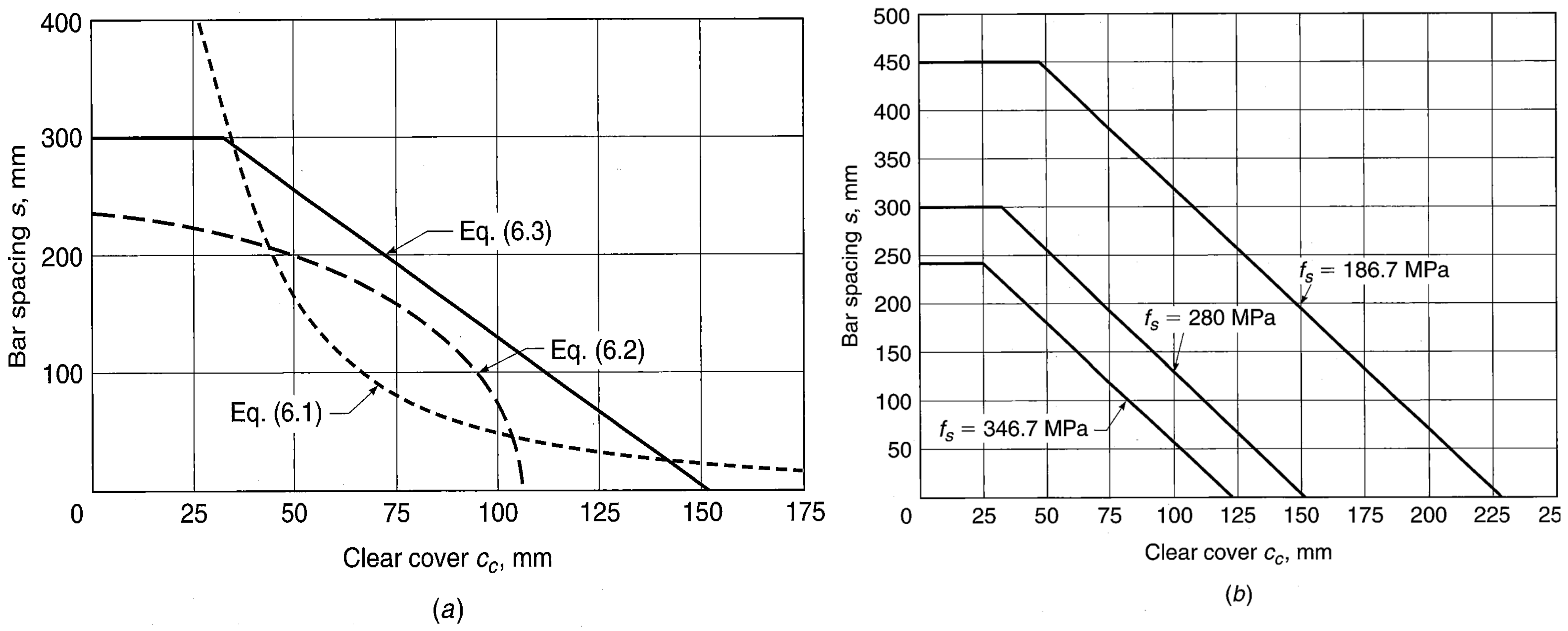
1) the bar stress under service conditions *fs,* and
2) the *clear cover* from the nearest surface in tension to the surface of the flexural tension reinforcement.

$$s = 380\left(\frac{280}{f_s}\right) - 2.5c_c \leq 300\left(\frac{280}{f_s}\right) \qquad (6\text{-}3)$$

- For the case of beams with Grade 420 reinforcement and 50 mm clear cover to the primary reinforcement, with f_s **= 280 MPa**, the maximum bar spacing is 250 mm.



- The choice of clear cover C_c , rather than the cover to the center of the bar d_c, was made to simplify design, since this allows *s* to be independent of bar size.
- As a consequence, maximum crack widths will be somewhat greater for larger bars than for smaller bars.
- In equation (6-3) for s, the ACI Code sets an upper limit on *s* of $300\left(\frac{280}{f_s}\right)$.
- The stress *fs* is calculated by dividing the service load moment by the product of the area of reinforcement and the internal moment arm.
- Alternatively, the ACI Code permits f_s to be taken as two-thirds of the specified yield strength, $(\frac{2}{3}f_y)$.
- For members with only a single bar, the width of the extreme tension face may not exceed *s*, in accordance with ACI Code 24.3.3.



Maximum bar spacing versus clear cover: (*a*) Comparison of Eqs. (6.1), (6.2), and (6.3) for w_c = 0.4 mm, f_s = 280 MPa, β = 1.2, bar size = No. 8 (No. 25) and

(*b*) Eq. (6.3) for f_s = 186, 280, and 346.7 MPa corresponding to 2/3 fy for Grades 40 (280), 60 (420), and 75 (520) reinforcement, respectively.

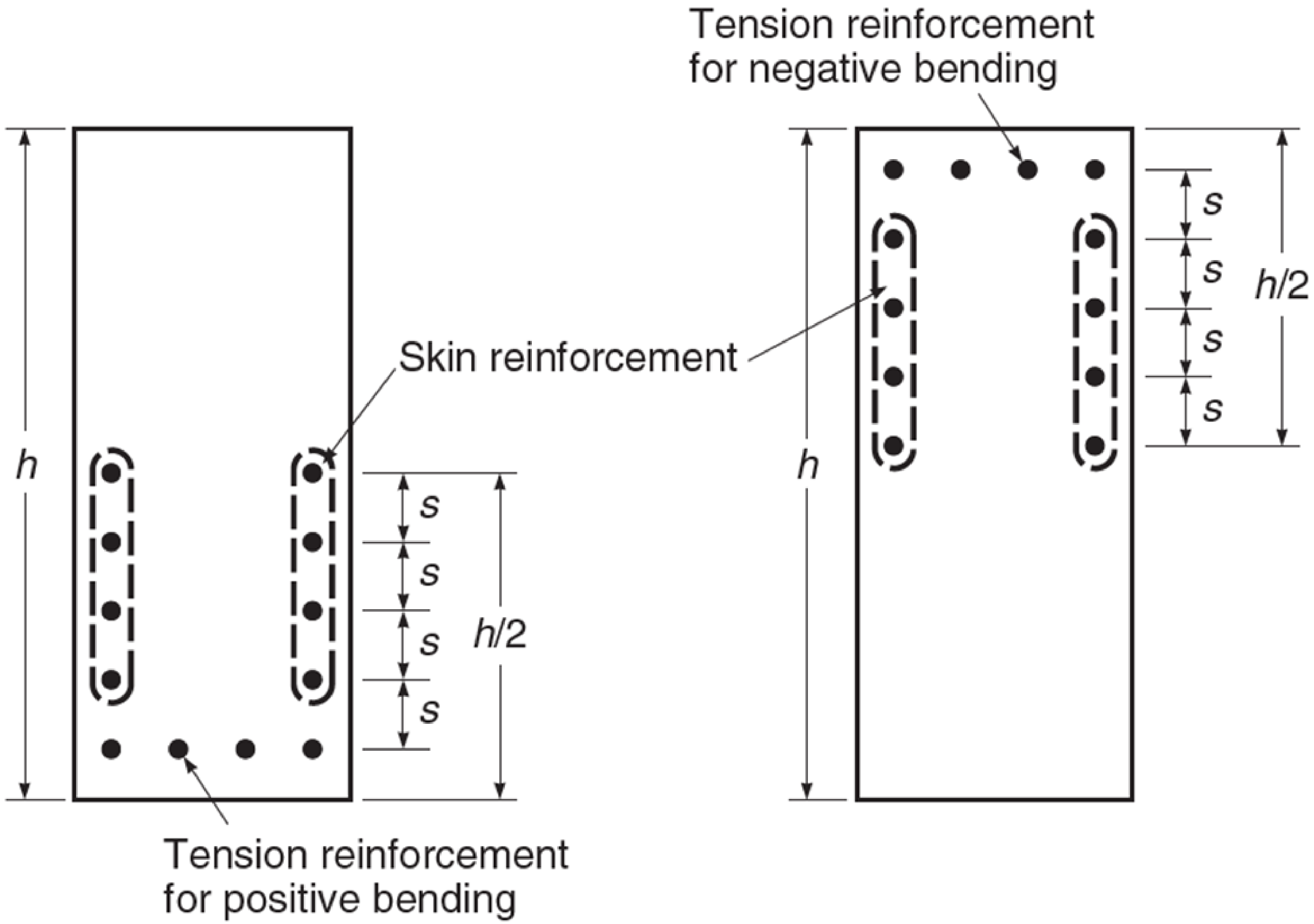
Table 24.3.2—Maximum spacing of bonded reinforcement in non-prestressed and Class C prestressed one-way slabs and beams

Reinforcement type	Maximum spacing s	
Deformed bars or wires	Lesser of:	$380\left(\frac{280}{f_s}\right)-2.5c_c$
		$300\left(\frac{280}{f_s}\right)$
Bonded prestressed reinforcement	Lesser of:	$\left(\frac{2}{3}\right)\left[380\left(\frac{280}{\Delta f_{ps}}\right)-2.5c_c\right]$
		$\left(\frac{2}{3}\right)\left[300\left(\frac{280}{\Delta f_{ps}}\right)\right]$
Combined deformed bars or wires and bonded prestressed reinforcement	Lesser of:	$\left(\frac{5}{6}\right)\left[380\left(\frac{280}{\Delta f_{ps}}\right)-2.5c_c\right]$
		$\left(\frac{5}{6}\right)\left[300\left(\frac{280}{\Delta f_{ps}}\right)\right]$

Distribution of reinforcement in T-beams:

- When concrete T beam flanges are in tension, as in the negative-moment region of continuous T beams, concentration of the reinforcement over the web may result in excessive crack width in the overhanging slab, even though cracks directly over the web are fine and well distributed.
- To prevent this, the tensile reinforcement should be distributed over the width of the flange, rather than concentrated in the width of the web.
- However, because of shear lag, the outer bars in such a distribution would be considerably less highly stressed than those directly over the web, producing an uneconomical design.
- As a reasonable compromise, ACI Code requires that the tension reinforcement in such cases be distributed over the effective flange width or a width equal to one-tenth the span, whichever is smaller.
- If the effective flange width exceeds one-tenth of the span, some longitudinal reinforcement must be provided in the outer portions of the flange.
- The amount of such additional reinforcement is left to the discretion of the designer; it should at least be the equivalent of temperature

- For beams with relatively deep webs, some reinforcement should be placed near the vertical faces of the web to control the width of cracks in the concrete tension zone above the level of the main reinforcement.
- Without such steel, crack widths in the web wider than those at the level of the main bars have been observed.
- According to ACI Code, if the total depth of the beam h exceeds 900 mm, longitudinal “skin” reinforcement must be uniformly distributed along both side faces of the member for a distance of $(h/2)$ from the tension face, as shown in figure below.
- The spacing s between longitudinal bars or wires is as specified in Eq. (6-3).
- The size of the bars or wires is not specified, but as indicated in ACI Commentary (No. 10 to No. 16) bars or welded wire reinforcement with a minimum area of 210 mm^2 per meter of depth are typically used.
- The contribution of the skin steel to flexural strength is usually disregarded, although it may be included in the strength calculations if a strain compatibility analysis is used to establish the stress in the skin steel at the flexural failure load.



Skin reinforcement for flexural members with total depth *h* greater than 900 mm.

Concrete Protection for Reinforcement:

- To provide the steel with adequate concrete protection against fire and corrosion, the designer must maintain a certain minimum thickness of concrete cover outside of the outermost steel.
- The thickness required will vary, depending upon the type of member and conditions of exposure.
- Non-prestressed cast-in-place concrete members shall have specified concrete cover for reinforcement at least that given in Table 20.5.1.3.1.
- In general, the centers of main flexural bars in beams should be placed 50 to 75 mm from the top or bottom surface of the beam to furnish at least 40 mm of clear cover for the bars and the stirrups.
- In slabs, 25 mm to the center of the bar is ordinarily sufficient to give the required 20 mm cover.

Table 20.5.1.3.1—Specified concrete cover for cast-in-place nonprestressed concrete members

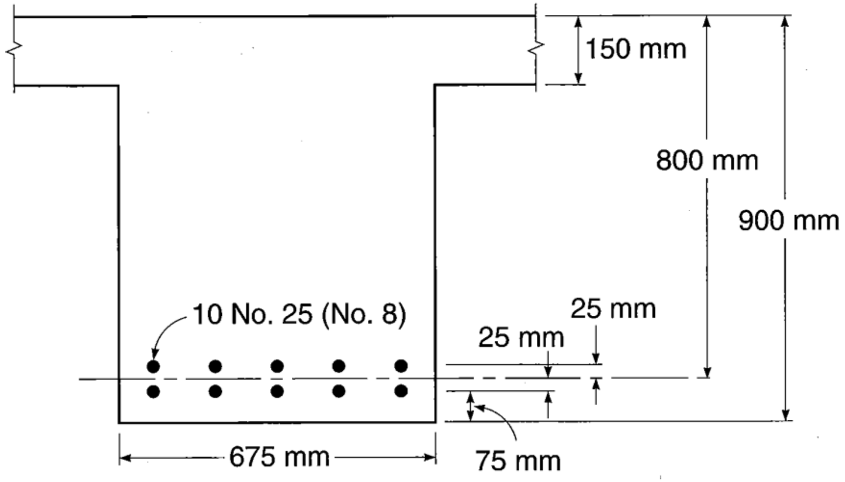
Concrete exposure	Member	Reinforcement	Specified cover, mm
Cast against and permanently in contact with ground	All	All	75
Exposed to weather or in contact with ground	All	No. 19 through No. 57 bars	50
		No. 16 bar, MW200 or MD200 wire, and smaller	40
Not exposed to weather or in contact with ground	Slabs, joists, and walls	No. 43 and No. 57 bars	40
		No. 36 bar and smaller	20
	Beams, columns, pedestals, and tension ties	Primary reinforcement, stirrups, ties, spirals, and hoops	40

EXAMPLE: Check crack control criteria. The figure shows the main flexural reinforcement at midspan for a T-girder in a high-rise building that carries a service load moment of 975 kN-m. The clear cover on the side and bottom of the beam stem is 60 mm. Determine if the beam meets the crack control criteria in the ACI Code.

Solution. Since the depth of the beam equals but does not exceed900 mm, skin reinforcement is not needed. To check the bar spacing criteria, the steel stress can be estimated closely by taking the internal lever arm equal to the distance $d - \frac{h_f}{2}$:

$$f_s = \frac{M_s}{A_s(d - h_f/2)} = \frac{975 \times 10^6}{5100 \times 725} = 263 \text{ MPa}$$

(Alternately, the ACI Code permits using fs = 2/3 f_y giving 280 MPa.)



Using f_s in Eq. (6.3) gives

$$s = 380\left(\frac{280}{f_s}\right) - 2.5c_c = 380\left(\frac{280}{263}\right) - 2.5 \times 60 = 255 \text{ mm}$$

By inspection, it is clear that this requirement is satisfied for the beam. If the results had been unfavorable, a redesign using a larger number of smaller-diameter bars would have been indicated.

Redistribution of moments

References:

1. Nilson (14th ed) chapter 6 & 12
2. Nadhim Hasson chapter 11
3. Design of Reinforced Concrete McCormac ch 14

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M.Sc. Course – structures

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Limit Analysis

- Most reinforced concrete structures are designed for moments, shears, and axial forces found by elastic theory with methods.
- On the other hand, the actual proportioning of members is done by strength methods, with the recognition that inelastic section and member response would result upon overloading.
- In other words, reinforced concrete design today, the current practice is to use elastic methods to analyze structures loaded with factored or ultimate loads.
- Such a procedure probably doesn’t seem quite correct but it does yield satisfactory results.
- Clearly this is an inconsistent approach to the total analysis-design process, although it can be shown to be both safe and conservative.
- A beam or frame so analyzed and designed will not fail at a load lower than the value calculated in this way.

- On the other hand, it is known that a continuous beam or frame normally will not fail when the nominal moment capacity of just one critical section is reached.
- A *plastic hinge* will form at that section, permitting large rotation to occur at essentially constant resisting moment and thus transferring load to other locations along the span where the limiting resistance has not yet been reached.
- As loading is further increased, additional plastic hinges may form at other locations along the span and eventually result in collapse of the structure, but only after a significant *redistribution of moments* has occurred.
- The ratio of negative to positive moments found from elastic analysis is no longer correct, for example, and the true ratio after redistribution depends upon the flexural strengths actually provided at the hinging sections.

- Recognition of redistribution of moments can be important because it permits a more realistic appraisal of the actual load-carrying capacity of a structure, thus leading to improved economy.
- In addition, it permits the designer to modify, within limits, the moment diagrams for which members are to be designed.
- Certain sections can be deliberately under-reinforced if moment resistance at adjacent critical sections is increased correspondingly.
- Adjustment of design moments in this way enables the designer to reduce the congestion of reinforcement that often occurs in high-moment areas, such as at the beam-column joints.

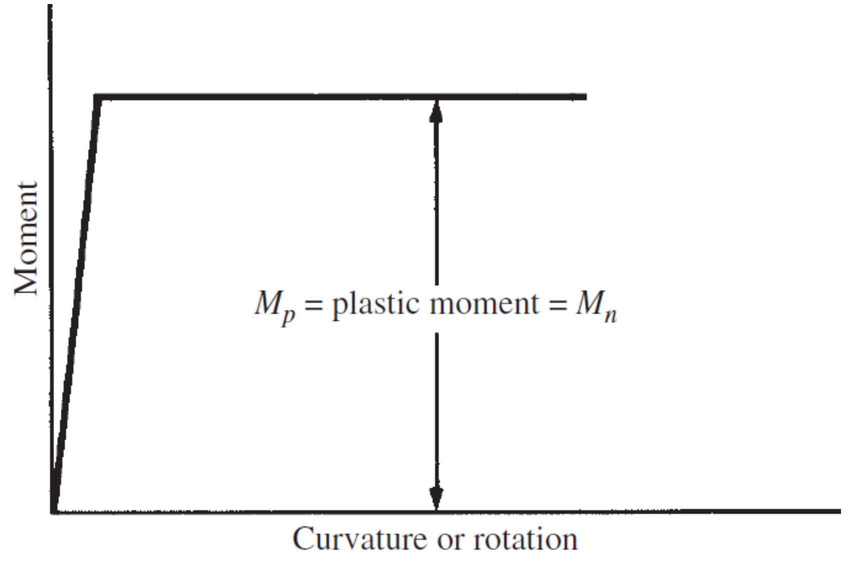
- The formation of plastic hinges is well established by tests on continuous beams.
- The beam continued to carry increasing load well beyond the load that produced first yielding at the supports.
- An extreme deflections and sharp changes in slope of the member axis that are usually noticed only slightly before final collapse.

Motivations of using Limit Analysis:

1. The *inconsistency* of the present approach to the total analysis-design process,
2. The possibility of using the *reserve strength* of concrete structures resulting from moment redistribution, and
3. The opportunity to *reduce steel congestion* in critical regions.

- For beams and frames, ACI Code 6.6.5 permits limited redistribution of moments, depending upon the strain in the tensile steel.
- For slabs, which generally use very low reinforcement ratios and consequently have great ductility, plastic design methods are especially suitable.

- The basic assumption used for limit design of reinforced concrete structures and for plastic design of steel structures is the ability of these materials to resist a so-called yield moment while an appreciable increase in local curvature occurs.
- In effect, if one section of a statically indeterminate member reaches this moment, it begins to yield but does not fail.
- Rather, it acts like a hinge (called a *plastic hinge*) and throws the excess load off to sections of the members that have lesser stresses.
- It is assumed that an ideal plastic material, such as a ductile structural steel, is involved.
- The relationship of moment to the resulting curvature of a short length of a ductile steel member is shown in figure.



Moment–curvature relationship for an ideal plastic material.

Limit design versus Plastic design:

- It might be useful to attempt to distinguish between the terms "*plastic design*" as used in structural steel and "*limit design*" as used in reinforced concrete.
- In structural steel, plastic design involves both:

(a) the increased resisting moment of a member after the extreme fiber of the member is stressed to its yield point and

(b) the redistribution or change in the moment pattern in the member.

- Load and resistance factor design [LRFD] is a steel design method that incorporates much of the theory associated with plastic design.
- In reinforced concrete, the increase in resisting moment of a section after part of the section has been stressed to its yield point has already been accounted for in the strength design procedure.
- Therefore, limit design for reinforced concrete structures is concerned only with the change in the moment pattern after the steel reinforcing at some cross section is stressed to its yield point.

MOMENT VS. CURVATURE FOR REINFORCED CONCRETE SECTIONS:

- The relation between moment applied to a given beam section and the resulting curvature, through the full range of loading to failure, is important in several contexts:

1) It is basic to the study of member ductility,
2) understanding the development of plastic hinges, and
3) accounting for the redistribution of elastic moments that occurs in most reinforced concrete structures before collapse.

- The basic assumption used for limit design of reinforced concrete structures and for
- plastic design of steel structures is the ability of these materials to resist a so-called yield
- moment while an appreciable increase in local curvature occurs. In effect, if one section of
- a statically indeterminate member reaches this moment, it begins to yield but does not fail.
- Rather, it acts like a hinge (called a *plastic hinge*) and throws the excess load off to sections
- of the members that have lesser stresses.

- The curvature is defined as the angle change per unit length at any given location along the axis of a member subjected to bending loads:

$$\emptyset = \frac{1}{r} \qquad (1)$$

where ϕ = unit curvature and

r = radius of curvature.

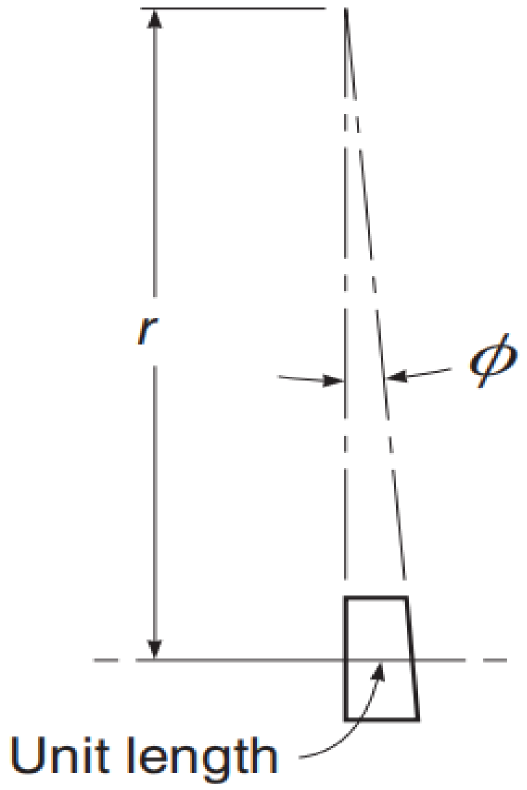


Figure (1): Unit curvature resulting from bending of beam section.

- The stress-strain relationships for steel and concrete, represented in idealized form are as shown.

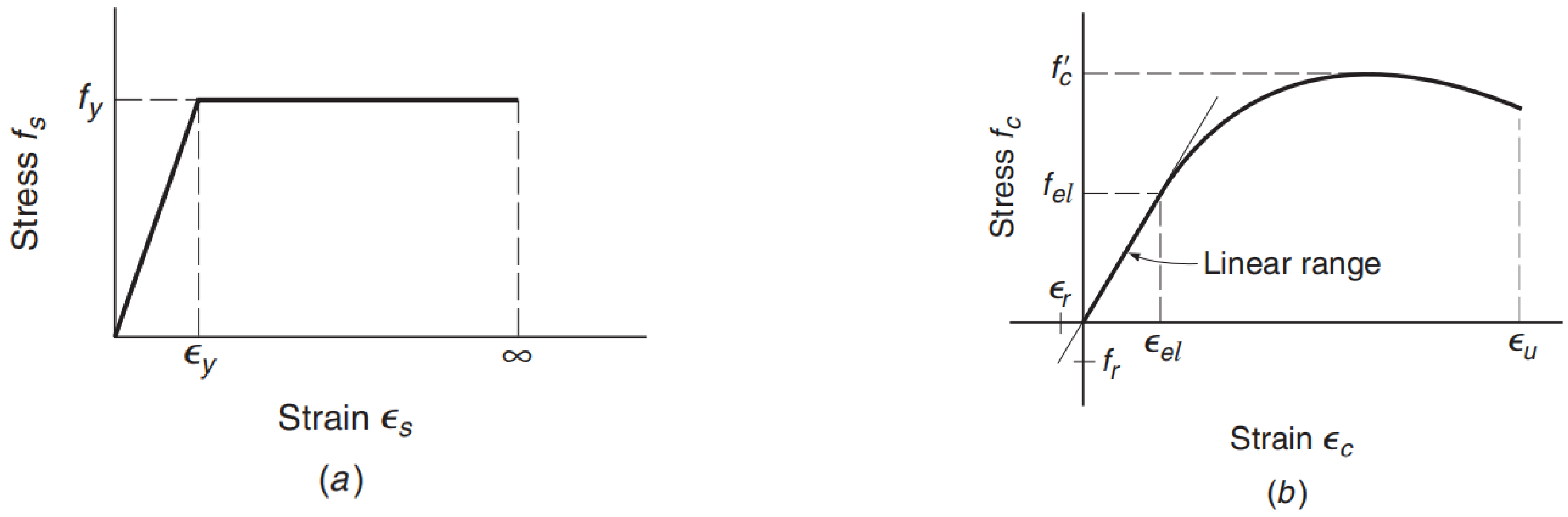


Figure (2): Idealized stress-strain curves: (*a*) steel and (*b*) concrete.

I) The un-cracked Stage:

- The transformed cross-section of a rectangular, tensile reinforced beam in the uncracked elastic loading stage, with steel represented by the equivalent concrete area *nA*.
- The neutral axis, a distance c_1 below the beam's top surface, can easily be found.
- In the limiting case, the concrete stress at the tension face st equal to the modulus of rupture f_r.

$$\epsilon_r = \frac{f_r}{E_c}$$

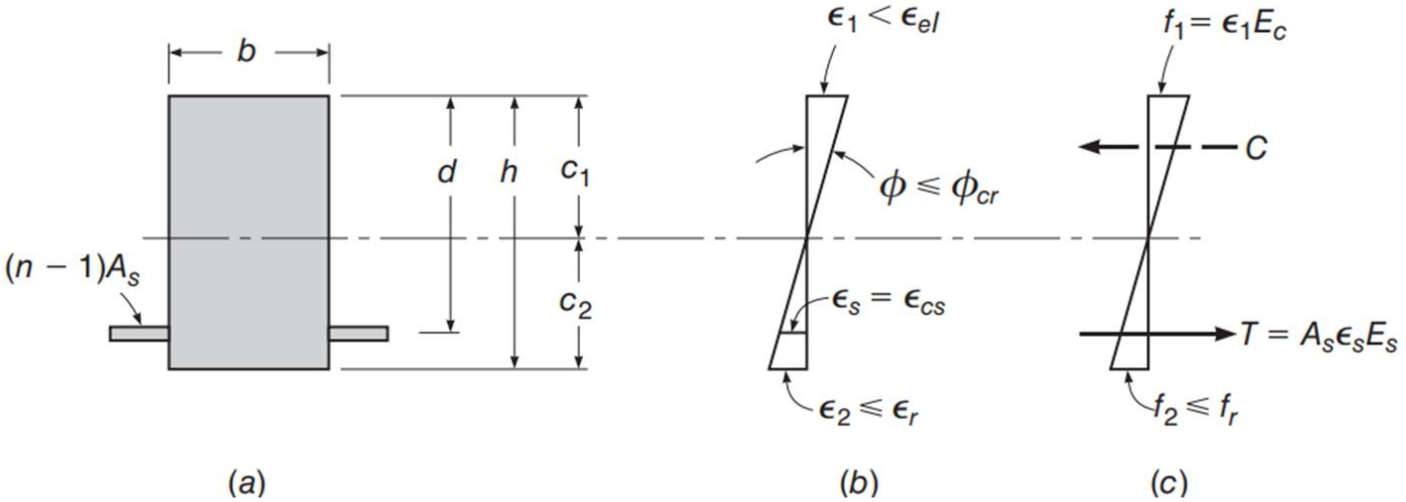


Figure (3): Uncracked beam in the elastic range of loading:
(*a*) transformed cross-section; (*b*) strains; and (*c*) stresses and forces.

- The steel is well below yield at this stage, which can be confirmed by computing, from the strain diagram, the steel strain is the same as strain in concrete at level of steel, i.e.

$$\epsilon_s = \epsilon_{cs}$$

where ϵ_{cs} is the concrete strain at the level steel.

- It is easily confirmed, also, that the maximum concrete compressive stress will be well below the proportional limit.
- The curvature is

$$\phi_{cr} = \frac{\epsilon_1}{c_1} = \frac{\epsilon_r}{c_2}$$

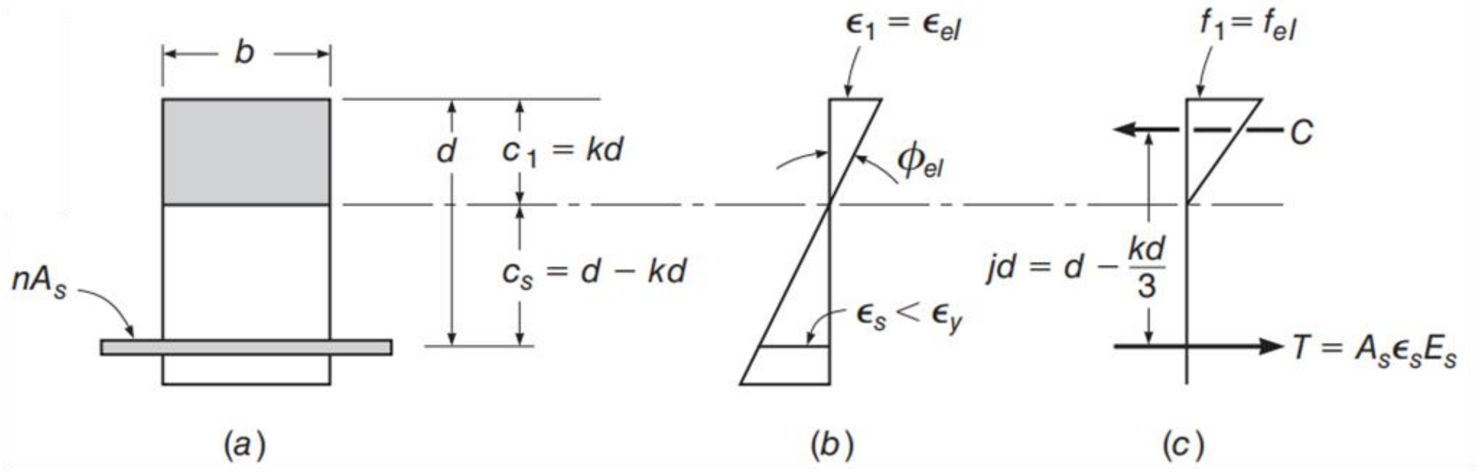
and the corresponding moment is

$$M_{cr} = \frac{f_r I_{ut}}{c_2}$$

where I_{ut} is the moment of inertia of the uncracked transformed section.

II. The cracked elastic stage:

- When tensile cracking occurs at the section, the stiffness is immediately reduced, and curvature increases with no increase in moment.
- The analysis now is based on the cracked transformed section with steel represented by the transformed area *nA* and tension concrete deleted.



Figure(4): Cracked beam in the elastic range of material response: (*a*) transformed cross-section; (*b*) strains; and (*c*) stresses and forces.

- The cracked, elastic neutral axis distance $c_1 = kd$ is easily found by the usual methods.
- In the limiting case, the concrete strain just reaches the proportional limit, and the steel is still below the yield strain.
- The curvature is easily computed by

$$\phi_{el} = \frac{\epsilon_1}{c_1} = \frac{\epsilon_{el}}{c_1}$$

- and the corresponding moment is

$$M_{el} = \frac{1}{2} f_{el}\, kj\, bd^2$$

III. The cracked inelastic stage:

- Next, in the cracked, inelastic stage of loading, the concrete is well into the inelastic range, although the steel has not yet yielded.
- The neutral axis depth c_1 is less than the elastic *kd* and is changing with increasing load as the shape of the concrete stress distribution changes and the steel stress changes.

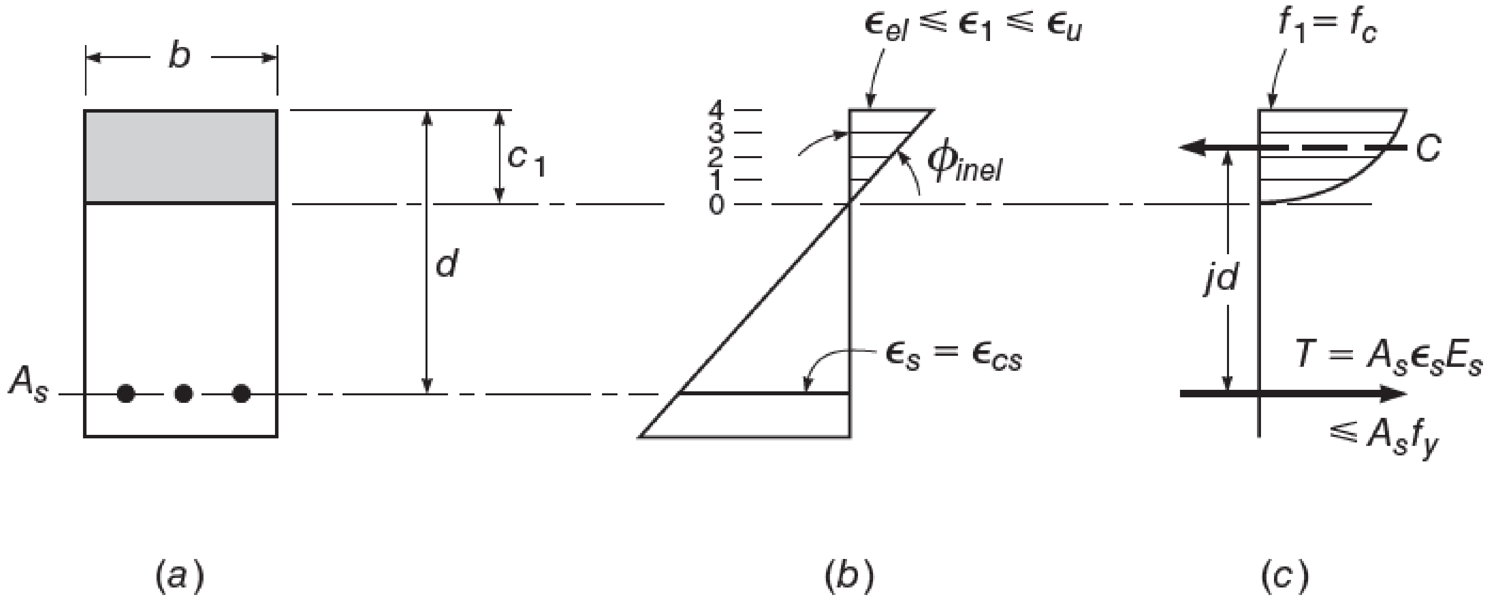


Figure (5): Cracked beam with concrete in the inelastic range of loading: (a) cross section; (b) strains; and (c) stresses and forces.

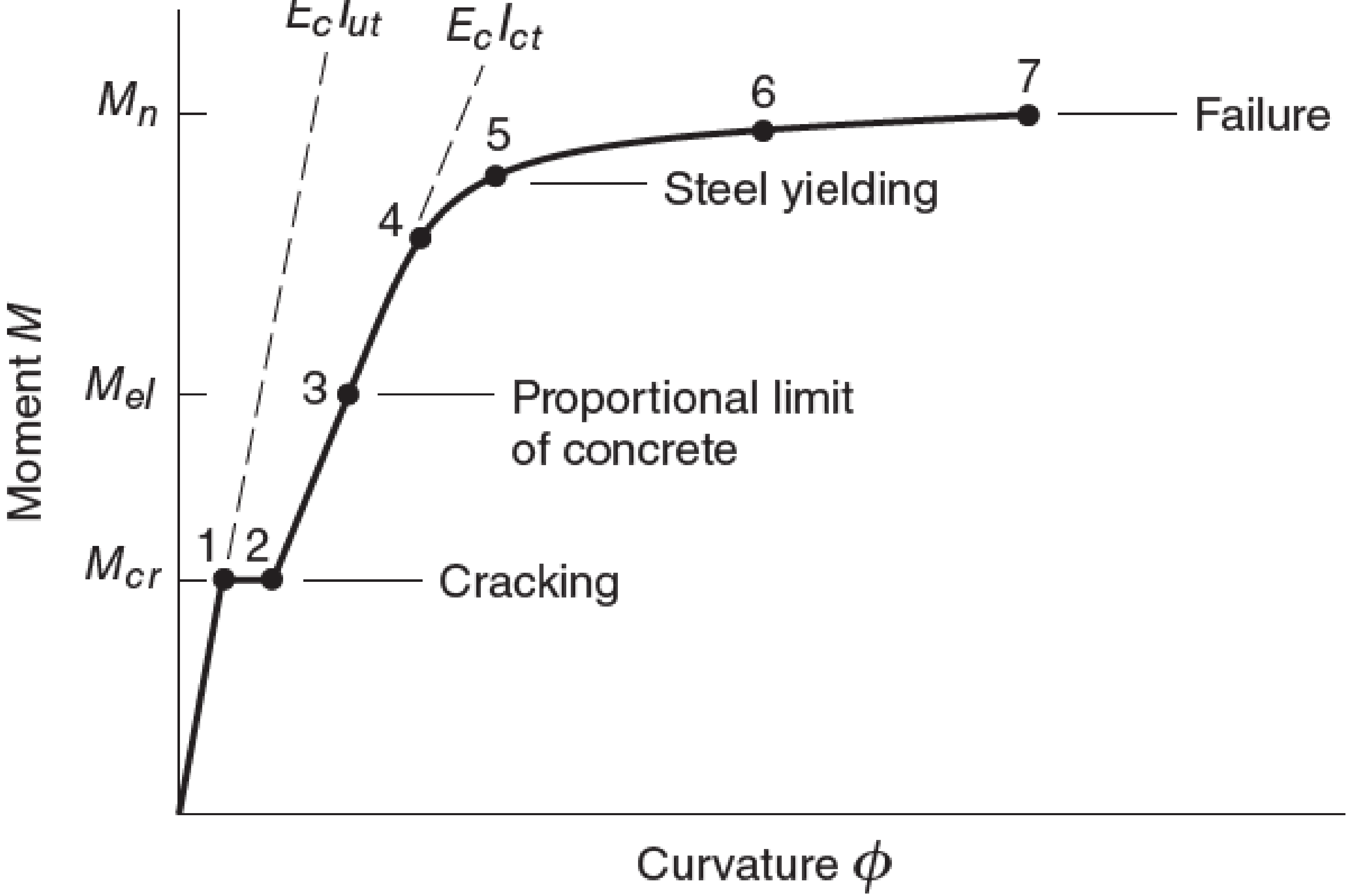


Figure (6): Moment-curvature relation for tensile-reinforced beam.

Plastic Hinges and Collapse Mechanisms:

- If a short segment of a reinforced concrete beam is subjected to a bending moment, curvature of the beam axis will result, and there will be a corresponding rotation of one face of the segment with respect to the other.
- It is convenient to express this in terms of an angular change per unit length of the member.
- The relation between moment and angle change per unit length of beam, or curvature, at a reinforced concrete beam section subject to tensile cracking was developed previously.
- Methods were presented there by which the theoretical moment-curvature graph might be drawn for a given beam cross section, as in Fig. 6.

- The actual moment-curvature relationship measured in beam tests differs somewhat from that shown in Fig. 6.
- The difference is mainly because, from tests, curvatures are calculated from average strains measured over a finite gage length, usually about equal to the effective depth of the beam.
- In particular, the sharp increase in curvature upon concrete cracking shown in Fig. 6 is not often seen because the crack occurs at only one discrete location along the gage length.
- Elsewhere, the uncracked concrete shares in resisting flexural tension, resulting in what is known as tension stiffening, which tends to reduce curvature.
- Furthermore, the exact shape of the moment-curvature relation depends strongly upon the reinforcement ratio as well as upon the exact stress-strain curves for the concrete and steel.

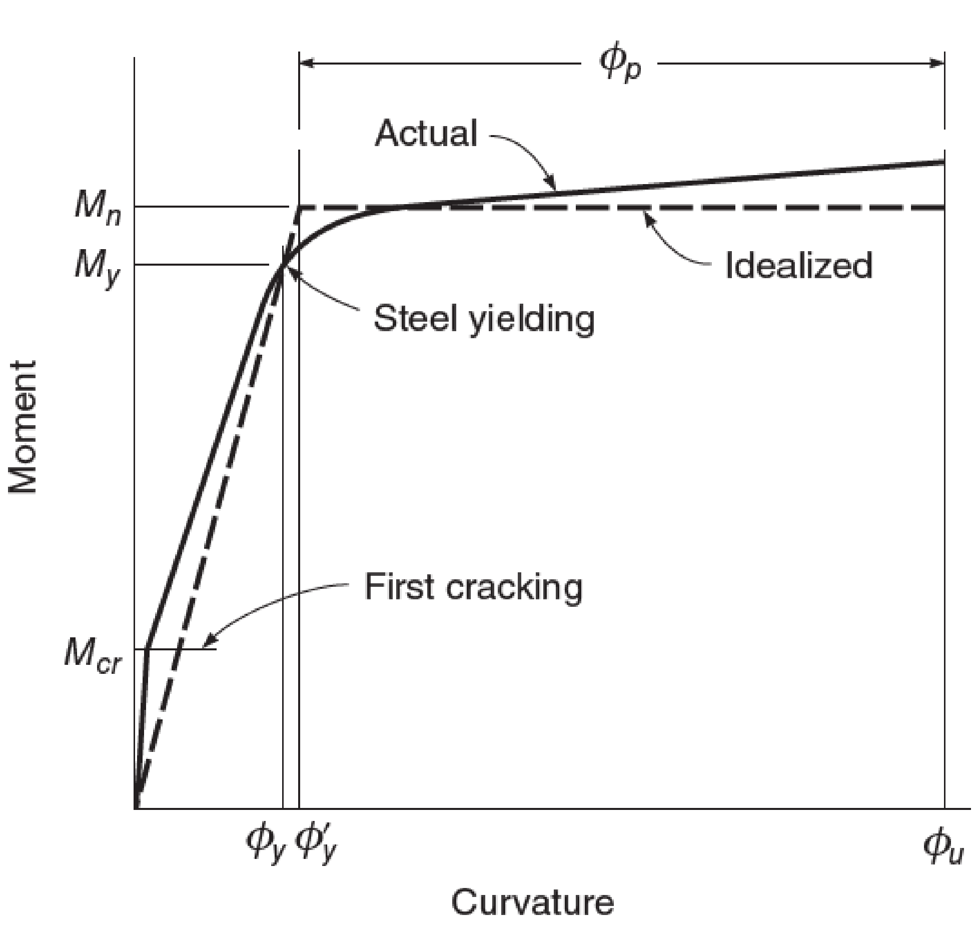


Figure (7): A simplified moment-curvature diagram for an actual concrete beam section having a tensile reinforcement ratio equal to about one half the balanced value.

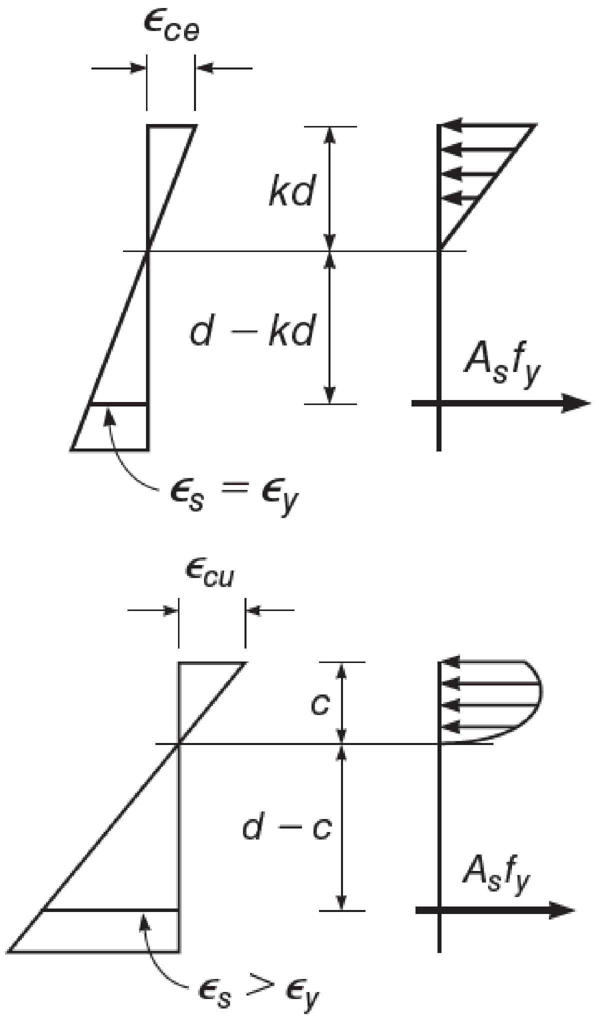


Figure (8): Strains and stresses at initiation of yield and at incipient failure.

- The diagram shown in Fig. 7 is linear up to the cracking moment M_{cr} , after which a nearly straight line of somewhat flatter slope is obtained.
- At the moment that initiates yielding M_y, the curvature starts to increase disproportionately.
- Further increase in applied moment causes extensive inelastic rotation until, eventually, the compressive strain limit of the concrete is reached at the ultimate rotation $\emptyset_u$.
- The maximum moment is often somewhat above the calculated flexural strength M_n , due largely to strain hardening of the reinforcement.

Moment-curvature relation in limit analysis:

- The nominal moment capacity M_n , based on Fig. 8, is calculated by the usual expression

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) = A_s f_y \left(d - \frac{\beta_1 c}{2}\right)$$

- For purposes of limit analysis, the M – ϕ curve is usually idealized, as shown by the dashed line in Fig. 7.
- The slope of the elastic portion of the curve is obtained with satisfactory accuracy using the moment of inertia of the cracked transformed section.
- After the nominal moment M_n is reached, continued plastic rotation is assumed to occur with no change in applied moment.
- The elastic curve of the beam will show an abrupt change in slope at such a section.
- The beam behaves as if there were a hinge at that point.
- However, the hinge will not be “friction-free,” but will have a constant resistance to rotation.

- If such a plastic hinge forms in a determinate structure, uncontrolled deflection takes place, and the structure will collapse.

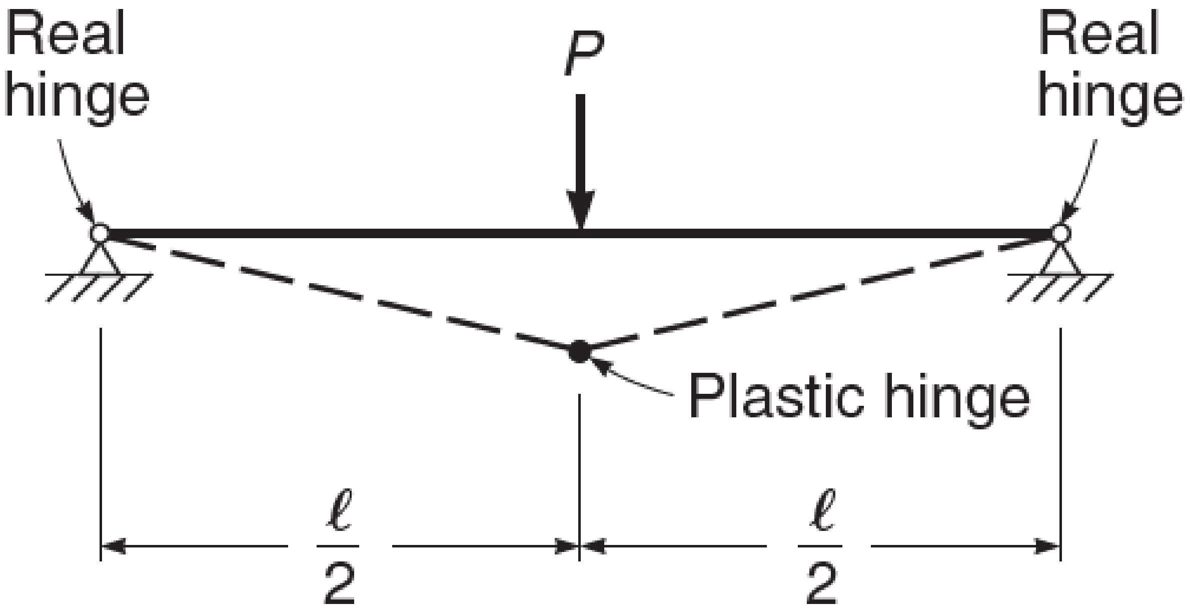


Figure (9): Statically indeterminate member after the formation of plastic hinge.

- The resulting system is referred to as a "**mechanism**", an analogy to linkage systems in mechanics.
- Generalizing, one can say that a statically determinate system requires the formation of only one plastic hinge to become a mechanism.

- This is not so for indeterminate structures, in which, stability may be maintained even though hinges have formed at several cross sections.
- The formation of such hinges in indeterminate structures permits a redistribution of moments within the beam or frame.

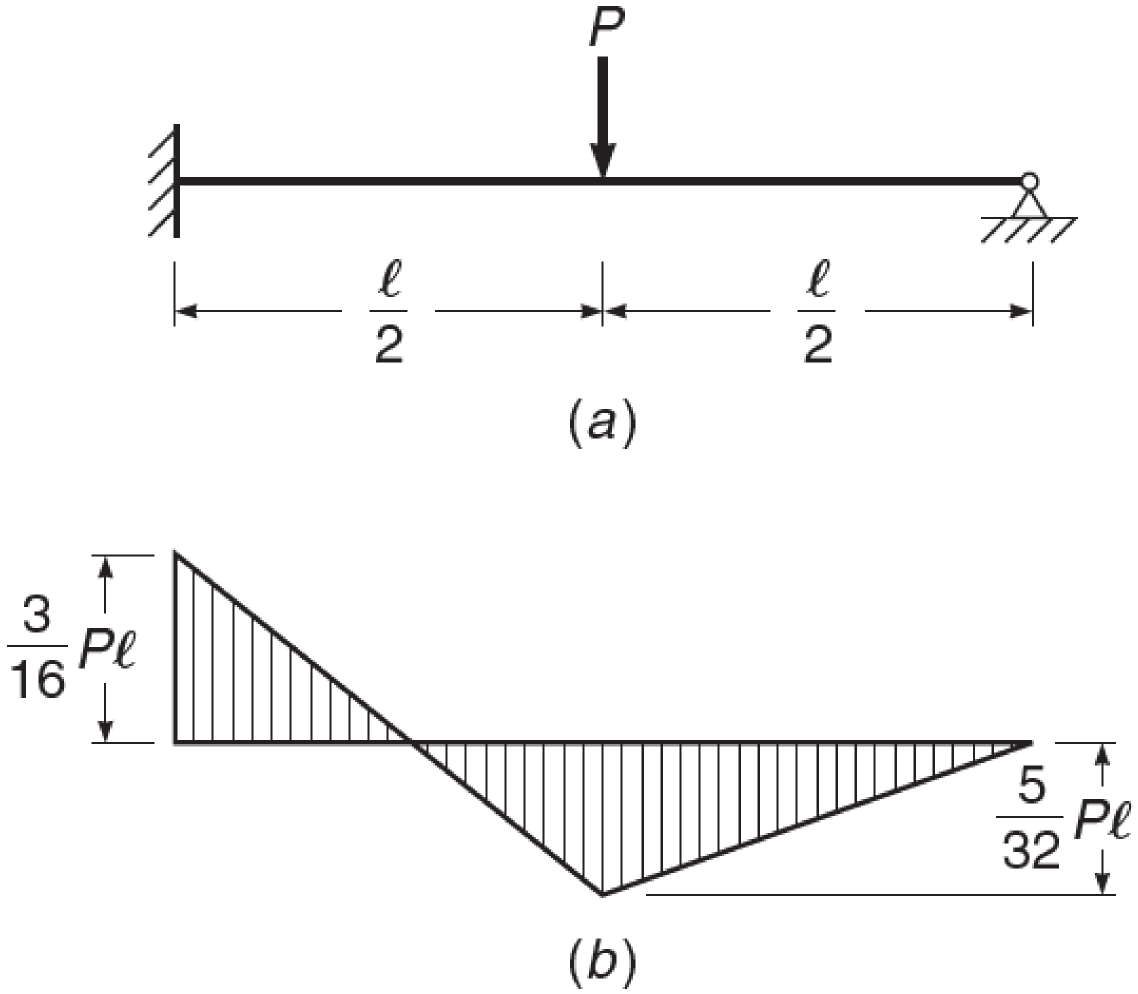


Figure (10)

- Let assume for simplicity that the indeterminate beam of Fig. 10 is symmetrically reinforced, so that the negative bending capacity is the same as the positive.

- Let the load P be increased gradually until the elastic moment at the fixed support, $\frac{3}{16}Pl$, is just equal to the plastic moment capacity of the section M_n .

$$P = P_{el} = \frac{16}{3}\frac{M_n}{\ell} = 5.33\frac{M_n}{\ell} \quad \text{(a)}$$

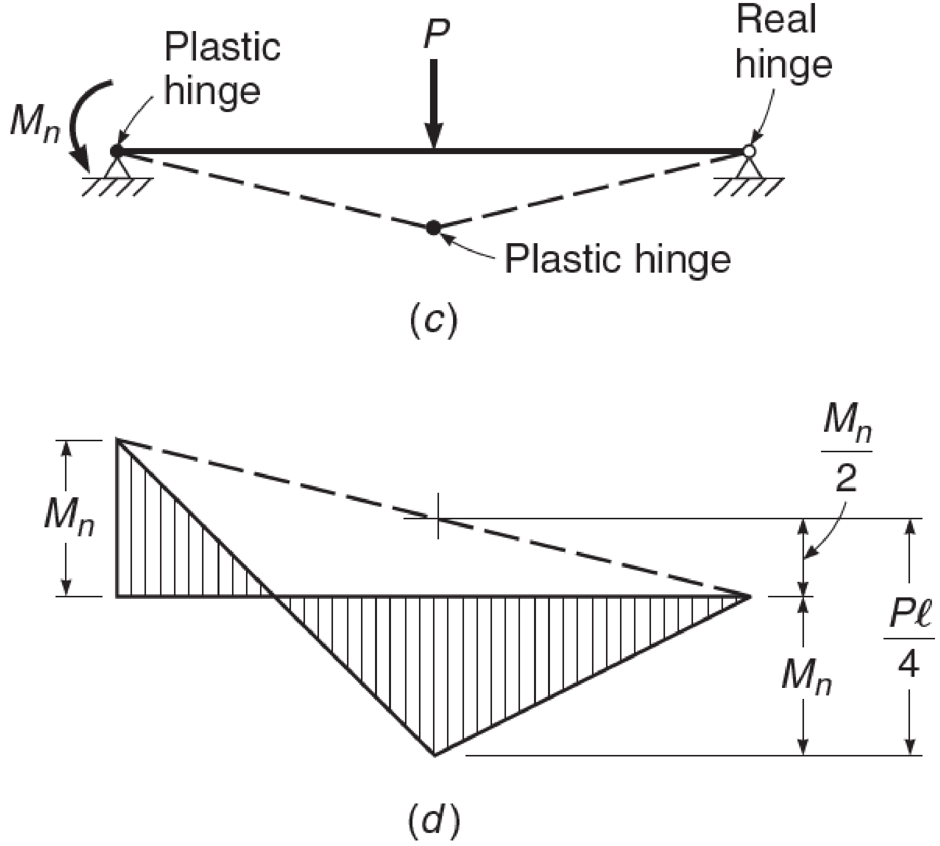


Figure (10): continued

- At this load, the positive moment under the load is $\frac{5}{32}Pl$.
- The beam still responds elastically everywhere but at the left support.
- At that point the actual fixed support can be replaced for purposes of analysis with a plastic hinge offering a known resisting moment M_n .
- Because a redundant reaction has been replaced by a known moment, the beam is now determinate.
- The load can be increased further until the moment under the load also becomes equal to M_n , at which load the second hinge forms.
- The structure is converted into a mechanism, as shown in Fig. 10 c, and collapse occurs.
- The moment diagram at collapse load is shown in Fig. 10 d.

- The magnitude of load causing collapse is easily calculated from the geometry of Fig. 10d:

$$M_n + \frac{M_n}{2} = \frac{P\ell}{4}$$

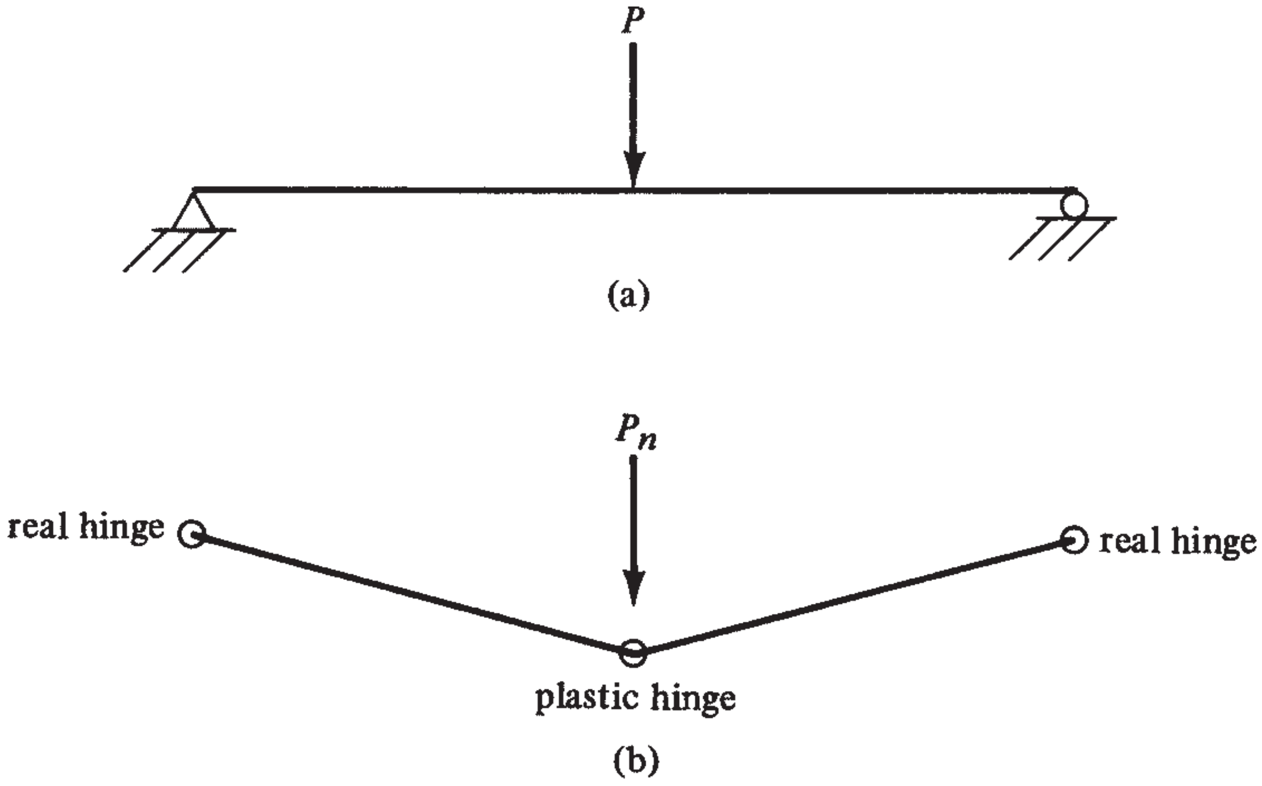
- from whuch

$$P = P_n = \frac{6M_n}{\ell} \qquad \text{(b)}$$

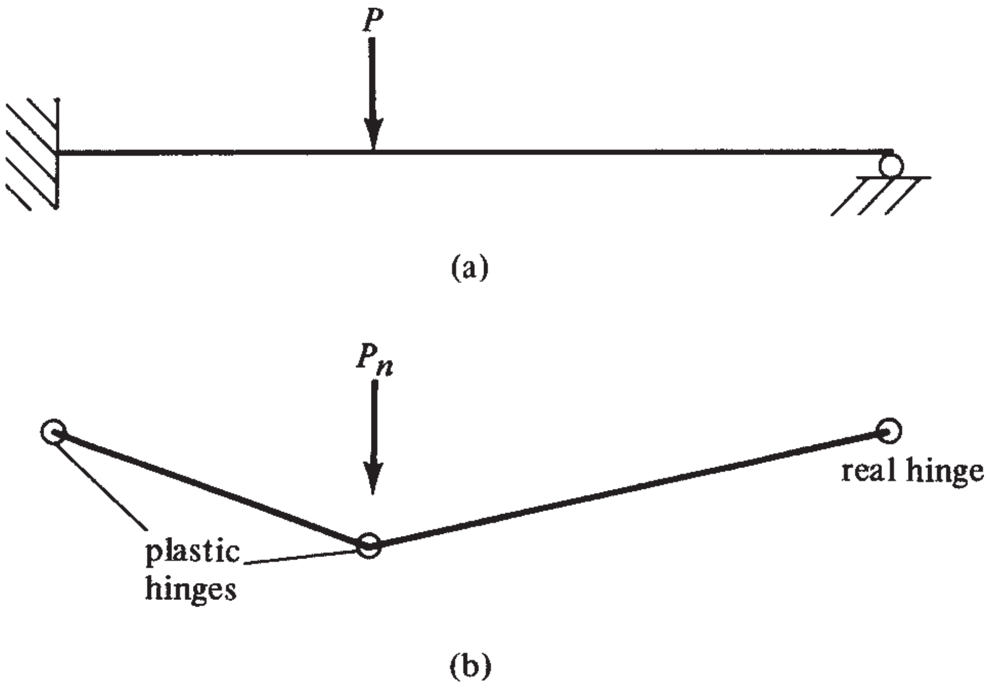
- By comparison of Eqs. (b) and (a), it is evident that an increase in P of 12.5% is possible, beyond the load that caused the formation of the first plastic hinge, before the beam will actually collapse.
- Due to the formation of plastic hinges, a redistribution of moments has occurred such that, at failure, the ratio between the positive moment and negative moment is equal to that assumed in reinforcing the structure.

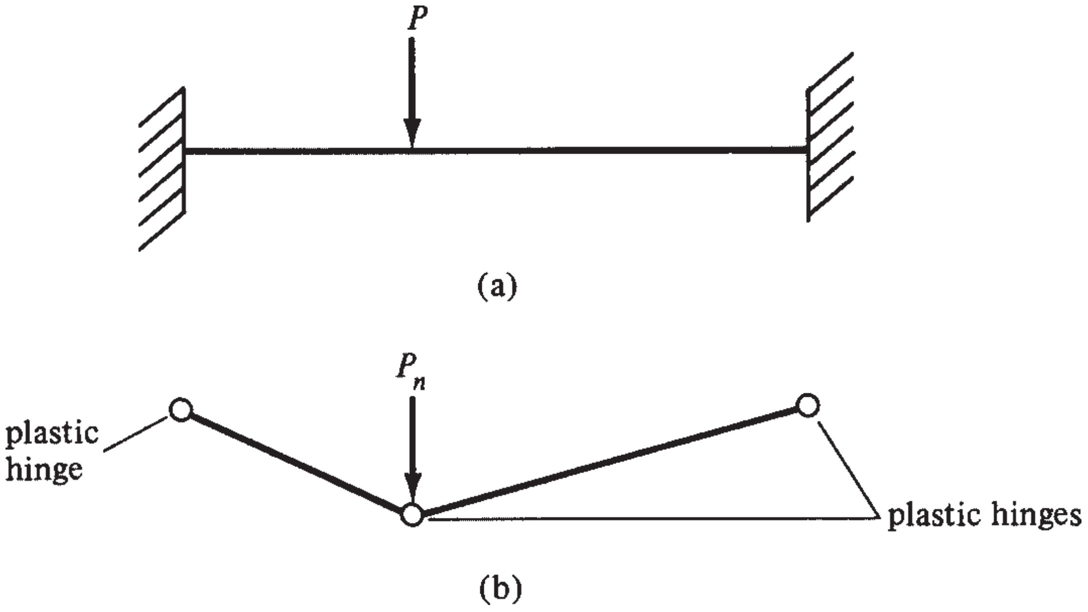
The Collapse Mechanism:

- To understand moment redistribution in steel or reinforced concrete structures, it is necessary first to consider the location and number of plastic hinges required to cause a structure to collapse.
- A statically determinate beam will fail if one plastic hinge develops.
- Should the load be increased until a plastic hinge is developed at the point of maximum moment (underneath the load in this case), an unstable structure will have been created.
- Any further increase in load will cause collapse.



- For a statically indeterminate structure to fail, it is necessary for more than one plastic hinge to form.
- The number of plastic hinges required for failure of statically indeterminate structures will be shown to vary from structure to structure but can never be less than two.
- The propped cantilever can fail after the formation of two plastic hinges.
- The fixed-end beams shown cannot fail unless the three plastic hinges shown in the figure are developed.

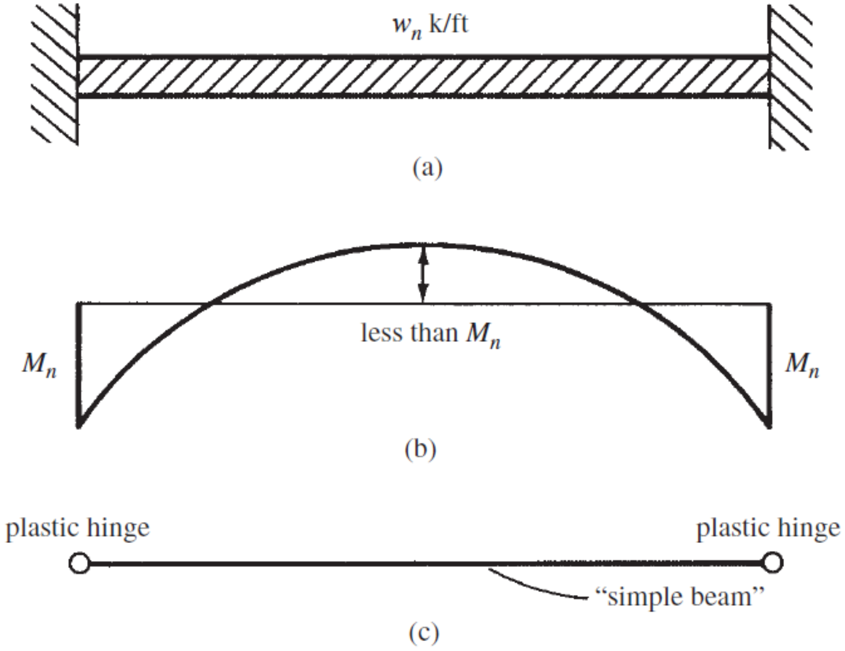


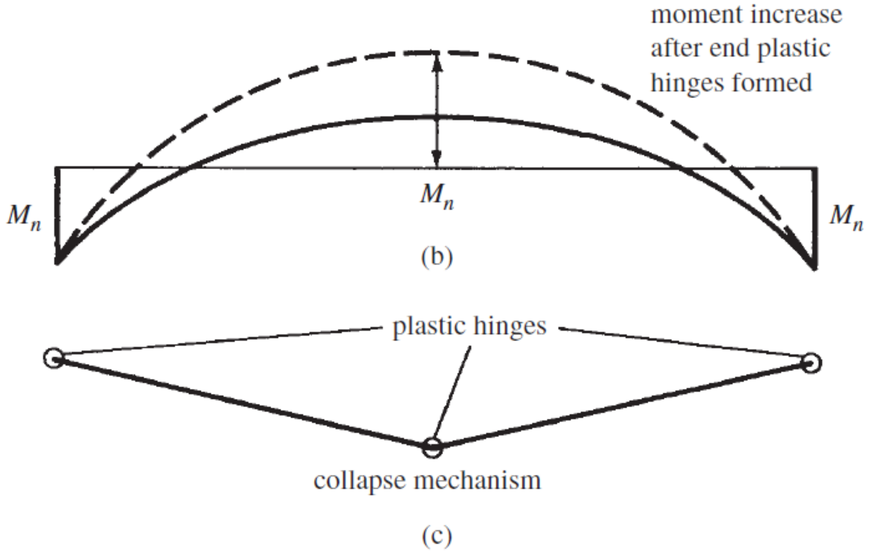


Plastic Analysis by the Equilibrium Method:

- To analyze a structure plastically, it is necessary to compute the plastic or ultimate moments of the sections, to consider the moment redistribution after the ultimate moments develop, and finally, to determine the ultimate loads that exist when the collapse mechanism is created.
- The method of plastic analysis known as the *equilibrium method* will be illustrated in this section.
- The fixed-end beam shown is considered first.
- It is desired to determine the value of w_n , the theoretical ultimate load the beam can support.
- The maximum moments in a uniformly loaded fixed-end beam in the elastic range occur at the fixed ends, as shown in the figure.

- If the magnitude of the uniform load is increased, the moments in the beam will be increased proportionately until a plastic moment eventually develops at the beam ends.
- Should the loads be further increased, the beam will be unable to resist moments larger than M_n at its end points.
- Those points will rotate through large angles, and thus the beam will be permitted to deflect more and allow the moments to increase out in the span.
- Although the plastic moment has been reached at the ends and plastic hinges are formed, the beam cannot fail because it has, in effect, become a simple end-supported beam for further load increases.
- The load can now be increased on this "simple" beam, and the moments at the ends will remain constant; however, the moment out in the span will increase.
- The load can be increased until the moment at midspan point reaches the plastic moment.
- When this happens, a third plastic hinge will have developed and a mechanism will have been created, permitting collapse.





- One method of determining the value of w_n is to take moments at the centerline of the beam (knowing the moment there is M_n at collapse).

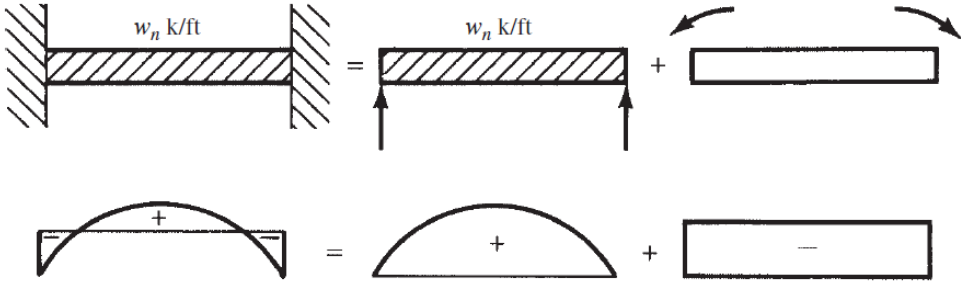
$$M_n = -M_n + \left(w_n \frac{\ell}{2}\right)\left(\frac{\ell}{2} - \frac{\ell}{4}\right) = -M_n + \frac{w_n \ell^2}{8}$$

$$w_n = \frac{16 M_n}{\ell^2}$$

- The same value could be obtained by considering the diagrams.
- You will remember that a fixed-end beam can be replaced with a simply supported beam plus a beam with end moments.
- Thus, the final moment diagram for the fixed-end beam equals the moment diagram if the beam had been simply supported plus the end moment diagram.

$$2M_n = \frac{w_n \ell^2}{8}$$

$$M_n = \frac{w_n \ell^2}{16}$$



Rotation Requirements:

- There is a direct relation between the amount of redistribution desired and the amount of inelastic rotation.
- In general, the greater the modification of the elastic moment ratio, the greater the required rotation capacity to accomplish that change.
- To illustrate, if the beam of Fig. 10-a had been reinforced according to the elastic moment diagram of Fig. 10-b, no inelastic-rotation capacity at all would be required.
- The beam would, at least in theory, yield simultaneously at the left support and at midspan.
- On the other hand, if the reinforcement at the left support had been deliberately reduced (and the midspan reinforcement correspondingly increased), inelastic rotation at the support would be required before the strength at midspan could be realized.

- The amount of rotation required at plastic hinges for any assumed moment diagram can be found by considering the requirements of compatibility.
- The member must be bent, under the combined effects of elastic moment and plastic hinges, so that the correct boundary conditions are satisfied at the supports.
- Usually, zero support deflection is to be maintained.
- Moment-area and conjugate-beam principles are useful in quantitative determination of rotation requirements.
- In deflection calculations, it is convenient to assume that plastic hinging occurs at a point, rather than being distributed over a finite hinging length, as is actually the case.
- Consequently, in loading the conjugate beam with unit rotations, plastic hinges are represented as concentrated loads.

- Calculation of rotation requirements will be illustrated by the two-span continuous beam shown in Fig. 11-a.
- The elastic moment diagram resulting from a single concentrated load is shown in Fig. 11-b.
- The moment at support B is 0.096 Pℓ, while that under the load is 0.182 Pℓ. If the deflection of the beam at support C were calculated using the unit rotations equal to M/EI, based on this elastic moment diagram, a zero result would be obtained.

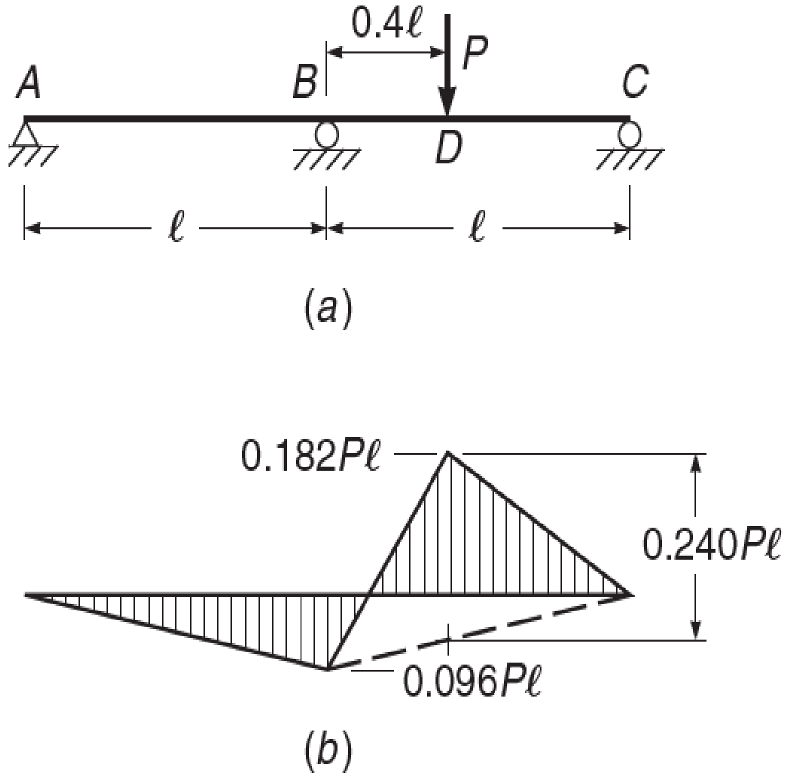


Figure (11)

- Figure 11-c shows an alternative, statically admissible moment diagram that was obtained by arbitrarily increasing the support moment from 0.096 Pℓ to 0.150 Pℓ.
- If the beam deflection at C were calculated using this moment diagram as a basis, a nonzero value would be obtained.
- This indicates the necessity for inelastic rotation at one or more points to maintain geometric compatibility at the right support.

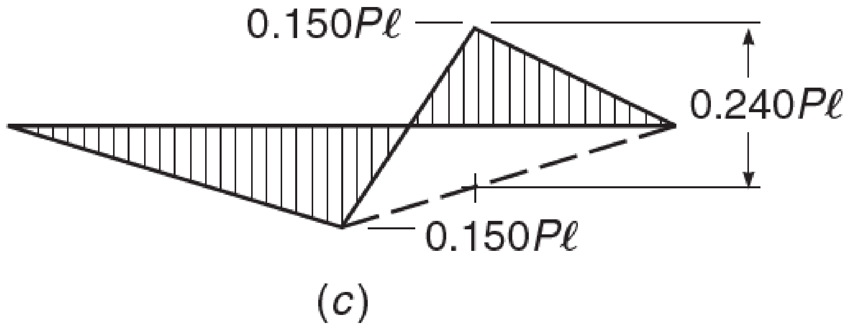


Figure (11): continued

- If the beam were reinforced according to Fig. 11-c, increasing loads would produce the first plastic hinge at D, where the beam has been deliberately made understrength.
- Continued loading would eventually result in formation of the second plastic hinge at B, creating a mechanism and leading to collapse of the structure.
- Limit analysis requires calculation of rotation at all plastic hinges up to, but not including, the last hinge that triggers actual collapse.
- Figure 11-d shows the M/EI load to be imposed on the conjugate beam of Fig. 11-e.
- Also shown is the concentrated angle change θ_d , which is to be evaluated.

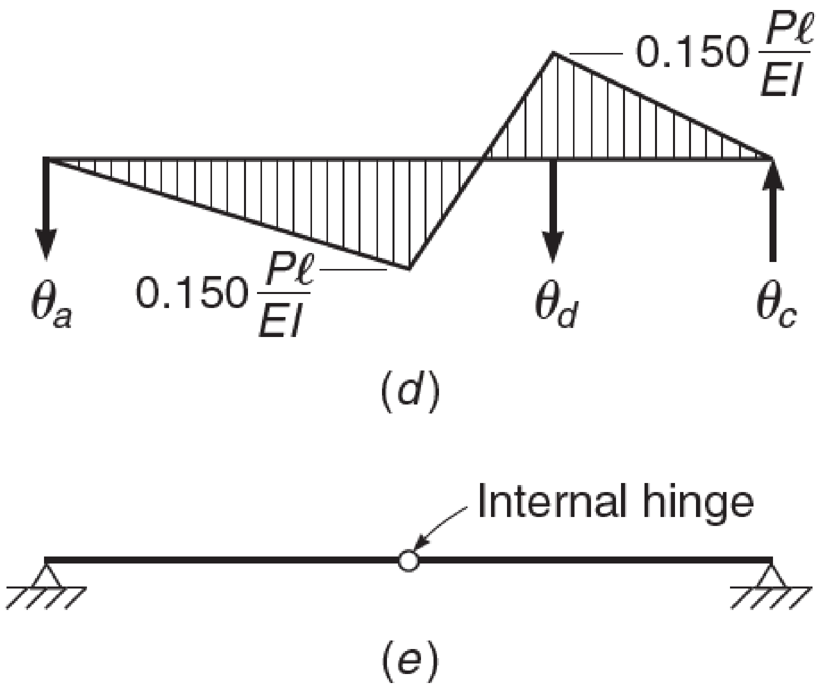


Figure (11): continued

- Starting with the left span, taking moments of the M/EI loads about the internal hinge of the conjugate beam at B, one obtains the left reaction of the conjugate beam (equal to the slope of the real beam):

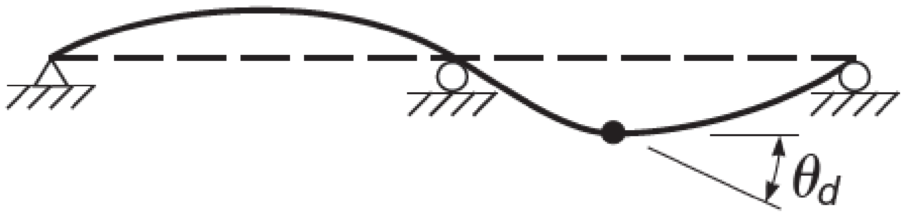
$$\theta_a = 0.025 \frac{P\ell^2}{EI}$$

- With that reaction known, moments are taken about the support C of the conjugate beam and set equal to zero to obtain:

$$\theta_d = 0.060 \frac{P\ell^2}{EI}$$

- This represents the necessary discontinuity in the slope of the elastic curve shown in Fig. 11-f to restore the beam to zero deflection at the right support.
- The beam must be capable of developing at least that amount of plastic rotation if the modified moment diagram assumed in Fig. 11-c is to be valid.

Figure (11): continued



(f)

Rotation Capacity:

- The capacity of concrete structures to absorb inelastic rotations at plastic-hinge locations is not unlimited.
- The designer adopting full limit analysis in concrete must calculate not only the amount of rotation required at critical sections to achieve the assumed degree of moment redistribution but also the rotation capacity of the members at those sections to ensure that it is adequate.

- Curvature at initiation of yielding is easily calculated from the elastic strain distribution shown in figure (8-a),

$$\phi_y = \frac{\epsilon_y}{d(1-k)} \qquad (3)$$

in which the ratio k establishing the depth of the elastic neutral axis.

- The curvature corresponding to the nominal moment can be obtained from the geometry of Fig. 8-b:

$$\phi_u = \frac{\epsilon_{cu}}{c} \qquad (4)$$

- Although it is customary in flexural strength analysis to adopt ε_{cu} = 0.003, for purposes of limit analysis a more refined value is needed.

- Extensive experimental studies indicate that the ultimate strain capacity of concrete is strongly influenced by:

a) the beam width b,

b) the moment gradient, and

c) the presence of additional reinforcement in the form of compression steel and confining steel (that is, web reinforcement).

- The last parameter is conveniently introduced by means of a reinforcement ratio ρ”, defined as the ratio of the volume of one stirrup plus the compressive steel volume within the concrete volume along the length of the member corresponding to one stirrup.
- On the basis of empirical studies, the ultimate flexural strain at a plastic hinge is

$$\epsilon_{cu} = 0.003 + 0.02\frac{b}{z} + \left(\frac{\rho'' f_y}{14.5}\right)^2 \qquad (5)$$

where z is the distance between points of maximum and zero moment

- Based on the equations (3) to (5), the inelastic curvature for the idealized relation shown in Fig. 7 is

$$\phi_p = \phi_u - \phi_y \frac{M_n}{M_y} \tag{6}$$

- This plastic rotation is not confined to one cross section but is distributed over a finite length referred to as the hinging length.
- The experimental studies upon which Eq. (5) is based measured strains and rotations in a length equal to the effective depth d of the test members.
- Consequently, ε_{cu} is an average value of ultimate strain over a finite length, and $\emptyset_p$, given by Eq. (6), is an average value of curvature.
- The total inelastic rotation θ_p can be found by multiplying the average curvature by the hinging length:

$$\theta_p = \left(\phi_u - \phi_y \frac{M_n}{M_y}\right) \ell_p \tag{7}$$

- On the basis of current evidence, it appears that the hinging length l_p in support regions, on either side of the support, can be approximated by the expression

$$\ell_p = 0.5d + 0.05z \qquad (8)$$

- in which z is the distance from the point of maximum moment to the nearest point of zero moment.

Moment Redistribution under the ACI Code:

- A limited amount of redistribution is permitted by ACI Code 6.6.5, depending upon a rough measure of available ductility, without explicit calculation of rotation requirements and capacities.

- The net tensile strain in the extreme tension steel at nominal strength ε_t , defined as

$$\epsilon_t = \epsilon_u \frac{d_t - c}{c} \qquad (9)$$

is used as an indicator of rotation capacity.

Limitations of ACI to redistribution of moments:

- According to ACI Code 6.6.5, reduction of moments at sections of maximum negative or maximum positive moment calculated by elastic theory shall be permitted for any assumed loading arrangement if (a) and (b) are satisfied:

(a) Flexural members are continuous;

(b) $\varepsilon_t \geq 0.0075$ at the section at which moment is reduced.

- Redistribution shall not exceed the lesser of $(1000\ \varepsilon_t)\%$ and 20%.
- The reduced moment shall be used to calculate redistributed moments at all other sections within the spans such that static equilibrium is maintained after redistribution of moments for each loading arrangement.
- Shears and support reactions shall be calculated in accordance with static equilibrium considering the redistributed moments for each loading arrangement.

Cases in which redistribution of moments is not permitted:

- According to ACI 6.6.5, the redistribution of moments is not allowed in the following cases:

1. Where approximate values for moments are used in accordance with ACI Code 6.5 (ACI moment coefficients)
2. where moments have been calculated in accordance with ACI Code 6.8 (inelastic second-order analysis), or
3. where moments in two-way slabs are determined using pattern loading specified in ACI Code 6.4.3.3 (using 75 percent of the factored live load).

- Redistribution for values of ε_t < 0.0075 is conservatively prohibited.
- The ACI Code provisions are shown graphically in Fig 12.
- The value of ρ corresponding to a given value of ε_t, and thus a given percentage change in moment, can be calculated using Eq. (10) below.

$$\rho = 0.85\beta_1 \frac{f'_c}{f_y} \frac{d_t}{d} \frac{\epsilon_u}{\epsilon_u + \epsilon_t} \qquad (10)$$

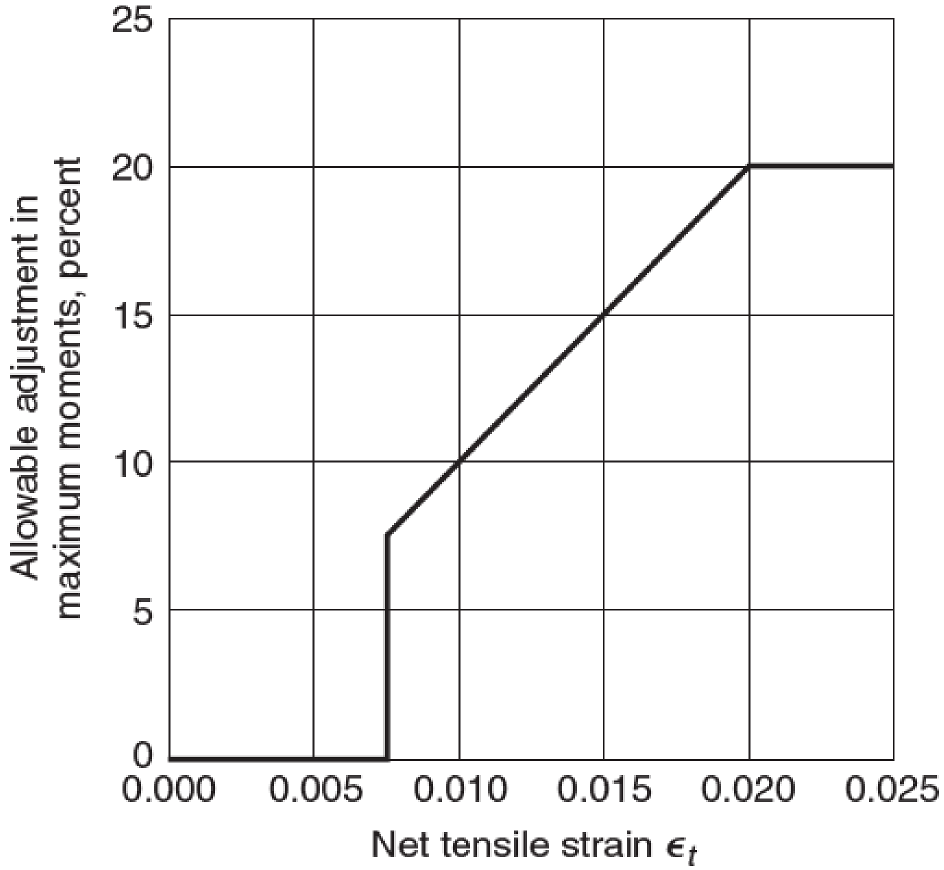
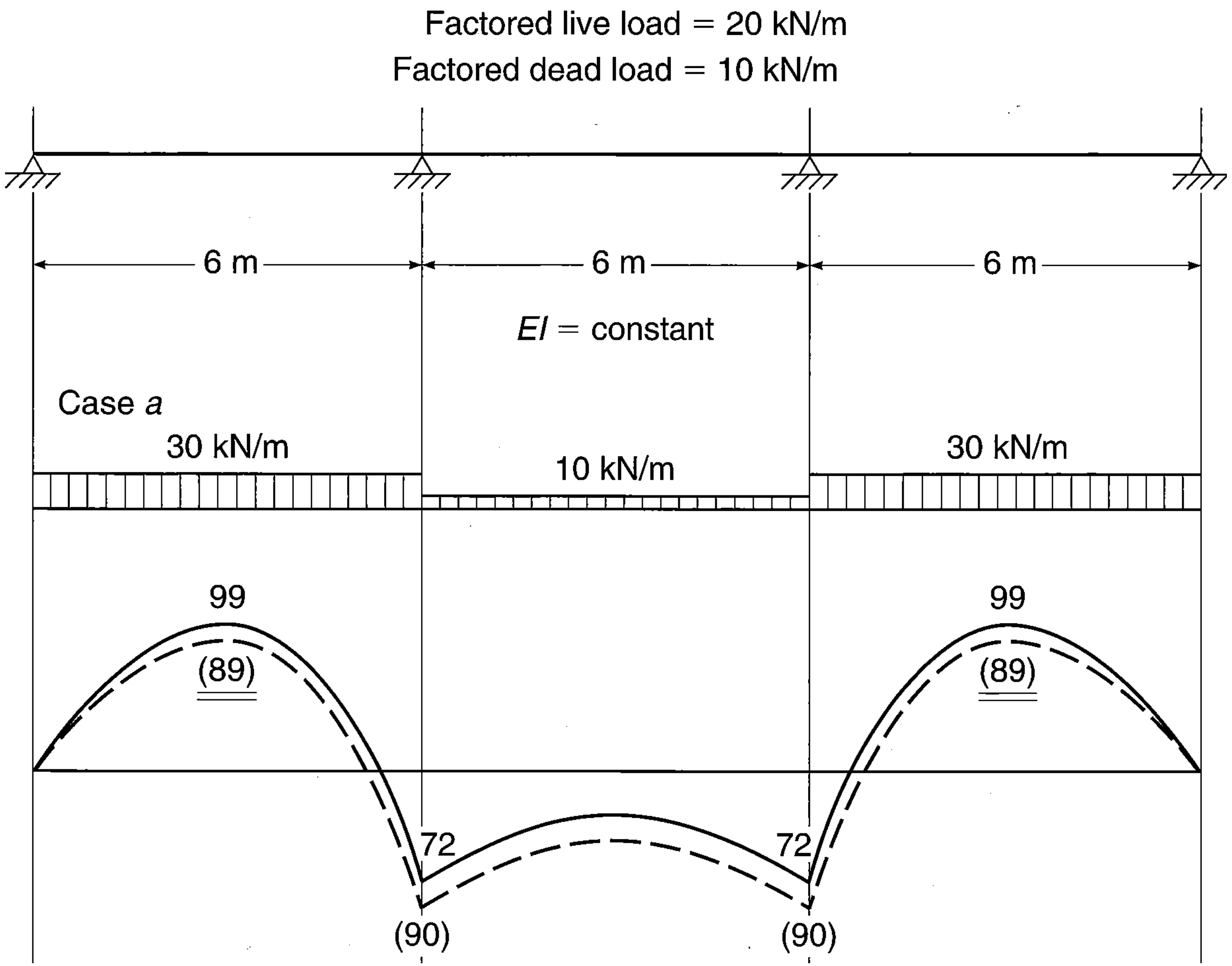
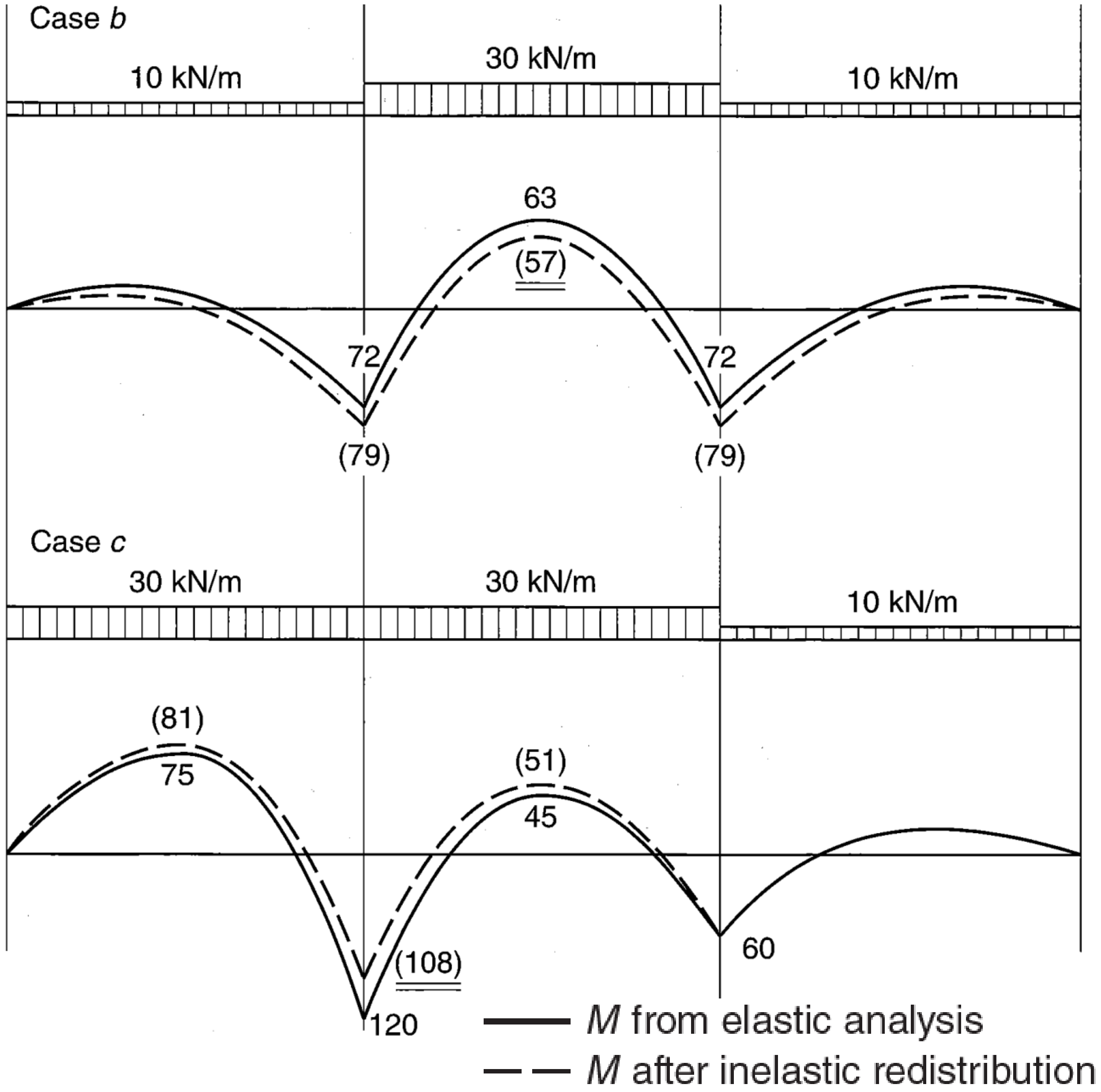


FIGURE 12: Allowable moment redistribution under the ACI Code.

- To demonstrate the advantage of moment redistribution when alternative loadings are involved, consider the continuous concrete beam.
- The beams has three spans.
- The dead load is 10 kN/m.
- The live load is 20 kN/m.
- To obtain maximum moments at all critical design sections, it is necessary to consider three alternative loadings.
- Case *a*, with live and dead load over exterior spans and dead load only over the interior span, will produce the maximum positive moment in the exterior spans.



- Case *b*, with dead load on exterior spans and dead and live load on the interior span, will produce the maximum positive moment in the interior span.
- The maximum negative moment over the interior support is obtained by placing dead and live load on the two adjacent spans and dead load only on the far exterior span, as shown in case *c*.



- It will be assumed for simplicity that a 10 percent adjustment of maximum negative and positive moments is permitted throughout.
- The other span moments are modified accordingly.
- An overall reduction in design moments through the entire three-span beam may be possible.
- Case *a*, for example, produces an elastic maximum span moment in the exterior spans of 99 kN.m.
- Corresponding to this is an elastic negative moment of 72 kN.m at the interior support.
- Adjusting the maximum positive moment downward by 10 percent, one obtains a positive moment of 89 kN.m, which results in an upward adjustment of the support moment to 90 kN.m.

Summery:

- The subject discusses the concepts of redistribution of moments and limit analysis in reinforced concrete structures. It highlights the importance of recognizing moment redistribution for improved structural economy and reduced reinforcement congestion. The text also differentiates between limit design in reinforced concrete and plastic design in steel structures, emphasizing the role of plastic hinges and the moment-curvature relationship in structural analysis. It further explains the conditions under which moment redistribution is permitted by the ACI Code and the limitations imposed.

Redistribution of moments

- Recognizing moment redistribution is crucial for realistic load-carrying capacity and improved economy.
- Redistribution allows modification of moment diagrams, reducing reinforcement congestion.
- Plastic hinges form at critical sections, allowing load transfer and moment redistribution before collapse.

Limit Analysis:

- Reinforced concrete structures are designed using elastic theory but proportioned by strength methods.
- Elastic methods yield satisfactory results despite being inconsistent with inelastic behavior.
- Plastic hinges form at critical sections, allowing moment redistribution and preventing premature failure.
- Redistribution of moments permits a more realistic appraisal of load-carrying capacity and reduces reinforcement congestion.

Motivations of using Limit Analysis

- Inconsistency in the current analysis-design process.
- Utilizing reserve strength from moment redistribution.
- Reducing steel congestion in critical regions.
- ACI Code 6.6.5 permits limited redistribution of moments based on tensile steel strain.
- Limit design methods are suitable for slabs with low reinforcement ratios.

Limit design versus Plastic design

- Plastic design in steel involves increased resisting moment and moment redistribution.
- Limit design in reinforced concrete focuses on moment redistribution after steel yields.

Moment vs. Curvature for Reinforced Concrete Sections

- Moment-curvature relationship is essential for studying ductility, plastic hinges, and moment redistribution.
- Curvature is the angle change per unit length due to bending.
- Stress-strain relationships for steel and concrete are idealized.
- Different stages of loading: un-cracked, cracked elastic, and cracked inelastic.

Plastic Hinges and Collapse Mechanisms

- Plastic hinges allow rotation and moment redistribution, preventing premature collapse.
- Indeterminate structures require multiple plastic hinges for collapse.
- Redistribution of moments occurs as plastic hinges form, altering the moment pattern.

Plastic Analysis by the Equilibrium Method

- Plastic analysis involves computing plastic moments, moment redistribution, and ultimate loads.
- The equilibrium method determines the ultimate load by considering moment redistribution and plastic hinge formation.
- Rotation requirements depend on the desired moment redistribution and inelastic rotation capacity.

Rotation Capacity

- Concrete structures have limited inelastic rotation capacity at plastic hinges.
- Rotation capacity must be calculated to ensure adequate moment redistribution.
- Curvature and ultimate strain are influenced by beam width, moment gradient, and additional reinforcement.

Moment Redistribution under the ACI Code

- ACI Code 6.6.5 permits limited redistribution based on tensile strain in the extreme tension steel.
- Redistribution is limited to the lesser of $(1000\,\varepsilon_t)\%$ and 20%.

Cases in which redistribution of moments is not permitted

- Redistribution is not allowed with approximate moment values, inelastic second-order analysis, or specific two-way slab loading patterns.
- Redistribution is prohibited for $\varepsilon_t < 0.0075$.

Strut and Tie model

University of Basrah – college of Engineering

Department of civil Engineering

Master of Science Course - STRUCTURES

Dr. Abdulamir Atalla Karim - 2024

<u>References:</u>

1. design of concrete structure by Nilson ch.10

2. Reinforced Concrete Mechanics and Design by JAMES K. WIGHT and JAMES G. MACGREGOR. chapter 17.

3. Design of Reinforced Concrete by McCormac Appendix C

4. ACI reinforced concrete design handbook Part 2.

INTRODUCTION:

- Reinforced concrete beam theory is based on:

1) Equilibrium,
2) Compatibility, and
3) The constitutive behavior of the materials, steel and concrete.

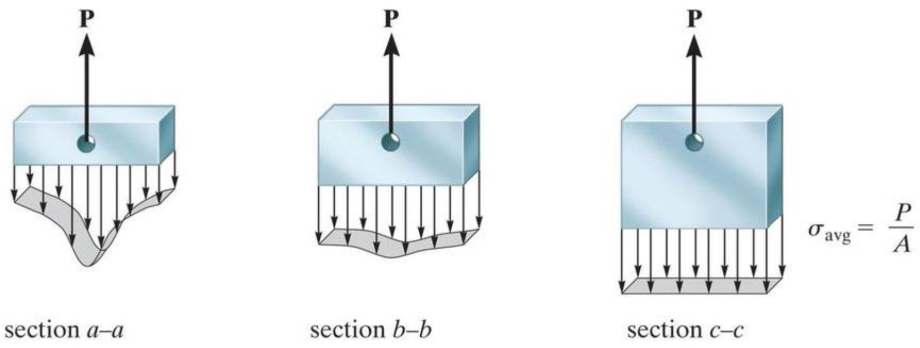
- Of particular importance is the assumption that **strain varies linearly** through the depth of a member and that, as a result, **plane sections remain plane**.
- This assumption is validated by St. Venant's principle, which stipulates that strains induced by discontinuities in load or in member cross section vary in an approximately linear fashion at distances greater than or equal to the greatest cross-sectional dimension *h* from the point of load application.

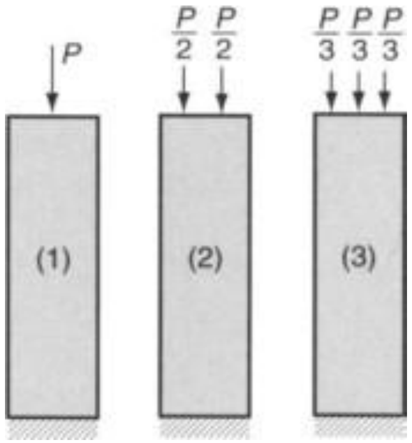
• **Venant’s Principle states that:**

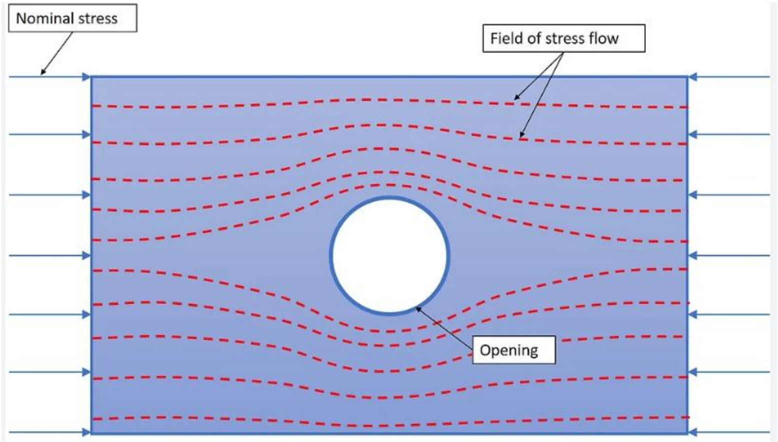
The stresses and strains in a body at points that are sufficiently remote from points of application of load depends only on the static resultant of the loads and not on the distribution of loads.

Saint-Venant’s Principle

If load is replaced by a statically equivalent load, only localized region is affected. At a distance larger than D, the stress and strain distributions remain the same.







- St. Venant's principle, however, does not apply at points closer than the distance *h* to discontinuities in applied load or geometry.
- This leads to the identification of *discontinuity regions* within reinforced concrete members near concentrated loads, openings, or changes in cross section.
- Because of their geometry, the full volume of deep beams and column brackets qualify as discontinuity regions.
- Thus, reinforced concrete structures may be divided into regions where beam theory is valid, often referred to as *B-regions*, and regions where discontinuities affect member behavior, known as *D-regions*.
- A number of D-regions are illustrated in Figure 1.

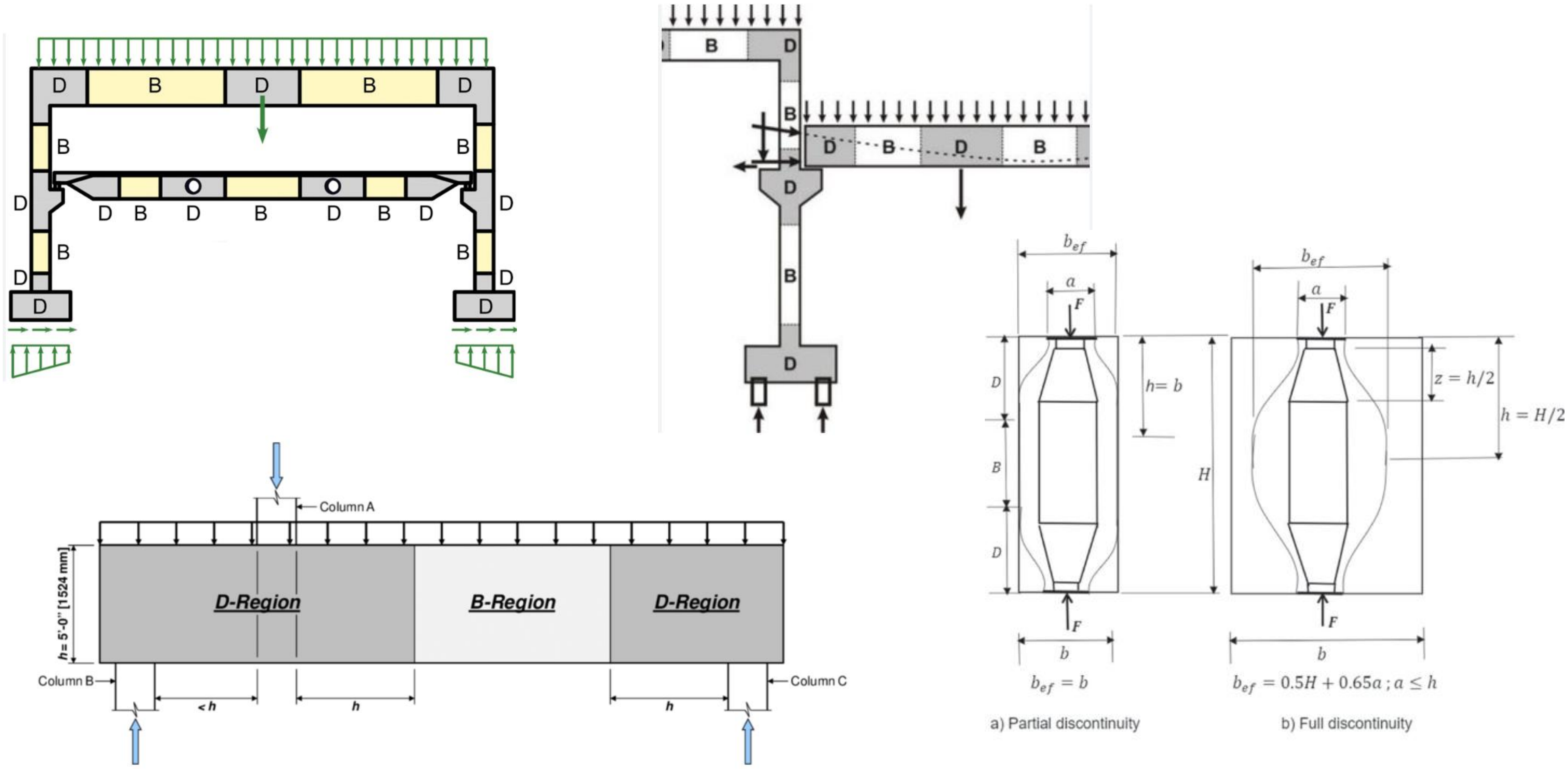
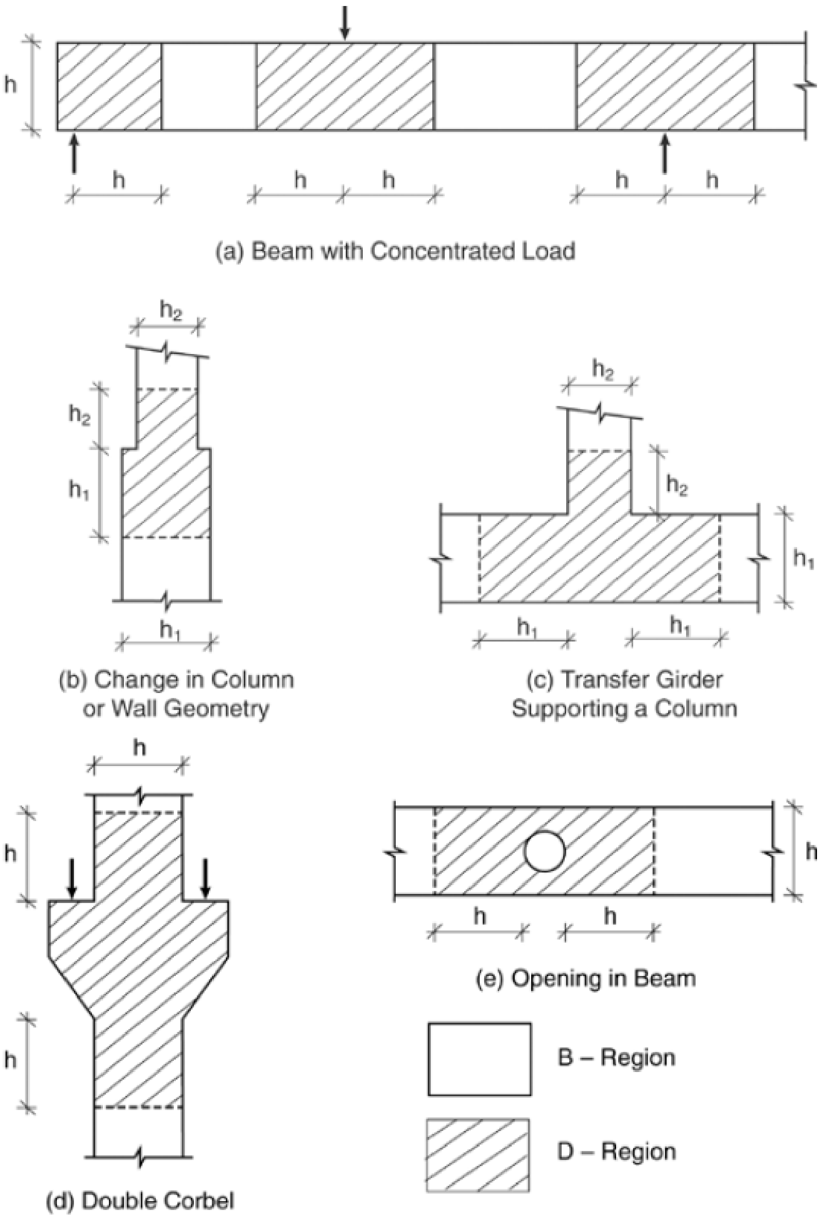


Figure (1): Geometric and load discontinuities for D-regions.



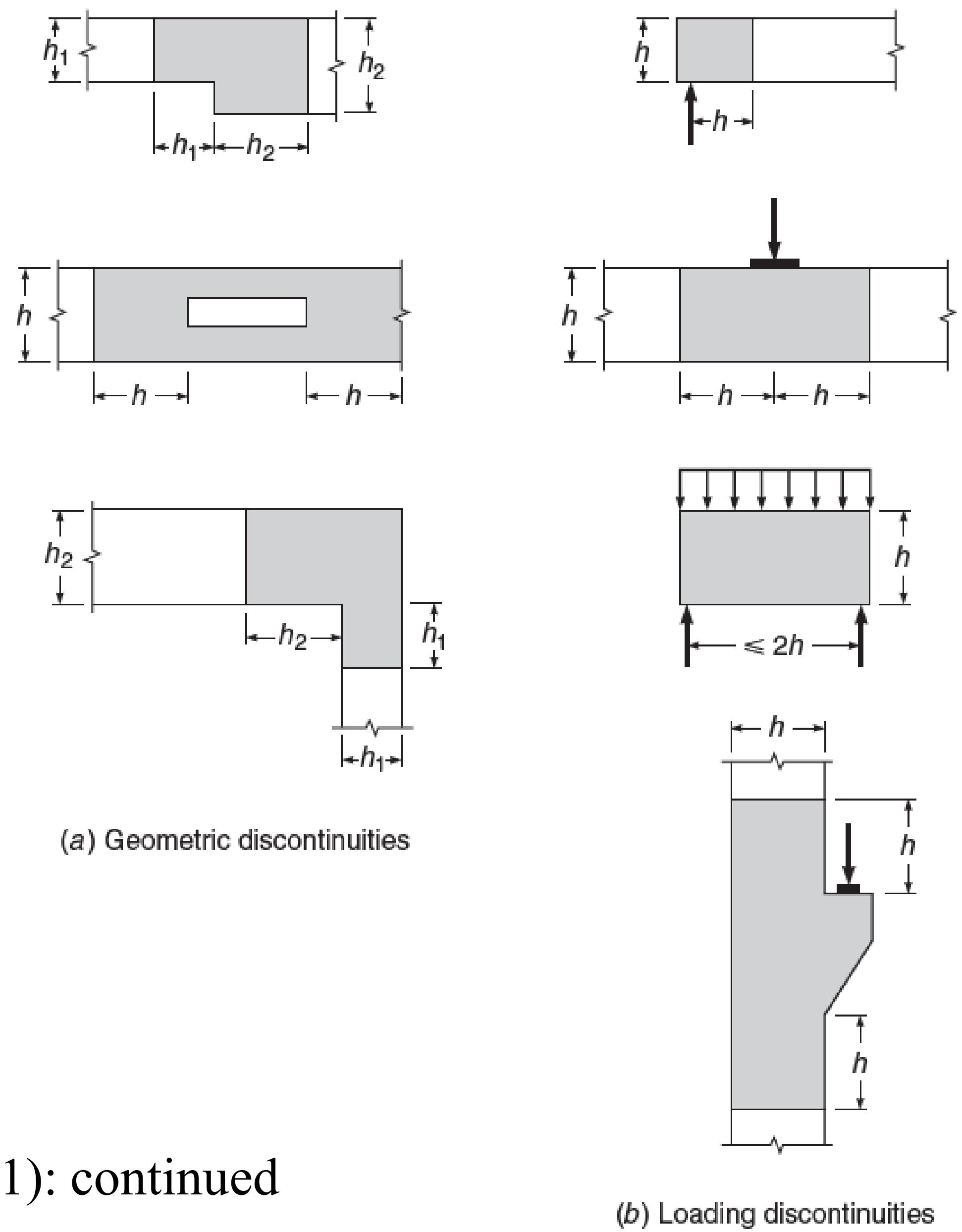


Figure (1): continued

- At low stresses, when the concrete is elastic and uncracked, the stresses within D-regions may be computed using finite element analysis or elasticity theory.
- When concrete cracks, the strain field is disrupted, causing a redistribution of the internal forces.
- Once this happens, it is possible to represent the internal forces within discontinuity regions using a statically determinate truss, referred to as a *strut-and-tie model*.
- This allows a complex design problem to be greatly simplified, producing a safe solution that satisfies statics.

Strut-and-tie model (STM):

- Strut-and-tie model (STM) is a tool for analysis, design, and detailing of structural concrete members.
- It is essentially a truss analogy.
- It is based on the fact that concrete is strong in compression and steel is strong in tension.
- Truss members that are in compression are made up of concrete, while truss members that are in tension consist of steel reinforcement.

- As shown in Fig. 2, strut-and-tie models consist of concrete compression *struts*, steel tension *ties*, and joints that are referred to as *nodal zones* (for consistency of presentation, struts are represented by dashed lines and ties are represented by solid lines).

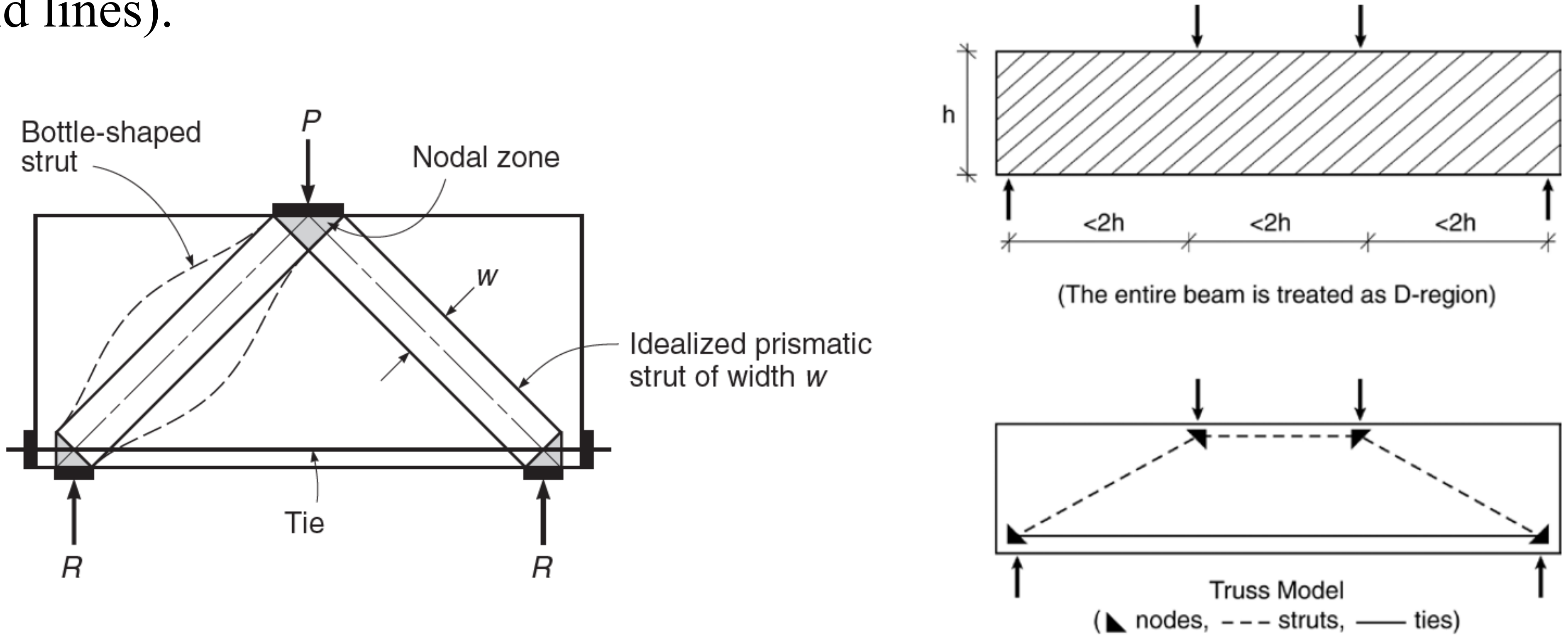


Figure (2): Strut-and-Tie model

B-regions and D-regions:

- STM calls for distinction of two types of zones in a concrete component depending on the characteristics of stress fields at each location.
- Thus, structural members are divided into B-regions and D-regions.
- B-regions represent portions of a member in which the "plane section" assumption of the classical beam theory can be applied with a sectional design approach.
- D-regions are all the zones outside the B-regions where cross-sectional planes do not remain plane upon loading.
- D-regions are typically assumed at portions of a member where discontinuities (or disturbances) of stress distribution occur because of concentrated forces (loads or reactions) or abrupt changes of geometry.

- Based on St. Venant’s Principle, the normal stresses (due to axial load and bending) approach quasi-linear distribution at a distance approximately equal to the larger of the overall height (h) and width of the member, away from the location of the concentrated force or geometric irregularity.
- While B-regions can be designed with the traditional methods (ACI 318), the STM was primarily introduced to facilitate design of regions, and can be extended to the B-regions as well.
- STM depicts the D-region of the structural member with a truss system consisting of compression struts and tension ties connected at nodes.
- This truss system is designed to transfer the factored loads to the supports or to adjacent B-regions.
- At the same time, forces in the truss members should maintain equilibrium with the applied loads and reactions.

DEVELOPMENT OF STRUT-AND-TIE MODELS:

- Strut-and-tie models evolved in the early 1980s in Europe.
- Their use is permitted by ACI Code 6.2.4 and defined in Chapter 23 of the Code.
- Strut-and-tie models divide members into D-regions and B-regions.
- Strut-and-tie models are applied within D-regions.
- Models consist of struts and ties connected at nodal zones that are capable of transferring loads to the supports or adjacent B-regions.
- The cross-sectional dimensions of the struts and ties are designated as thickness and width.
- Thickness b is perpendicular to the plane of the truss model, and width w is measured in the plane of the model, as shown in Fig. 2.

The Struts:

- A strut is an internal compression member.
- It may consist of a single element, parallel elements, or a fan-shaped compression field.

- Along its length, a strut may be rectangular or *bottle-shaped*, in which case the compression field spreads laterally between nodal zones.
- For design purposes, a strut is typically idealized as a prismatic member between two nodes.
- While not preferred, a strut can also be idealized as a uniformly tapered compression member if the design criteria require different widths at the two ends of the strut.
- The dimensions of the cross section of the strut are established by the contact area between the strut and the nodal zone.

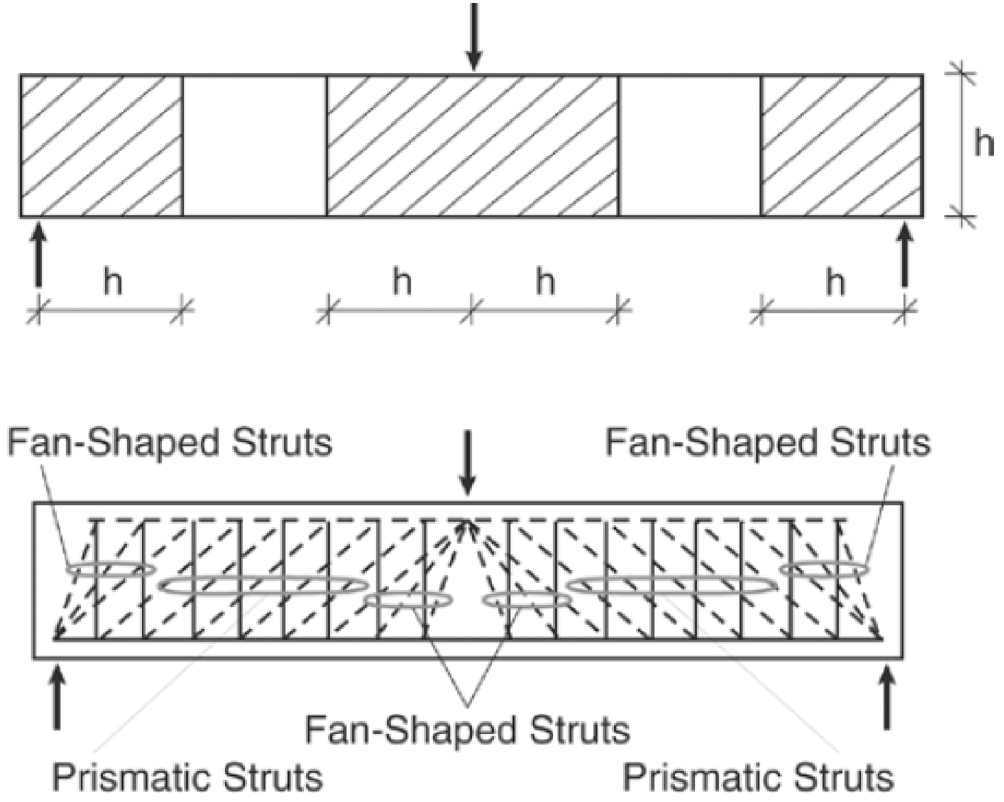


Figure 3: Prismatic and fan-shaped struts

- Bottle-shaped struts are wider at the center than at the ends and form when the surrounding concrete permits the compression field to spread laterally.
- As the compression zone spreads along the length of bottle-shaped struts, tensile stresses perpendicular to the axis of the strut may result in longitudinal cracking.

- For simplicity in design, bottle-shaped struts are idealized as having linearly tapered ends and uniform center sections.
- The linear taper is taken at a slope of 1:2 to the axis of the compression force, as shown in figure (4).
- The capacity of a strut is a function of the effective concrete compressive strength, which is affected by transverse stresses within the struts.
- Because of longitudinal splitting, bottle-shaped struts are weaker than rectangular struts, even though they possess a larger cross section at mid-length.

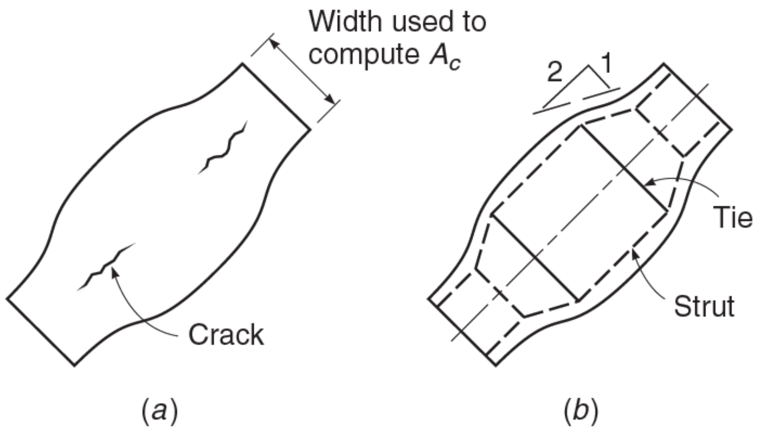


Figure (4): Bottle-shaped strut.

- Transverse reinforcement is designed to control longitudinal splitting and proportioned using a strut-and-tie model that forms within the strut element, as shown in Fig. (4-b).

The Ties:

- A tie is a tension member within a strut-and-tie model.
- Ties consist of reinforcement (prestressed or non-prestressed) plus a portion of the concrete that is concentric with and surrounds the axis of the tie.
- The surrounding concrete defines the tie area and the region available to anchor the tie.
- For design purposes, it is assumed that the concrete within the tie does not carry any tensile force.
- Even though the tensile capacity of the concrete is not used in design it assists in reducing tie deformation at service load.

Nodal Zones:

- Nodes are points within strut-and-tie models where the axes of struts, ties, and concentrated loads intersect.
- A nodal zone is the volume of concrete around a node where force transfer occurs.
- A nodal zone may be treated as **a single region** or may be subdivided into **two smaller zones** to equilibrate forces.
- For example, the nodal zone shown in Fig. (5) *a* is subdivided, as shown in Fig. (5) *b* , where two reactions *R* 1 and *R* 2 equilibrate the vertical components of strut forces *C* 1 and *C* 2 .

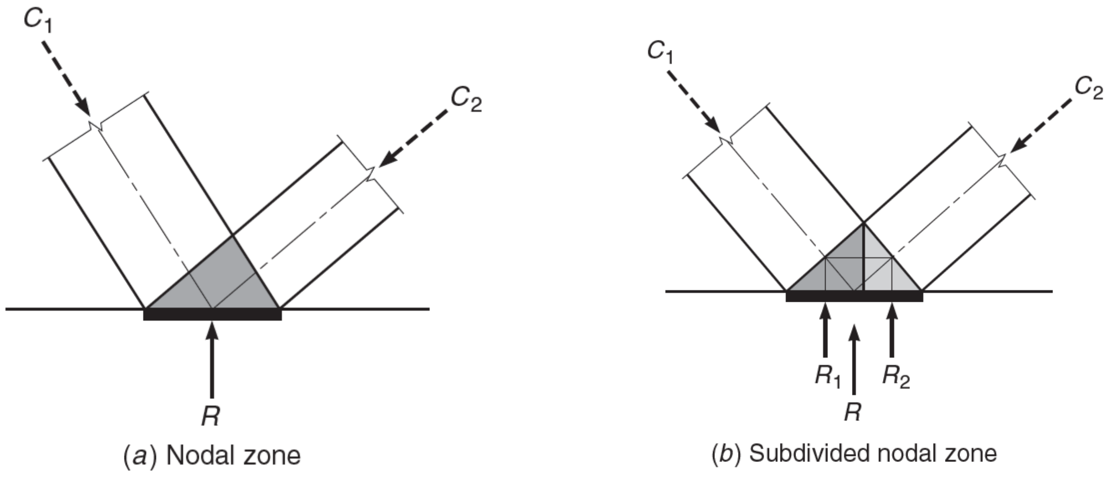


Figure (5): Subdivision of nodal zones

- The early STMs used hydrostatic nodal zones, which were lately superseded
- by extended nodal zones.
- The faces of a hydrostatic nodal zone are perpendicular
- to the axes of the struts and ties acting on the node, as depicted in
- Figure (5).
- The term hydrostatic refers to the fact that the in-plane stresses are the same in all directions.
- Note that in a true hydrostatic stress state, the out-of- plane stresses should also be equal.

- For equilibrium, at least three forces must act on a node.
- Nodes are classified by the sign of these forces (Fig. 6).
- Thus, a ***C-C-C* node** resists three compressive forces, and a ***C-C-T*** node resists two compressive forces and one tensile force, etc.

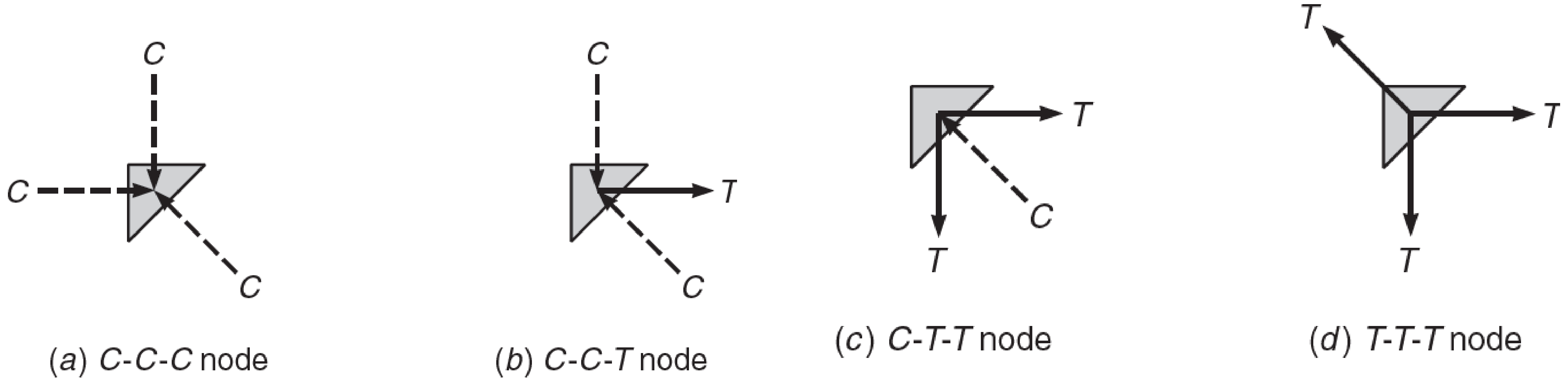


Figure (6): Classification of nodes

- Both tensile and compressive forces place nodes in compression because tensile forces are treated as if they pass through the node and apply a compressive force on the far side, or anchorage face.
- Thus, within the plane of a strut-and-tie model truss, nodal zones are considered to be in compression, as shown in Fig. (7).

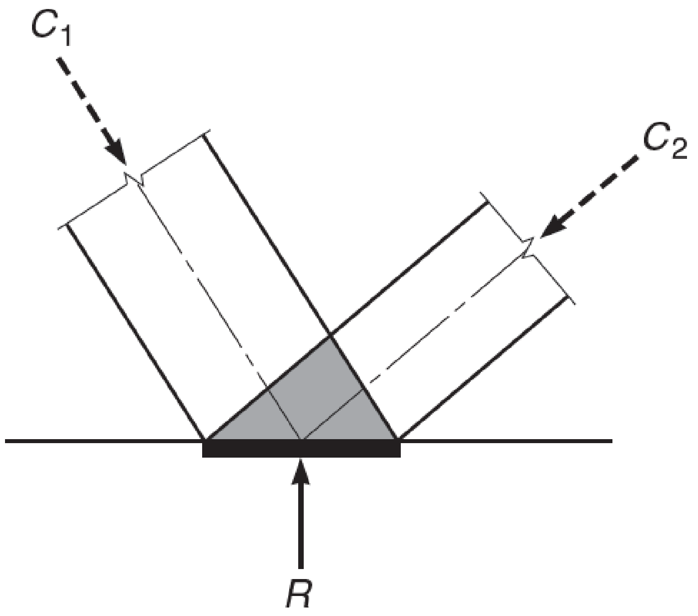


Figure (7): Nodal zone

- If the nodal zone dimensions w_{n1}, w_{n2}, and w_{n3} are proportional to the applied compressive forces, the state of stress becomes one of *hydrostatic* compression.
- The dimension of one side of a nodal zone is often determined based on the contact area of the load, such as a bearing plate, column base, or beam support.
- If a hydrostatic state of stress is desired, the dimensions of the remaining sides are selected to maintain a constant level of stress *p* within the node.
- By selecting nodal zone dimensions that are proportional to the applied loads, the stresses on the faces of the nodal zone are equal.
- If, instead, the dimensions are determined based on preselected strut dimensions, such as minimum width, the state of stress will no longer be hydrostatic.
- The decision to use a hydrostatic or a nonhydrostatic state of stress is made by the designer, with the former being more typical because the latter results in a more complex design.

- The length of a hydrostatic zone is often not adequate to allow for anchorage of tie reinforcement.
- For this reason, an *extended nodal zone*, defined by the intersection of the nodal zone and the associated strut (shown in light shading in Fig. 8-*a* and *b*), is used.
- It increases the length within which the tensile force from the tie can be transferred to the concrete and, thus, defines the available anchorage length for ties.
- Ties may be developed outside of the nodal and extended nodal zones if needed, as shown to the left of the node in Fig. 8-*b* .

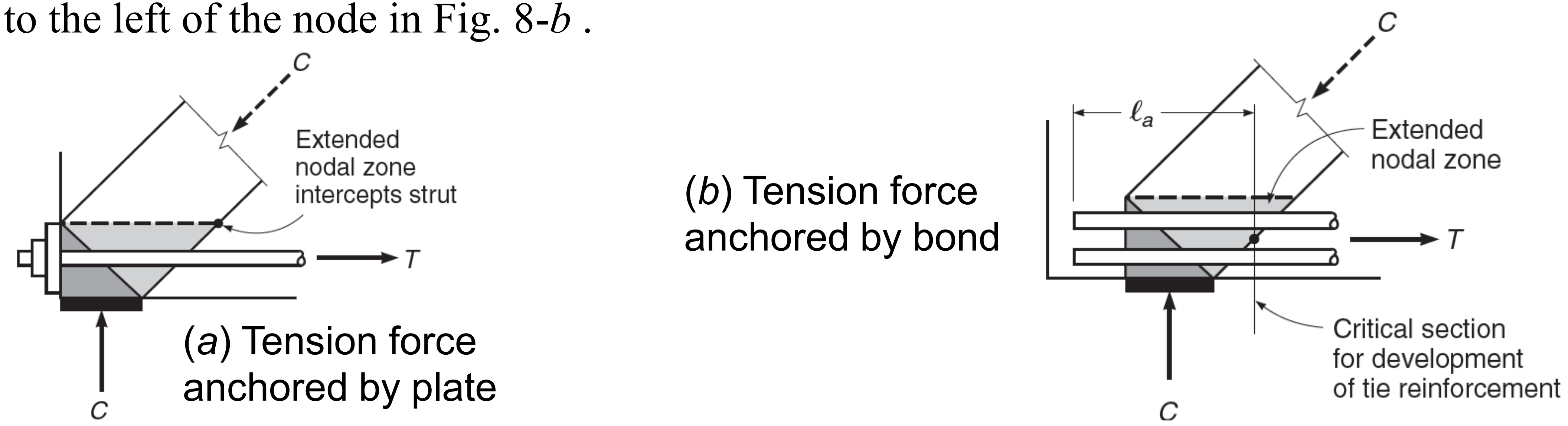


Figure (8): the extended nodal zones

STRUT-AND-TIE DESIGN METHODOLOGY:

- Strut-and-tie models are used in several ways during the design process.
- At the *conceptual* design level, sketching a strut-and-tie model provides insight into structural behavior and detailing requirements.
- Strut-and-tie models may be used to *validate* design details, such as for special reinforcement configurations.
- Finally, strut-and-tie models may form the basis for *detailed* design of a member.

Steps of using strut-and-tie models:

a) Define and isolate the D-regions.

b) Compute the force resultants on each D-region boundary.

c) Select a truss model to transfer the forces across a D-region.

d) Select dimensions for strut-and-tie nodal zones.

e) Verify the capacity of the node and the strut, the latter both at mid-length and at the nodal interface.

f) Design the ties and the tie anchorage.

g) Prepare design details and check minimum reinforcement requirements.

- According to ACI Code 23.3.1, design using a strut-and-tie model requires that:

For struts $\emptyset F_{ns} \geq F_{us}$ (1-a)

For ties $\emptyset F_{nt} \geq F_{ut}$ (1-b)

For nodes $\emptyset F_{nn} \geq F_{un}$ (1-c)

- where F_{ns}, F_{nt}, and F_{nn} = nominal capacity of strut, tie, and nodal zone, respectively.
- F_{us}, F_{ut}, and F_{un} = factored force acting in strut, tie, and nodal zone, respectively.
- ϕ = strength reduction factor of 0.75 for struts, ties, and nodes, as given in Table 21.2.1 of ACI:318-19.

Table 21.2.1 (of ACI-318)—Strength reduction factors ϕ:

Action or structural element		ϕ	Exceptions
(a)	Moment, axial force, or combined moment and axial force	0.65 to 0.90 in accordance with 21.2.2	Near ends of pretensioned members where strands are not fully developed, ϕ shall be in accordance with 21.2.3.
(b)	Shear	0.75	Additional requirements are given in 21.2.4 for structures designed to resist earthquake effects.
(c)	Torsion	0.75	—
(d)	Bearing	0.65	—
(e)	Post-tensioned anchorage zones	0.85	—
(f)	Brackets and corbels	0.75	—
(g)	Struts, ties, nodal zones, and bearing areas designed in accordance with strut-and-tie method in Chapter 23	0.75	—
(h)	Components of connections of precast members controlled by yielding of steel elements in tension	0.90	—
(i)	Plain concrete elements	0.60	—
(j)	Anchors in concrete elements	0.45 to 0.75 in accordance with Chapter 17	—

- In addition to strength criteria, service level performance must be considered in design because strut-and-tie models, which are based on strength, do not necessarily satisfy serviceability requirements.
- To this end, the spacing of reinforcement within ties should be checked with the provisions of ACI Code 24.3.2, i.e.

$$s = 380\left(\frac{280}{f_s}\right) - 2.5c_c \leq 300\left(\frac{280}{f_s}\right)$$

- ACI Code 9.9.2 limits the nominal shear strength of deep beams to

$$V_u \leq 0.83\emptyset\sqrt{f_c'}\, b_w\, d$$

- This limit applies to strut-and-tie models and should be checked prior to beginning a detailed design.

Force Resultants on D-region Boundaries:

- Once the D-region is defined, the next step involves determining the magnitude, location, and direction of the resultant forces acting on the D-region boundaries.
- These forces serve as input for the strut-and-tie model and assist in establishing the geometry of the truss model.
- When one face of a D-region is loaded with a uniform or linearly varying stress field, or when a face is loaded by bending of a concrete section, it may be necessary to subdivide the boundary into segments corresponding to struts or ties and then to compute the resultant force on each segment.
- For example, the distributed load along the top of the deep beam is represented by four concentrated loads, and the stresses at the beam-column interface are represented by concentrated reactions.
- The moments at the faces of the beam-column joint are represented by couples consisting of tensile and compressive forces acting at the interfaces between the members and the joint.

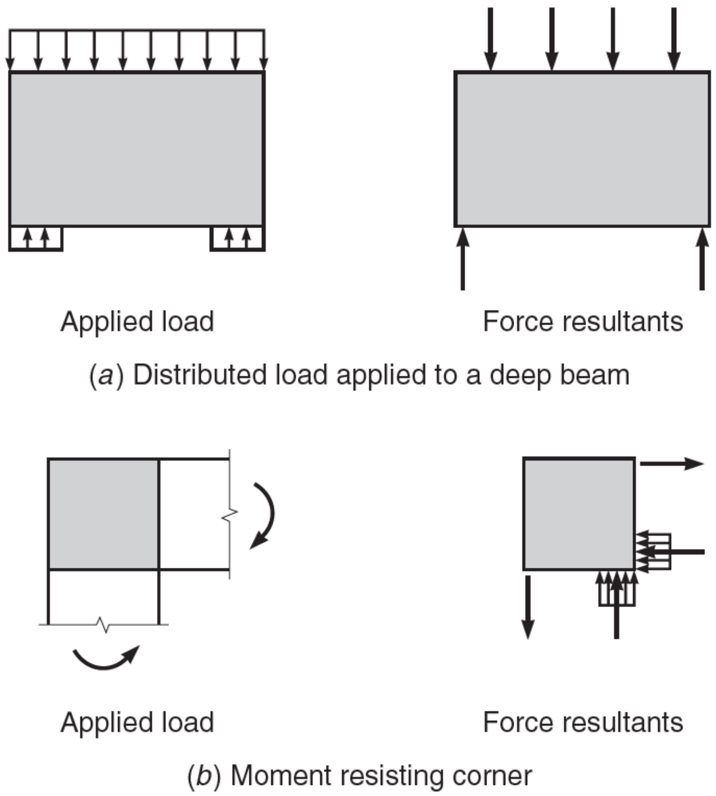


Figure (9): Resolution of forces in a D-region.

The Truss Model:

- The truss representing the strut-and-tie model must fit within the envelope defined by the D-region.
- The selection of struts and ties is made at the discretion of the designer, and, therefore, multiple solutions are possible.
- The axes of the truss members are chosen to coincide with the centroids of the tension and compression fields, and the geometry so established is used to compute the forces in the members.
- The layout of a truss model is constrained by the geometric requirement that struts must intersect only at nodal zones, and ties may cross struts.
- Because struts are typically much stiffer than ties, a model with a minimum number of tension ties is generally preferred.
- An effective model will represent a minimum energy distribution through the D-region; that is, within the model, forces should follow the stiffest load path.

- Alternative truss models for a deep beam are compared in Fig. (10).
- Figure (10-a) shows a deep beam subjected to a concentrated load at midspan.
- Figure (10-b) shows the preferred strut-and-tie model for this beam and loading condition.
- In this case, struts carry the load directly to nodal regions at the supports, which are, in turn, connected by a single tension tie.
- The model in Fig. (10-c) shows an ineffective load path, with a single strut carrying the load to a node at the bottom of the beam that is supported by two diagonal tension ties that are, in turn, supported by vertical struts over the supports.

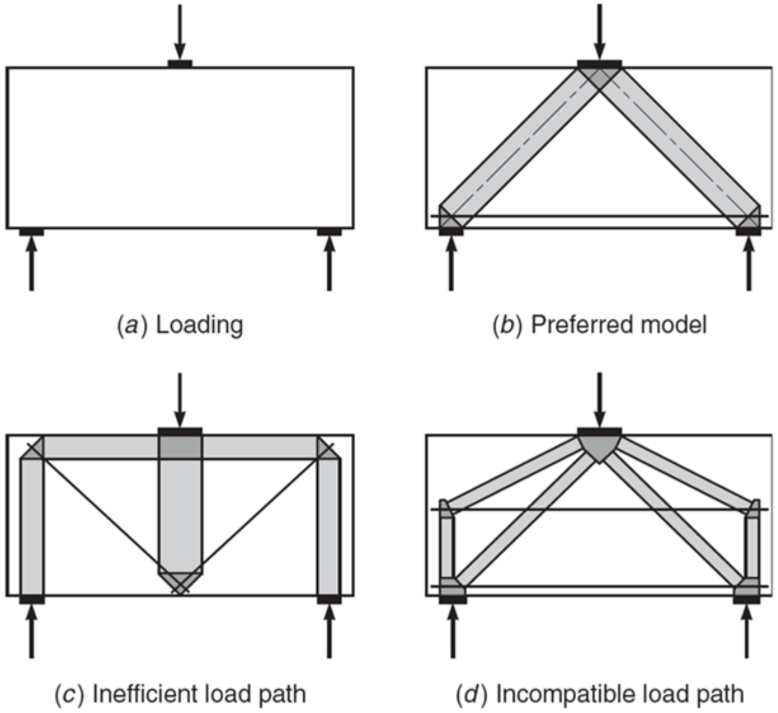


Figure (10): Alternatives for a deep beam truss model

- In this instance, the number of transfer points and tension ties is greater, as is the flexibility of the truss, indicating a solution that is much less effective than that shown in Fig. (10-b).

- Lastly, Fig. (10-d) illustrates a model with multiple struts and ties.
- This particular layout not only is unduly complex but also includes an upper tension tie that will be effective only after extensive yielding and possible failure of the lower tension tie.
- Theoretically, there may be a unique minimum energy solution for a strut-and-tie model.
- Practically, any model that satisfies equilibrium and pays attention to structural stiffness will prove satisfactory.
- Using the rationale just discussed allows the designer to select a logical model that effectively mobilizes ties and minimizes the potential for excessive cracking.
- Finite element analyses and solutions based on the theory of elasticity for the full structure can provide an indication of where maximum stresses occur.
- A truss model that provides struts in regions of high compression and ties in regions of high tension based on these analyses will, in general, provide an efficient load path.

Selecting Dimensions for Struts and Nodal Zones:

- The width of each truss member depends on the magnitude of the forces and the dimensions of the adjoining elements.
- An external element, such as a bearing plate or column, can serve to define a nodal zone.
- If the bearing area is too small, a high hydrostatic pressure results, and the corresponding width of the node or struts will not be sufficient to carry the applied load.
- The solution in this case is to increase the size of the bearing surface and, thus, reduce the contact pressures.
- Some designers intentionally select struts and nodes that are large enough to keep the compressive stresses low; in this case, only the tension ties require detailed design.
- To minimize cracking and to reduce complications that may result from incompatibility in the deformations due to struts shortening and ties elongating in nearly the same plane, the angle between struts and ties at a node should be at least 25 ° .
- Designers, however, often prefer to use an angle of at least 40 ° because forces in the struts and ties are often unacceptably high at lower angles.

ACI PROVISIONS FOR STRUT-AND-TIE MODELS:

- ACI Code Chapter 23 provides guidance for sizing struts, nodes, and ties.
- It addresses the performance of:

i. highly stressed compression zones that may be adjacent to or crossed by cracks in a member,

ii. the effect of stresses in nodal zones, and

iii. the requirements for bond and anchorage of ties.

- The effective compressive strength of concrete $0.85f_c'$ is modified by a factor β to account for the effects of:

a) cracks (caused by spreading compressive resultants) and

b) confining reinforcement in struts and the anchorage of ties in nodal zones.

- The balance of this section describes the steps needed to calculate the capacity
- A strength reduction factor ϕ = 0.75 is used for struts, ties, nodal zones, and bearing areas.

a. Strength of Struts:

- The strength of a strut is limited based on the strength of the concrete in the strut and the strength of the nodal zones at the ends of the strut.
- The nominal compressive strength of a strut F_{ns} is given as:

$$F_{ns} = f_{ce} A_{cs} \qquad (2)$$

Where: f_{ce} is the effective compressive strength of the concrete in a strut or nodal zone.

A_{cs} is the cross-sectional area at one end of the strut, which is equal to the product of the strut thickness and the strut width.

- The effective strength of concrete in a strut is:

$$f_{ce} = 0.85\, \beta_s\, f_c' \qquad (3)$$

- where β_s is a factor that accounts for the effects of cracking and confining reinforcement within the strut, with values ranging from 1.0 for a strut with a uniform cross sectional area over its length to 0.4 for struts in tension members or the tension flanges of members (Table 1).

Table (1-a)

Table 23.4.3(a)—Strut coefficient β_s

Strut location	Strut type	Criteria	β_s	
Tension members or tension zones of members	Any	All cases	0.4	(a)
All other cases	Boundary struts	All cases	1.0	(b)
	Interior struts	Reinforcement satisfying (a) or (b) of Table 23.5.1	0.75	(c)
		Located in regions satisfying 23.4.4	0.75	(d)
		Beam-column joints	0.75	(e)
		All other cases	0.4	(f)

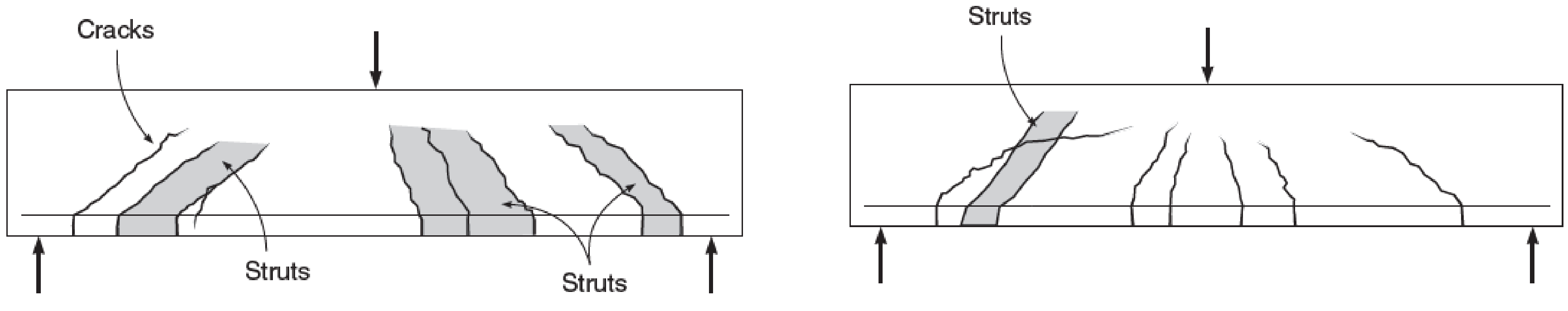
Table (1-b)

β_s values for strut strength

Condition	β_s
Strut with uniform cross section over its entire length	1.0
Strut with the width at midsection larger than the width at the nodes (bottle-shaped strut) and with reinforcement satisfying transverse requirements	0.75
Strut with the width at midsection larger than the width at the nodes (bottle-shaped strut) and reinforcement not satisfying transverse requirements	$0.60\lambda^{\dagger}$
Struts in tension members or in the tension flange of members	0.40
All other cases, Fig. 11	0.60λ

† λ equals 1.0 for normalweight concrete, and varies between 0.75 and 1.0 for various combinations of lightweight fine and coarse aggregates.

- Intermediate values include 0.75 for struts with a width at midsection that is larger than the width at the nodes (bottle-shaped struts) and crossed by transverse reinforcement to resist the transverse tensile force resulting from the compressive force spreading in the strut.
- A value of 0.60 λ is for bottle-shaped struts without the required transverse reinforcement, where λ is the correction factor related to the unit weight of concrete.
- A value of β_s = 0.60 λ for all other cases, as when parallel diagonal cracks divide the web struts or when diagonal cracks are likely to turn and cross a strut.



(*a*) Struts in beam with inclined cracks parallel to strut

(*b*) Struts crossed by skew cracks

Figure (11): Beam cracking conditions for β_s = 0.6 λ .

- Compression steel may be added to increase the strength of a strut, so that

$$F_{ns} = f_{ce}\, A_{cs} + A'_s f'_s \qquad (4)$$

where F_{ns} shall be evaluated at each end of the strut and taken as the lesser value;

A_{cs} is the cross-sectional area at the end of the strut under consideration;

A'_s is the area of compression reinforcement along the length of the strut; and

f'_s is the stress in the compression reinforcement at the nominal axial strength of the strut.

- It shall be permitted to take f'_s equal to f_y for Grade 280 or 420 reinforcement.
- For other grades f'_s is computed based on the strain in the concrete at peak stress.
- In accordance with ACI Code 23.6, compression reinforcement must be properly anchored, oriented parallel to the axis of the strut, located within the strut, and enclosed by ties or spirals, as required for columns.

- To design transverse reinforcement for bottle-shaped struts, ACI Code 23.5.3 permits the assumption that the compressive force in the struts spreads at a slope of two longitudinal to one transverse along the axis of the strut.

- For 0.85 $\beta_s\ f_c' \leq 42$ MPa, the ACI Code considers the transverse reinforcement requirement to be satisfied if the strut is crossed by layers of reinforcement that satisfy

$$\sum \frac{A_{si}}{b_s s_i} \sin \alpha_i \geq 0.003 \qquad (5)$$

- where A_{si} is the total area of reinforcement at spacing s_i in a layer of reinforcement with bars at an angle α_i to the axis of the strut and b_s is the thickness of the strut.

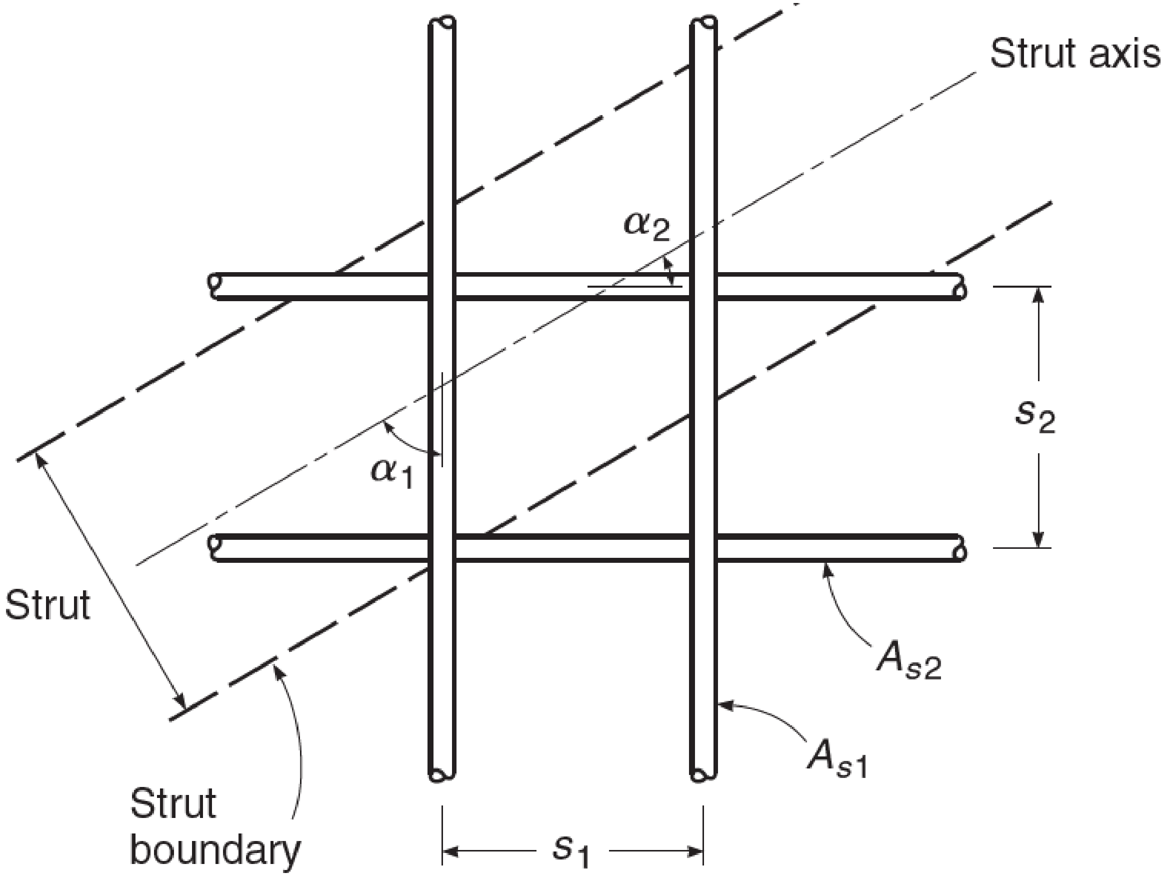


Figure (12): Details of reinforcement crossing a strut

- The reinforcement may be perpendicular to the strut axis or may be placed in an orthogonal grid pattern.
- The ACI Code provides no clear guidance to indicate when a strut should be considered as rectangular or bottle-shaped.
- Some researchers suggest that horizontal struts be represented as rectangular and inclined struts represented as bottle-shaped.
- Others simply assume a bottle-shaped strut will develop and use the lower values of for design β_s.

b. Strength of Nodal Zones:

- The nominal compressive strength of a nodal zone is

$$F_{nn} = f_{ce}\, A_{nz} \qquad (6)$$

- where f_{ce} is the effective strength of the concrete in the nodal zone,
- and A_{nz} is:

1) The area of the face of the nodal zone taken perpendicular to the line of action of the force from the strut or tie, or
2) the area of a section through the nodal zone taken perpendicular to the line of action of the resultant force on the section.

- The latter condition occurs when multiple struts intersect a node, as shown in Fig. 13 .

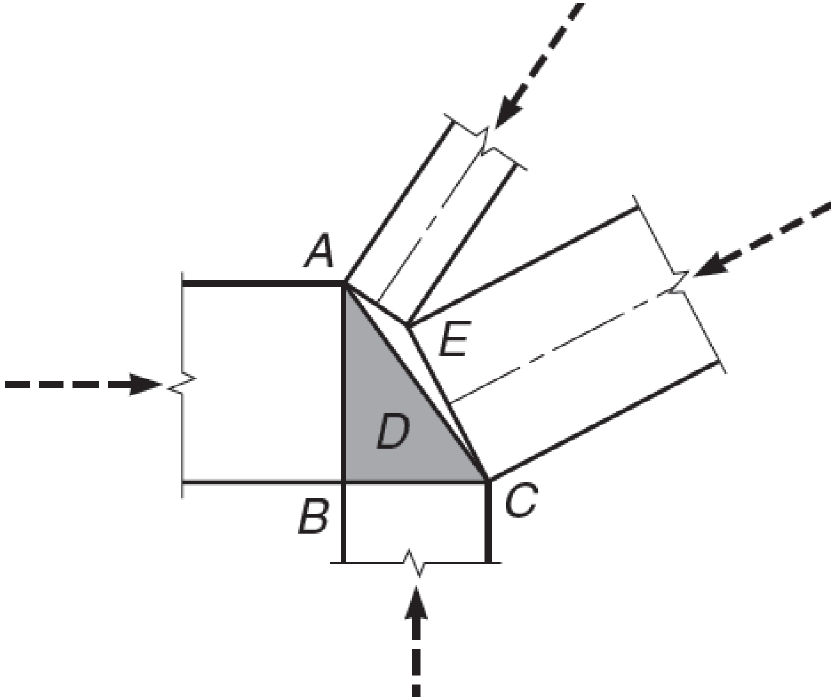


Figure (13): Intersection of multiple struts at nodal zone

- The effective concrete strength in a nodal zone is:

$$f_{ce} = 0.85\, \beta_n\, f_c' \qquad (7)$$

- where f_c' is the compressive strength of the concrete in the nodal zone, an
- β_n is a factor that reflects the degree of disruption in nodal zones due to the incompatibility of tensile strains in ties with compressive strains in struts, Table (2).
- ACI Code 23.9.3 permits the strength of a node to be increased above the value given in Eq. (7) if the node contains confining reinforcement and the effect of that reinforcement is demonstrated by tests and analysis.
- Unless compression reinforcement is used in the struts, the lower value of f_{ce} from Eqs. (3) and (7) governs and should be used to design both the node and the adjoining struts.

Table (2): Values of β_n for nodal zone

Nodal Zone Condition	Classification	β_n
Bounded by struts or bearing area	*C-C-C*	1.0
Anchoring one tie	*C-C-T*	0.80
Anchoring two or more ties	*C-T-T* or *T-T-T*	0.60

c. Strength of Ties:

- The nominal strength of ties F_{nt} is the sum of the strengths of the reinforcing steel and prestressing steel within the tie, i.e.

$$F_{nt} = A_{ts} f_y + A_{tp}(f_{pe} + \Delta f_p)$$

where A_{ts} = area of reinforcing steel

f_y = yield strength of reinforcing steel

A_{tp} = area of prestressing steel, if any

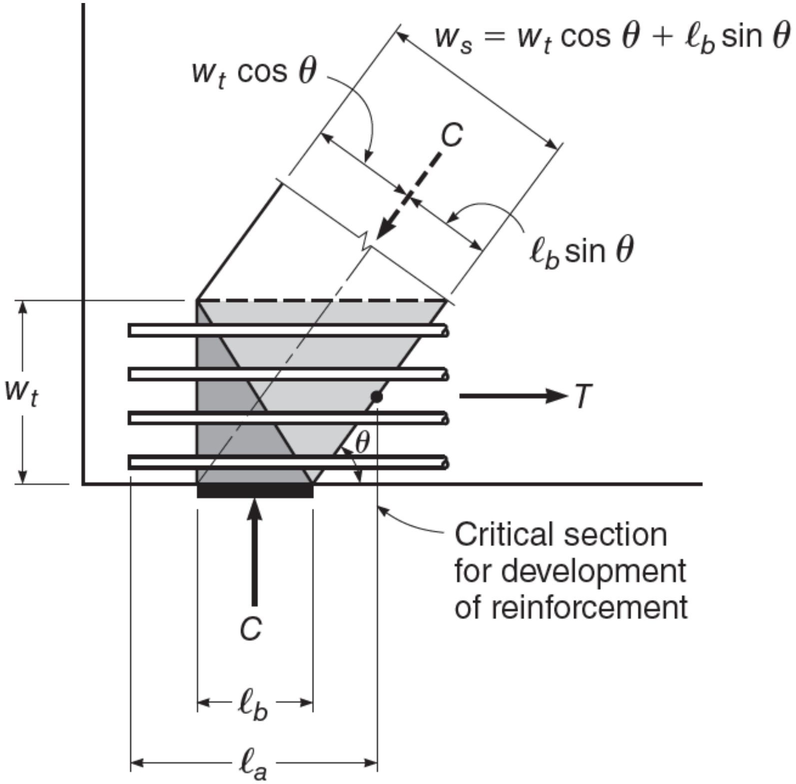
f_{pe} = effective stress in prestressing steel

Δf_p = increase in prestressing steel stress due to factored load

- The sum $f_{ps} + \Delta f_p$ must be less than or equal to the yield stress of the prestressing reinforcement f_{py}.
- A_{tp} is zero for non-prestressed members.
- The value of Δf_p may be found by analysis; or, in lieu of formal analysis, ACI Code 23.7.3 allows a value 420 MPa to be used for bonded tendons and 70 MPa to be used for unbonded tendons.

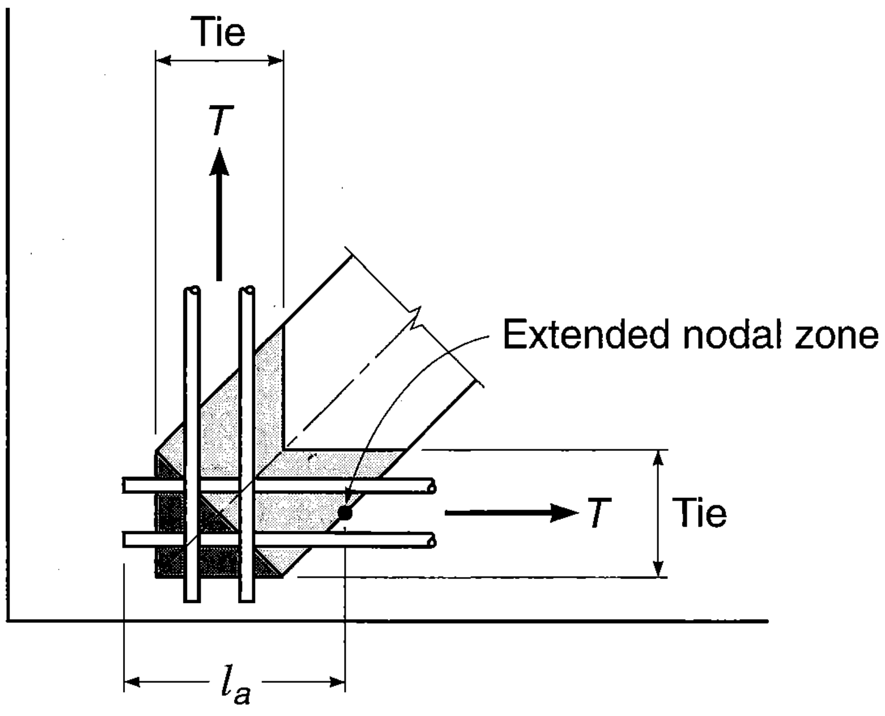
- The effective width of a tie w_t depends on the distribution of the tie reinforcement.
- If the reinforcement in a tie is placed in a single layer, the effective width of a tie may be taken as the diameter of the largest bars in the tie plus twice the cover to the surface of the bars.
- Alternatively, the width of a tie may be taken as the width of the anchor plates.
- The practical upper limit for tie width $w_{t,max}$ is equal to the width corresponding to the width of a hydrostatic nodal zone, given as:

$$w_{t,\max} = \frac{F_{nt}}{b_s f_{ce}}$$

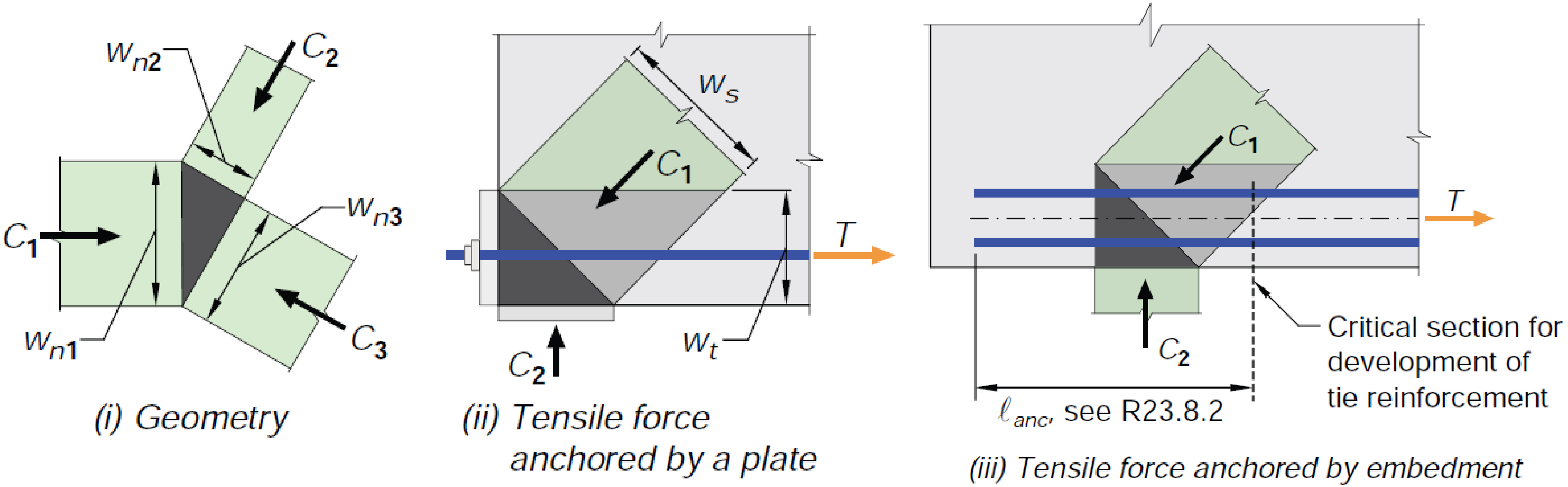


where f_{ce} is the effective nodal zone compressive stress given in Eq. (7) and
b_s is the thickness of the strut.

- Ties must be anchored before they leave the extended nodal zone at a point defined by the centroid of the bars in the tie and the extension of the outlines of either the strut or the bearing area.
- If the combined lengths of the nodal zone and extended nodal zone are inadequate to provide for development of the reinforcement, additional anchorage may be obtained by extending the reinforcement beyond the nodal zone, using 90 ° hooks, heads or mechanical anchors.

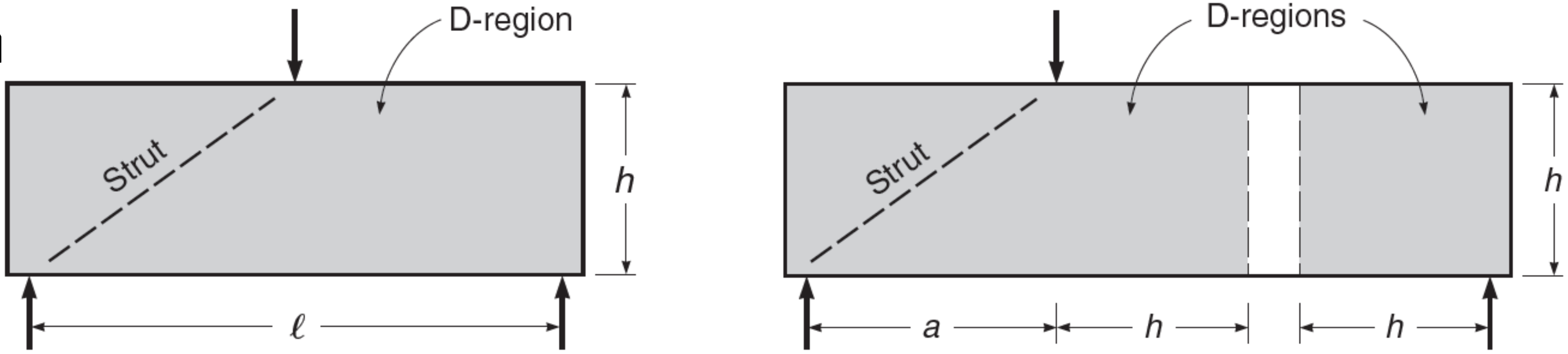


- If the tie is anchored with a 90 ° hook, the hooks should be confined by reinforcement extending into the beam from supporting members to avoid splitting of the concrete within the anchorage region.


w_{n2}
C_2
C_1
w_{n3}
w_{n1}
C_3
(i) Geometry
w_s
C_1
T
C_2
w_t
(ii) Tensile force anchored by a plate
C_1
T
C_2
Critical section for development of tie reinforcement
ℓ_{anc}, see R23.8.2
(iii) Tensile force anchored by embedment

ACI Shear Requirements for Deep Beams:

- Beams with clear spans less than or equal to 4 times the total member depth or with concentrated loads placed within twice the member depth of a support are classified as deep beams, according to ACI Code 9.9.
- Examples of deep beams are shown in Figure below.
- ACI Code 9.9.1 allows such members to be designed either by using a nonlinear analysis or by applying the strut-and-tie method of ACI Code Cha



(a) Deep beam with $\ell \leq 4h$

(b) Deep beam with $a \leq 2h$

Summary of The Strut-and-Tie Method:

Introduction

- The Strut-and-Tie analogy is an alternative method for designing reinforced concrete members with force and geometric discontinuities.
- The method is also very useful for designing deep beams for which the usual assumption of linear strain distribution is not valid.

Deep Beams:

- Section 10.7 of the ACI Code defines a deep beam as a member that:
- **(a)** Is loaded on one face and supported on the opposite face so that compression struts can develop between the load and the supports.
- **(b)** Has a clear span not more than four times its overall depth or that has regions where concentrated loads are located within two times the member depth from the support.
- *Transfer girders* are one type of deep beam that occur rather frequently. Such members are used to transfer loads laterally from one or more columns to other columns.
- Sometime bearing walls also exhibit deep beam action.
- Deep beams begin to crack at loads ranging from $\frac{1}{3}\ P_u$ to $\frac{1}{2}\ P_u$.
- As a result, elastic analyses are not of much value to us except in one regard: the cracks tell us something about the way the stresses that cause the cracks are distributed.
- In other words, they provide information as to how the loads will be carried after cracking.

Shear Span and Behavior Regions:

- The ratio of the shear span of a beam to its effective depth determines how the beam will fail when overloaded.
- The shear span for a particular beam is shown in Figure 1, where it is represented by the symbol *a*.
- This is the distance from the concentrated load shown to the face of the support.
- Should the beam be supporting only a uniform load, the shear span is the clear span of the beam.
- When shear spans are long, they are referred to as B-regions.
- These are regions for which the usual beam theory applies—plane sections remain plain before and after bending.

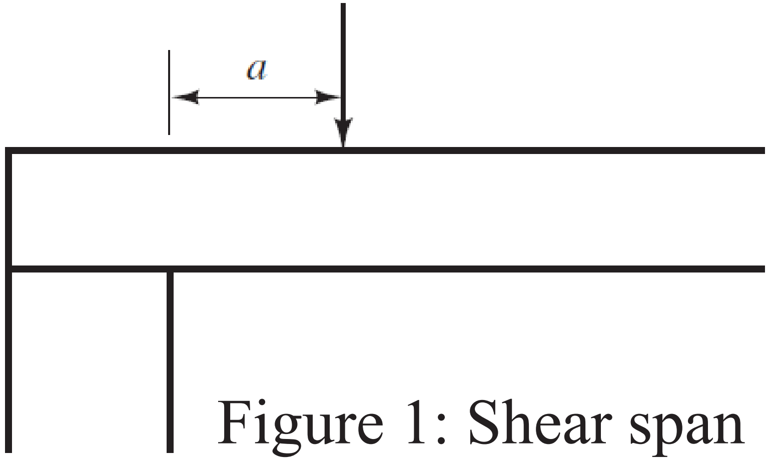
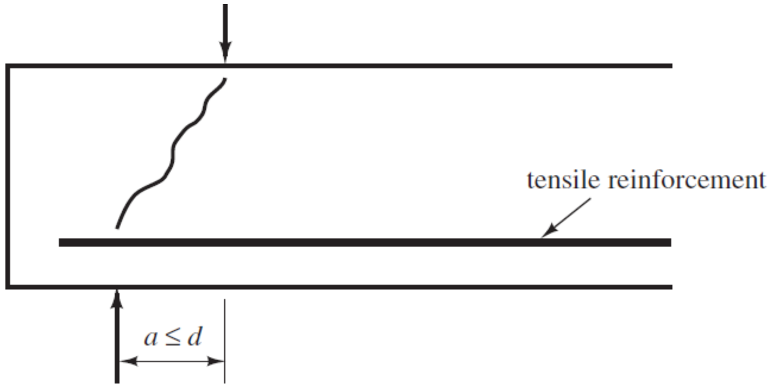


Figure 1: Shear span

- The letter B stands for beam or for Bernoulli (he is the one who presented the linear strain theory for beams).
- In some situations, the usual beam theory does not apply.
- When shear spans are short, loads are primarily resisted by arch action rather than beam action.
- Locations where this occurs are called D-regions.
- The letter D represents discontinuity or disturbance.
- In such regions, plane sections before bending do not remain plane after bending, and the forces obtained with the usual shear and moment diagrams and first-order beam theory are incorrect.
- D regions are those parts of members located near concentrated loads and reactions.
- They also include joints and corbels and other locations where sudden changes in member cross section occur, such as where holes are present in members.
- According to the St. Venant principle, local disturbances such as those caused by concentrated loads tend to dissipate within a distance approximately equal to the member depth.

Truss Analogy:

- If shear spans are very short, inclined cracks extending from the concentrated loads to the supports tend to develop.
- In effect, the flow of horizontal shear from the longitudinal reinforcement to the compression zone has been interrupted.
- As a result, the behavior of the member has been changed from that of a beam to that of a tied arch where the reinforcing bars act as the ties of an arch.



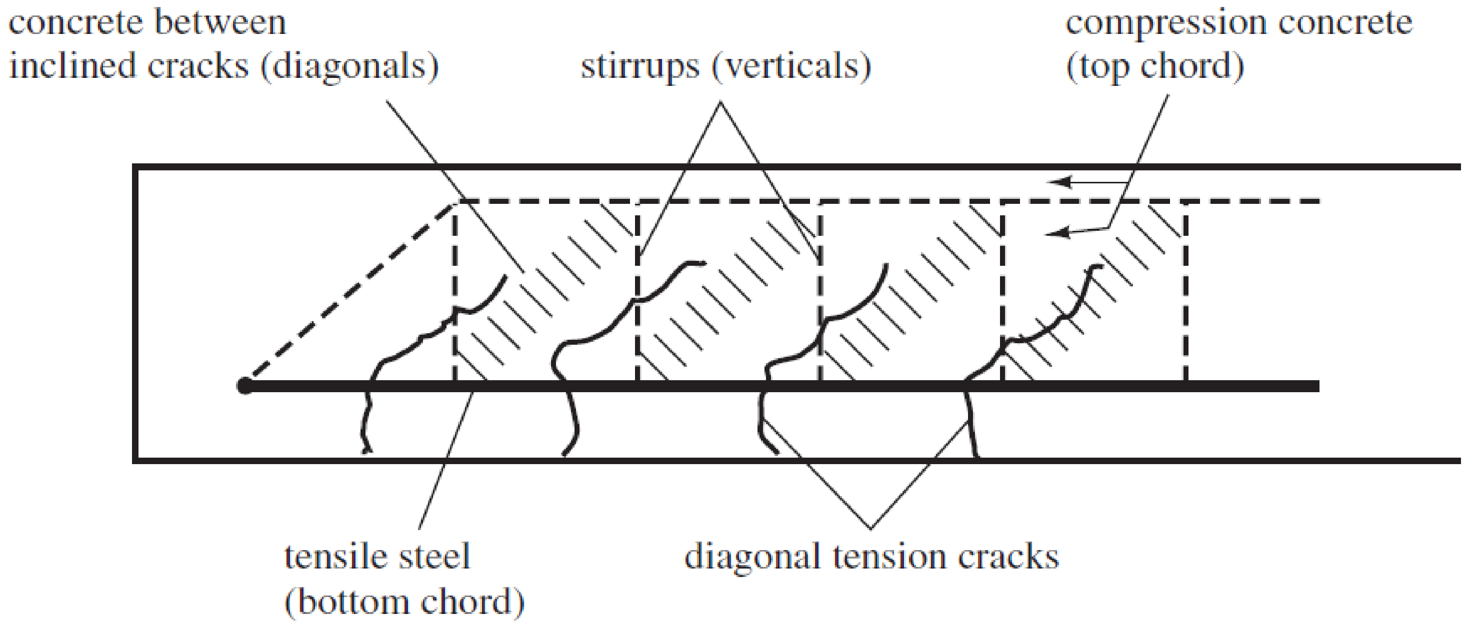
- According to the description of reinforced

concrete beams by Ritter-Morsch with the truss analogy method, a reinforced concrete beam with shear reinforcement behaves much like a statically determinate
parallel chord truss with pinned joints.

- The concrete compression block is considered to be the

top chord of the fictitious "truss," while the tensile reinforcement is considered to act as the bottom chord.

- The “truss” web is said to consist of the stirrups acting as vertical tension members, while the portions of the concrete between the diagonal cracks are assumed to act as diagonal compression members.
- Such a “truss” is shown in figure below, in which, the compression concrete and the stirrups are shown with dashed lines.
- These lines represent the estimated centers of gravity of those forces.
- The tensile forces are represented with solid lines because those forces clearly act along the reinforcing bar lines.



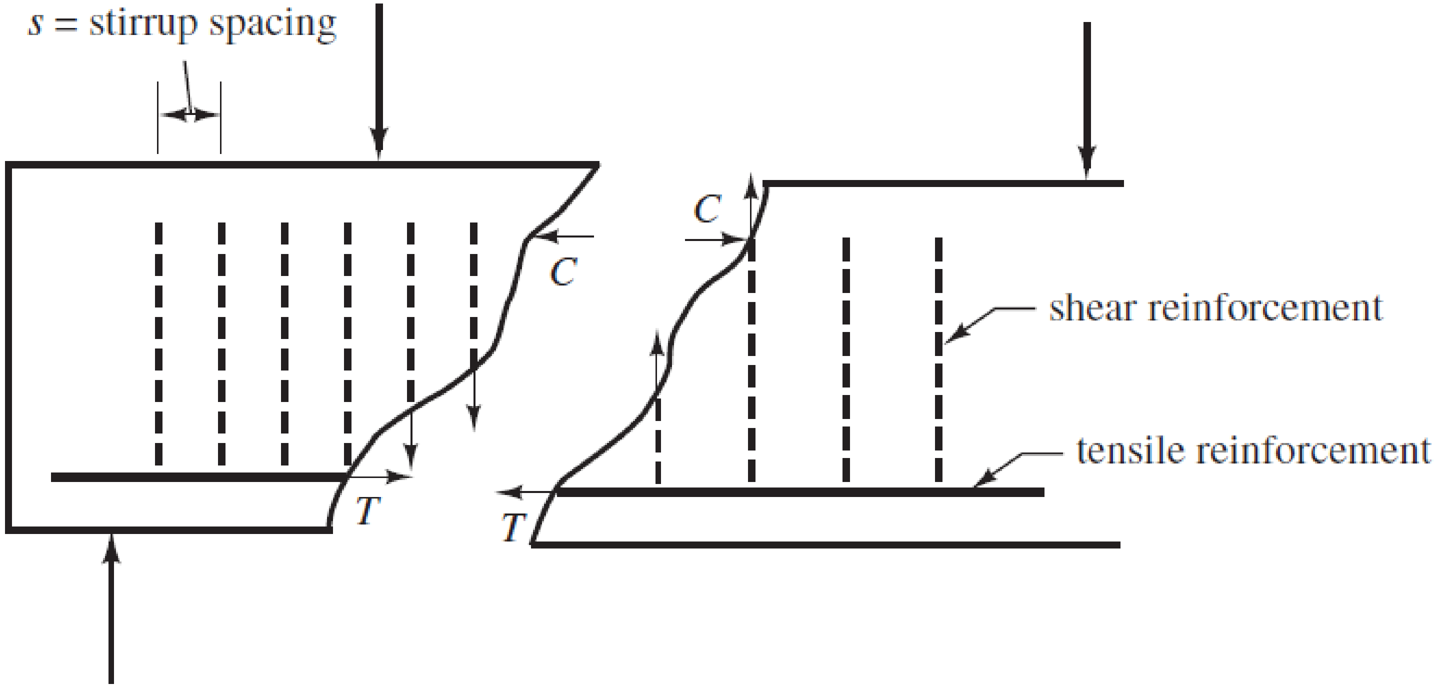
Terminology:

- A *strut-and-tie model* is a truss model of a D region where the member is represented by an idealized truss of struts and ties.
- A *tie* is a tension member in a strut-and-tie model.
- A *strut* is a compression member in a strut-and-tie model that represents the resultant of the compression field.
- A *node* in a strut-and-tie model is the point in a joint where the struts, ties, and concentrated forces at the joint intersect.
- The *nodal zone* is the volume of concrete around a node that is assumed to transfer the forces from the struts and ties through the node.

Selecting a Truss Model:

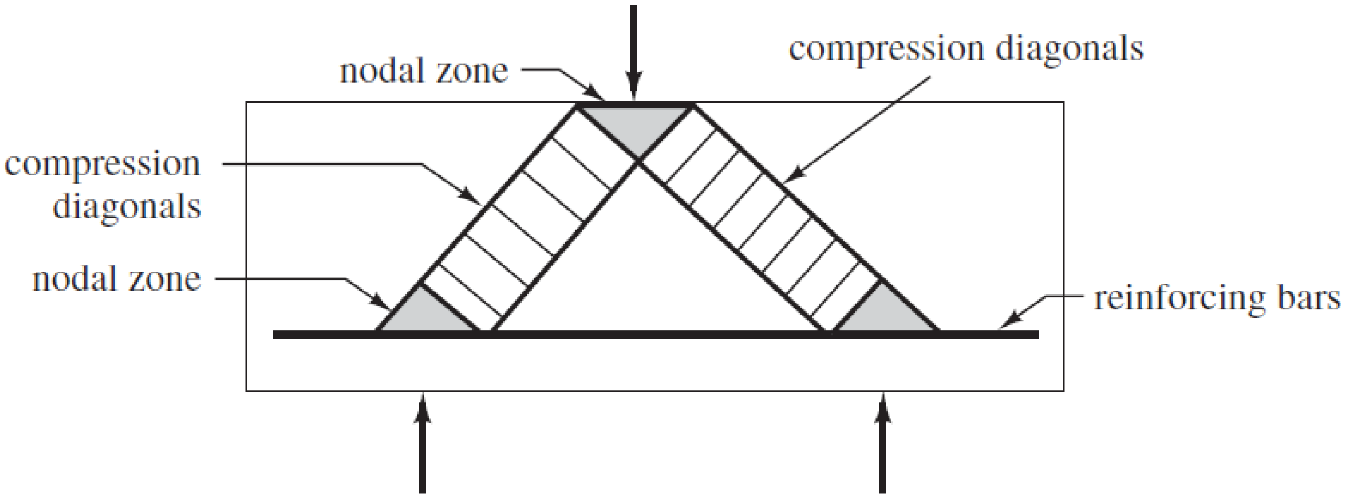
- When the strut-and-tie method is used for D regions, the results are thought to be more conservative but more realistic than the results obtained with the usual beam theory.
- To design for a D region of a beam, it is necessary to isolate the region as a free body, determine the forces acting on that body, and then select a system or truss model to transfer the forces through the region.
- Once the D region has been identified and its dimensions have been determined, it is assumed to extend a distance h on each side of the discontinuity or to the face of the support if that value is less than the depth.
- The stresses on the boundaries of the region are computed with the usual expression for combined axial load and bending, $P/A \pm Mc/I$.
- The resulting values must be divided by the capacity reduction factor φ for shearing forces (0.75) to obtain the required nominal stresses.

- The designer needs to represent the D regions of members, which fail in shear, with some type of model before beginning the design.
- The model selected for beams with shear reinforcement is the truss model, as it is the best one available at this time.
- For this discussion, the beam shown in figure is considered.
- The internal and external forces acting on this beam, which is assumed to be cracked, are shown.



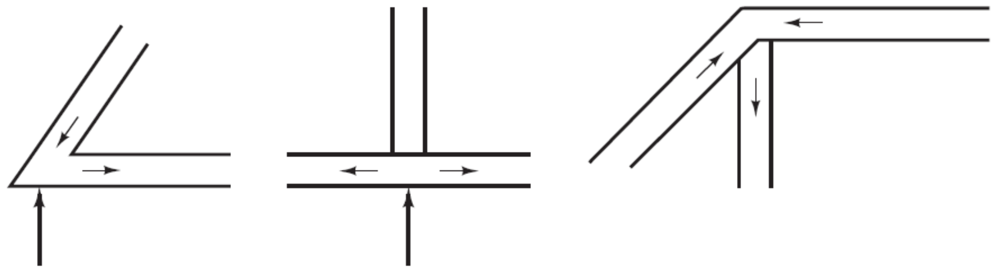
- To select a strut-and-tie model for such a beam, all the stirrups cut by the imaginary section are lumped into one.
- In a similar fashion, the concrete parallel to a diagonal is also lumped together in one member.

- With the strut-and-tie method, forces are resisted by an idealized internal truss such as the triangular one sketched in figure below.
- The members and joints of this truss are designed so that they will be able to resist the computed forces.
- **The truss selected must, of course, be smaller than the beam that encloses it**, and any reinforcing steel must be given adequate cover.
- For a first illustration, a short deep beam supporting a concentrated load is shown in figure below.

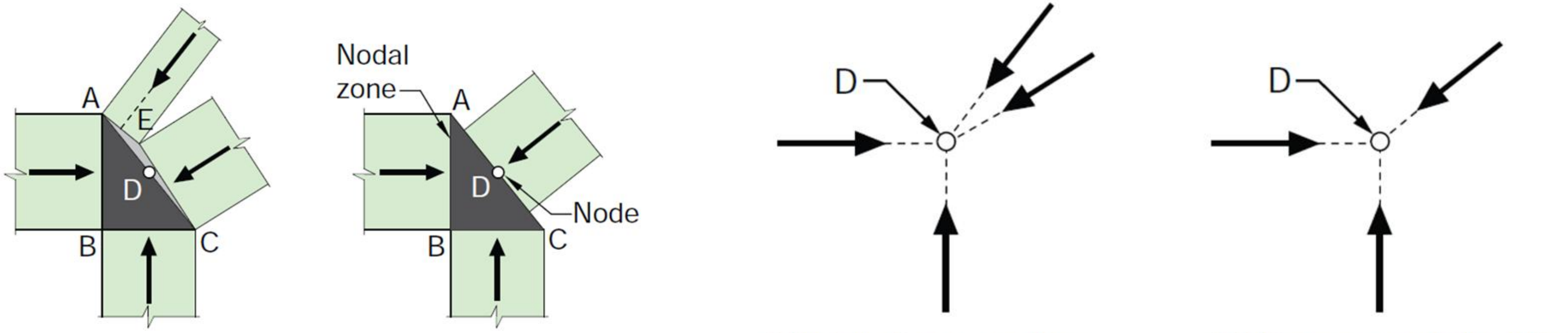


A short deep beam with the truss model shown.

- Various types of nodes are shown.
- It should be observed that there have to be at least three forces at each joint for equilibrium.



- This is the number of forces necessary for static equilibrium as well as the largest number that can occur in a state of determinate static equilibrium.
- If more than three forces meet at a joint when a truss is laid out, the designer will need to make combinations of them in some way so that only three forces are considered to meet at the node.



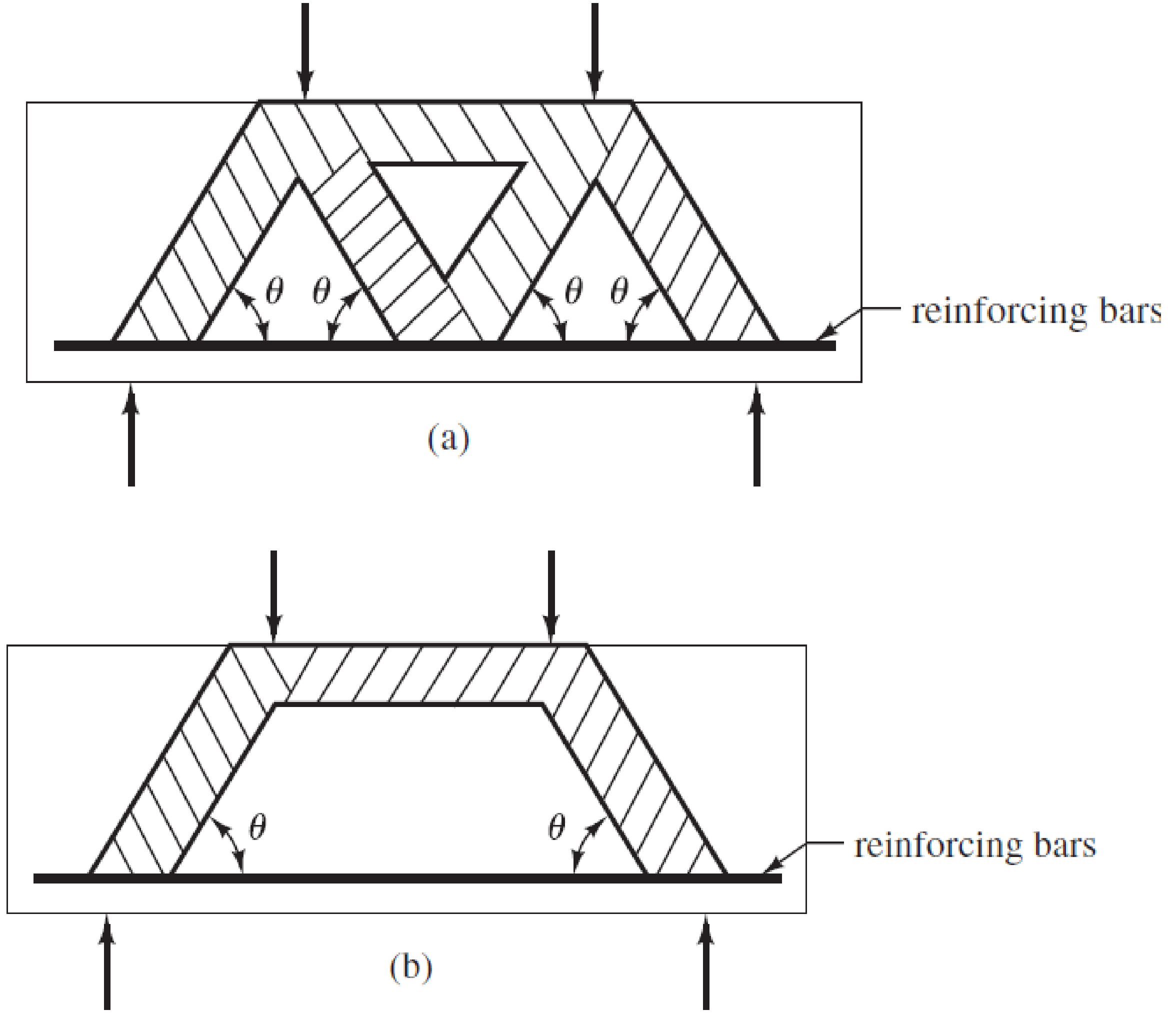
(a) Struts A-E and C-E may be replaced by A-C

(b) Three struts acting on a nodal zone

(c) Four forces acting on node D

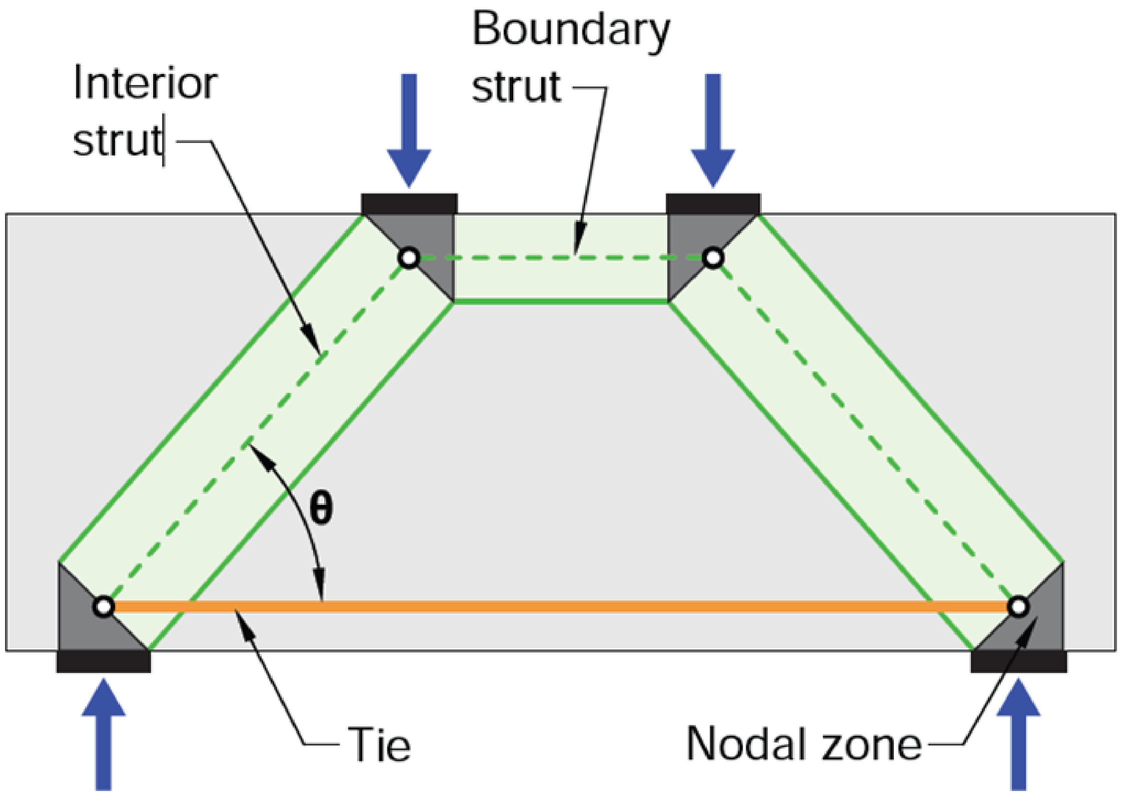
(d) Forces on right side of node shown in (c) resolved

- Two possible strut-and-tie models for a deep beam supporting two concentrated loads are shown in figure.
- In part (a) of the figure, four forces meet at the location of each concentrated load.
- As such, we cannot determine all of the forces.
- An alternative truss is shown in part (b) of the figure in which only three forces meet at each joint.
- It can be seen that the assumptions of the paths of the forces involved in the trusses described might vary quite a bit among different designers.
- As a result, there is no one correct solution for a particular member designed by the strut-and-tie method.



Angles of Struts in Truss Models:

- To lay out the truss, it is necessary to establish the slope of the diagonals (angle θ that is measured from the tension chord—the tension reinforcement).
- According to Schlaich and Weischede, the angle of stress trajectories varies from about 68 to 55°.
- A common practice, and is usually satisfactory, is to assume a 2 vertical to 1 horizontal slope for the struts.
- This will result in a value of $\theta = 63.43°$.
- The dimensions selected for the truss model must fit into the D region involved, so the angles may need to be adjusted.



Application of STM on Deep Beams:

- While there are a number of possible applications for a strut-and-tie model, ACI Code 9.9 and 13.2 specifically allow deep beam and foundation design to be completed with this method.
- Deep beams represent one of the principal applications of strut-and-tie models, since the alternative under ACI Code 9.9 is a nonlinear analysis.

Procedure of solution:

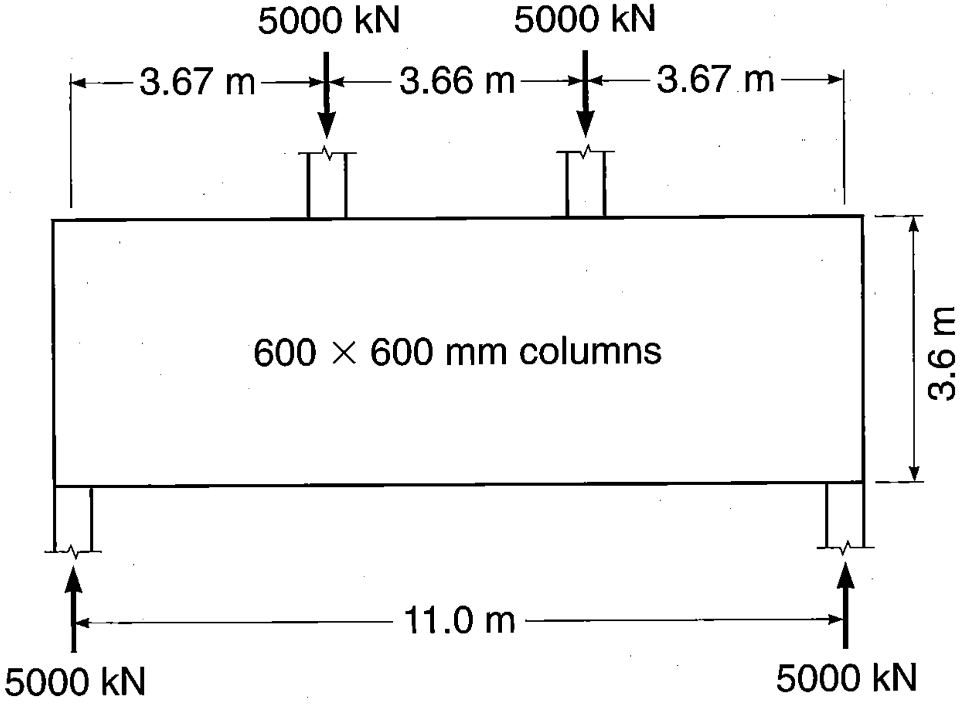
1. Based on assumed distance between horizontal strut and tie, find angle θ.
2. Find the internal forces Fi in struts and ties intersecting in each node (joint).
3. Determine the stress on nodal zone from, p= column load / bearing area.
4. Determine the width of each struts and ties intersecting at this nodal zone from:

   width=force in the member/(thickness × stress)

   $w = F_i/(b \times p)$

5. Revised distances and angle are calculated.
6. New internal forces are determined.
7. Steps 5 and 6 are repeated until convergent values are obtained.
8. The geometry of each member is selected and the stresses are checked.

EXAMPLE 1: A transfer girder is to carry two 600 mm square columns, each with factored loads of 5000 kN located at the third points of its 11 m span, as shown in Figure. The beam has a thickness of 600 mm and a total height of 3.6 m. Design the beam for the given loads, ignoring the self-weight, using $f_c' = 35\ MPa$ and $f_y = 420\ MPa$.



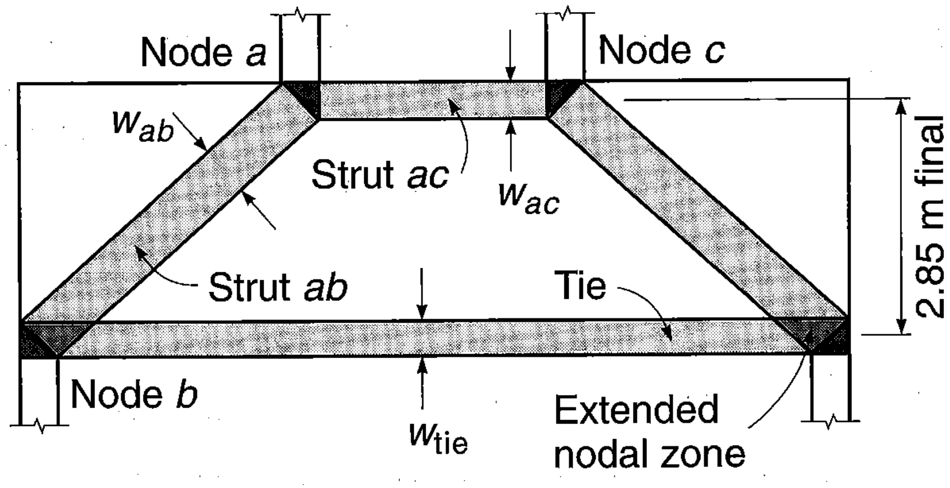
Solution: The span-to-depth ratio for the beam is 3, thereby qualifying it as a deep beam.

- A strut-and-tie solution will be used.
- The entire beam is within a distances not more than h from loads or supports.
- So the entire structure may be characterized as a D-region.
- The thickness of the struts and ties is equal to the thickness of the beam b = 600 mm.

1) checking for shear:

- maximum applied shear force= 5000 kN.
- the maximum design shear capacity is
- $\emptyset V_n = 0.83\emptyset\sqrt{f_c'}\ b_w\ d$
- Assuming effective depth d= 0.9 h
- d= 0.9×3.6 = 3.24 m
- $\emptyset V_n = 0.83 \times 0.75 \times \sqrt{35}\ \times 600 \times 3.24$ = 7159 kN > 5000 kN o.k.

- Based on the beam geometry and loading, a single truss is sufficient to carry the column loads, as shown .
- The truss has a trapezoidal shape. This is an acceptable solution since the nodes are not true pins and instability within the plane of the truss is not a concern in a strut-and- tie model.
- The truss geometry is established by the assumed intersection of the struts and ties and used to determine the angle θ.
- The reactions at supports are: R1 = R2 = 5000 kN



Assume center-to-center distance between the horizontal strut and the tie = 0.8 h= 0.8×3.60 = 2.88 m

Thus the diagonal struts ab form at an angle $\theta = tan^{-1}(2.88/3.67)$

$\theta = \ 38.12\ degree$

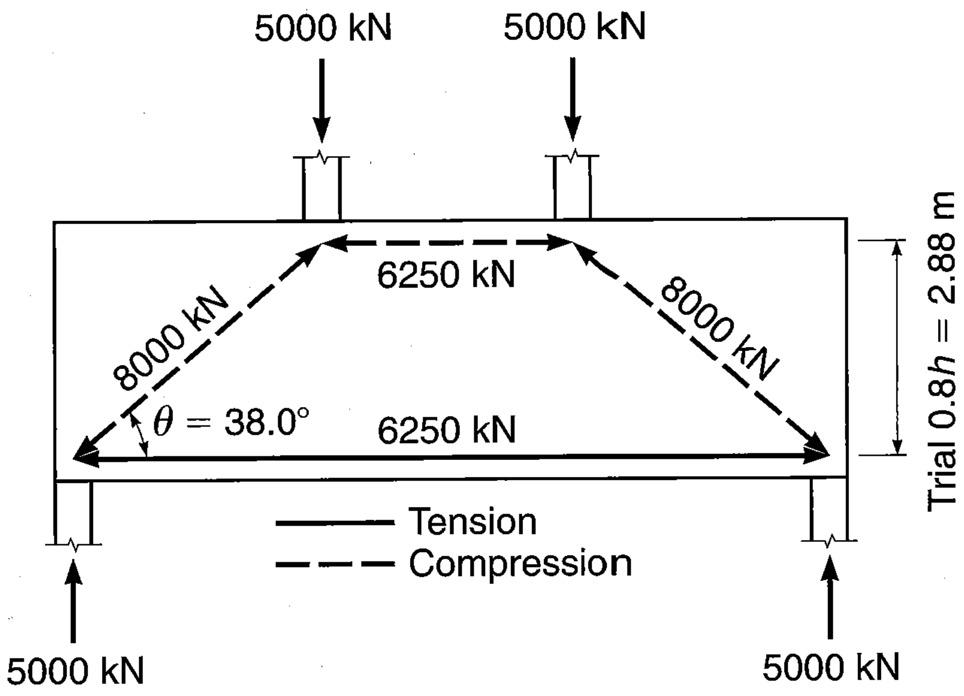
$At\ joint\ a: \sum F_y = 0\ ,\ F_{ab} sin(38.12) = 5000,\qquad F_{ab} =$ 8100 kN (compression)

also, $F_{bc} = \ F_{ab}\ cos(38.12) = \ 6372\ kN$ *(compression)*

$From\ joint\ b: \sum F_x = 0,\ F_{bd} =$ *6372 kN* (tension)

The truss model

- Based on the beam geometry and loading, a single truss is sufficient to carry the column loads.
- The truss has a trapezoidal shape. This is an acceptable solution since the nodes are not true pins and instability within the plane of the truss is not a concern in a strut-and-tie model.
- The truss geometry is established by the assumed intersection of the struts and ties and used to determine ϑ .



Selecting dimensions for strut and nodal zones

As mentioned in Section 10.2c, two approaches are used to select strut and nodal zone sizes: selection based on the geometry of the load-bearing transfer elements to maintain a constant level of stress p and selection based on the minimum strut width w. For this example, the first approach will be used. The nodal stress p is determined by the average stress under the columns. Thus, $p = 5000/(600 \times 600) = 13.9$ MPa. Because this is a C-C-C node, β_n is 1.0, and the design strength of the node is $\phi f_{ce} = \phi 0.85\beta_n f_c' = 0.75 \times 0.85 \times 1.0 \times 35 = 22.3$ MPa, which is greater than the demand from the column. This result implies that smaller truss member sizes may be possible, a point that is discussed at the end of this example. The width of strut ac, found using p, is

$$w_{ac} = F_{ac}/(b_s \times p) = 6250 \times 10^3/(600 \times 13.9) = 749 \text{ mm}$$

Similarly, $w_{ab} = 959$ mm and $w_{\text{tie}} = 749$ mm. The center-to-center distance between the horizontal strut ac and the tie is $3.600 - 0.749 = 2.851$ m, or $0.79h$. The angle θ between the diagonal strut ab and the tie is thus 38.3°. Using an angle of 38.0° gives a revised force in strut ab of 8121 kN. Similarly, the revised force in strut ac and the tie is 6345 kN, while the widths are revised to $w_{ab} = 974$ mm, and w_{ac} and $w_{\text{tie}} = 761$ mm. (*Note*: After iteration, the actual angle becomes 38.2°. The value of 38.0° is conservative and is, thus, retained.)

Capacity of struts

The horizontal strut *ac* will be assumed to have a uniform cross section, while the diagonal struts will be considered as bottle-shaped because of the greater width available. Strut capacity is given in Eqs. (10.2) and (10.3), which, when combined, give $\phi F_{ns} = \phi 0.85\beta_s f_c' w_i b_s$, where w_i is the width of the strut and β_s is 1.0 for a rectangular strut. For strut *ac*,

$$\phi F_{ns} = 0.75 \times 0.85 \times 1.0 \times 35 \times 761 \times 600 \times 10^{-3} = 10{,}190 \text{ kN} > 6345 \text{ kN}$$

Therefore, strut *ac* is adequate. Similarly, for strut *ab*,

$$\phi F_{ns} = 0.75 \times 0.85 \times 0.75 \times 35 \times 974 \times 600 \times 10^{-3} = 9780 \text{ kN} > 8121 \text{ kN}$$

From Eqs. (10.6) and (10.7), the capacity of the nodal zone is $\phi F_{nn} = \phi 0.85\beta_n f_c' w_i b_s$. At *a*, a *C-C-C* node, $\beta_n = 1.0$, and at *b*, a *C-C-T* node, $\beta_n = 0.80$. Thus, the capacity of strut *ab* is established at node *b* with $\beta_n = 0.80$ and

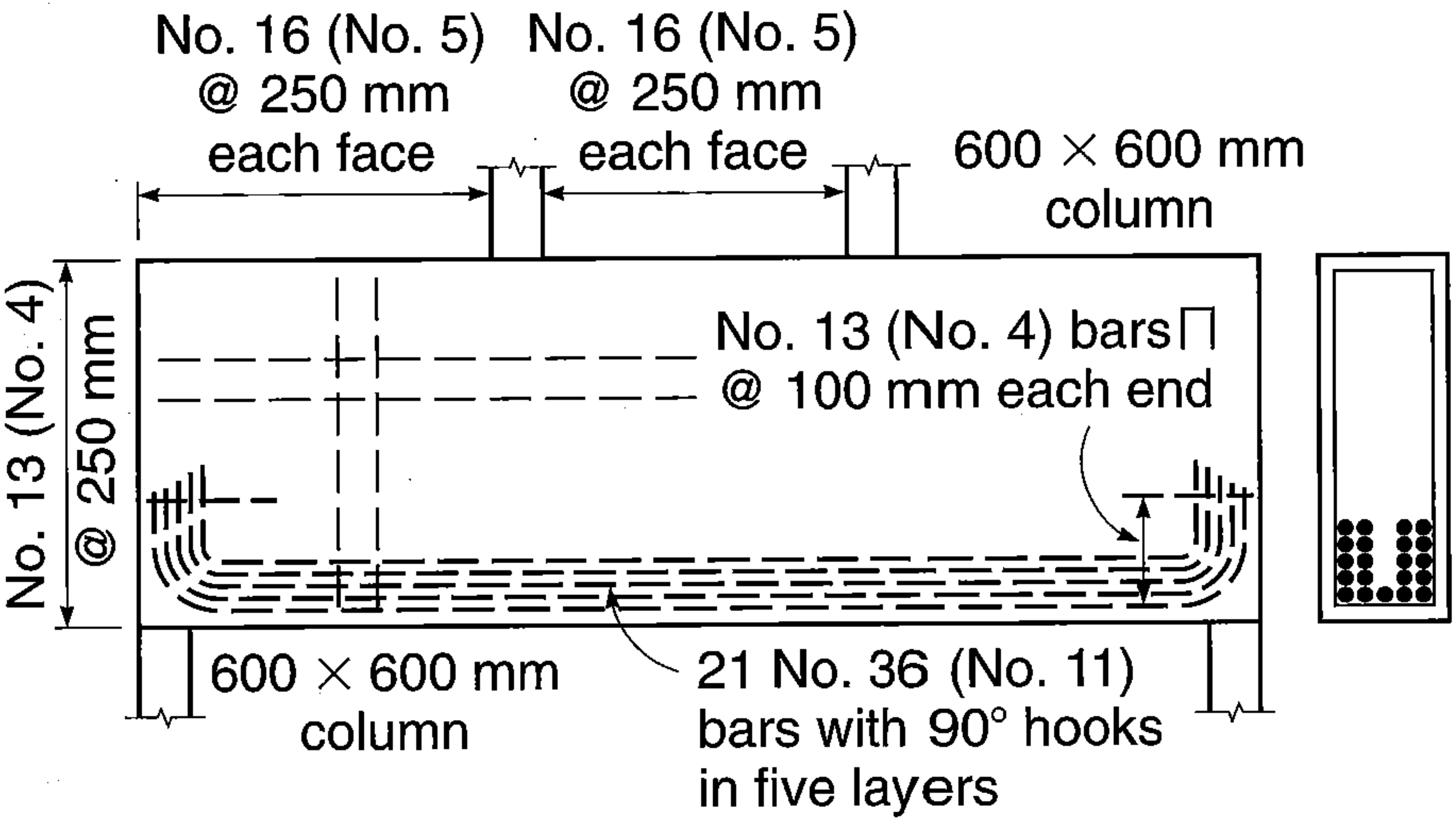
$$\phi F_{nn} = 0.75 \times 0.85 \times 0.80 \times 35 \times 974 \times 600 \times 10^{-3} = 10{,}430 \text{ kN} \geq 8121 \text{ kN}$$

Similarly, the nodal end capacity of strut *ac* is 10,190 kN with $\beta_n = 1.0$. The capacity at the end of the struts and at the nodes exceeds the factored loads, and thus, the struts are adequate.

Design ties and anchorage

The tie design consists of three steps: selection of the area of steel, design of the anchorage, and validation that the tie fits within the available tie width. The steel area is computed as $A_{ts} = F_{tu}/\phi f_y = 6345 \times 10^3/(0.75 \times 420) = 20{,}140\ \text{mm}^2$. This is satisfied by using 21 No. 36 (No. 11) bars, having a total area of $A_{ts} = 20{,}120\ \text{mm}^2$. Placing the bars in one layer of five bars and four layers, while allowing for 60 mm clear cover to the bottom of the beam and 112 mm clear spacing between layers, results in a total tie width of $5 \times 35.8 + 4 \times 112 + 2 \times 60 =$ 747 mm, closely matching the tie dimension.

The anchorage length l_d for No. 36 (No. 11) bars (from Table A.10 in Appendix A) is $42d_b = 1.617$ m. The length of the nodal zone and extended nodal zone is $600 + 0.5 \times 761 \cot 38.0° = 1.087$ m, which is less than l_d. The beam geometry does not allow the tie reinforcement to extend linearly beyond the node; therefore, 90° hooks or mechanical anchors are required on the No. 36 (No. 11) bars. Placement details are covered in the next section. Allowing 40 mm cover on the sides, No. 16 (No. 5) transverse and horizontal reinforcement, and 65 mm spacing between No. 36 (No. 11) bars, five No. 36 (No. 11) bars require a total thickness of $b_{\text{reqd}} = 2 \times 40 + 4 \times 15.9 + 4$ spaces @ $65 + 5$ bars $\times\ 35.8 = 582.6$ mm, which fits within the 600 mm beam thickness.

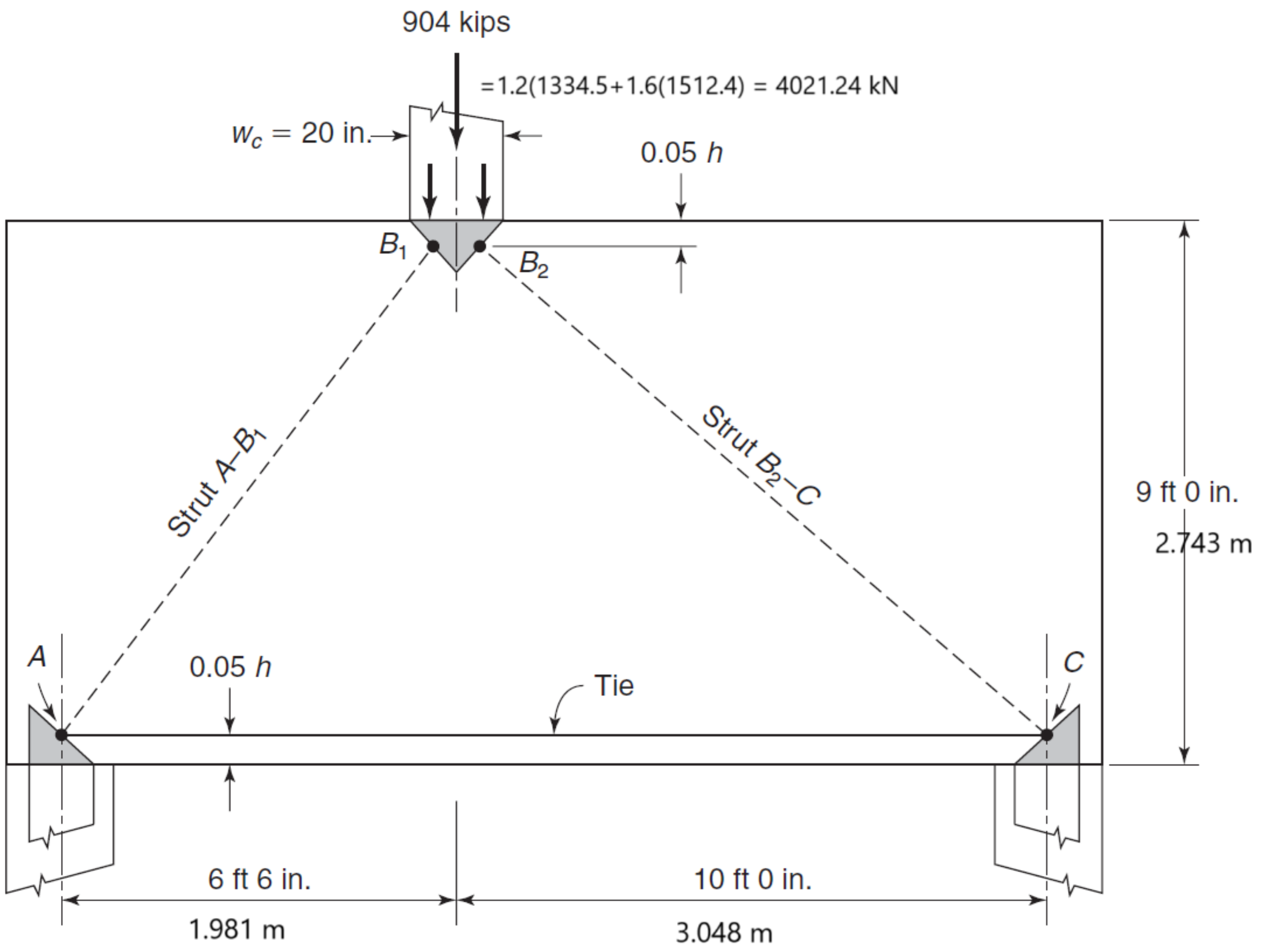

No. 16 (No. 5) @ 250 mm each face
No. 16 (No. 5) @ 250 mm each face
600 × 600 mm column
No. 13 (No. 4) @ 250 mm
No. 13 (No. 4) bars ⊓ @ 100 mm each end
600 × 600 mm column
21 No. 36 (No. 11) bars with 90° hooks in five layers

Simplified tension development in bar diameters l_d/d_b for uncoated bars and normalweight concrete

	f_y, MPa	No. 19 (No. 6) and Smaller[a] f'_c, MPa 28	35	42	No. 22 (No. 7) and Larger f'_c, MPa 28	35	42
(1) Bottom bars							
Spacing, cover and ties as per Case *a* and *b*	280	25	23	21	31	28	25
	420	38	34	31	47	42	38
	520	47	42	38	58	52	47
Other cases	280	38	34	31	48	43	39
	420	57	51	46	72	65	59
	520	70	63	57	89	80	73
(2) Top bars							
Spacing, cover and ties as per Case *a* and *b*	280	33	29	27	40	36	33
	420	49	44	40	61	54	50
	520	61	54	50	75	67	61
Other cases	280	49	44	40	63	56	51
	420	74	66	60	94	84	77
	520	91	82	75	116	104	95

EXAMPLE: Design a deep beam to support an un-factored column load of 300 kips (1334.5 kN) dead load and 340 kips (1512.4) live load from a 20-in.-by-20-in. column (508mm×508mm).

The axes of the supporting columns are 6.5 ft (1.98m) and 10 ft (3.05m) from the axis of the loading column as shown. The supporting columns are also 20 in. (508 mm) square. Use 4000-psi (27.6 MPa) normal-weight concrete and Grade-60 (414 MPa) steel.



1. Select a Strut-and-Tie Model—Single-Span Deep Beam.

(a) Select shape and flow of forces: The strut-and-tie model is shown.

It consists of two inclined struts, three nodal zones, and one tie. We shall assume that a portion of the column load equal to the left reaction flows down through strut A-B to the reaction at A. The rest of the column load is assumed to flow down strut B–C to the reaction at the right end of the beam.

(b) Compute the factored load and the reactions—single-span deep beam: The factored load from the column is

$$P_u = 1.2D + 1.6L = 1.2 \times 300 + 1.6 \times 340 = 904 \text{ kips}$$

The reactions are 548 kips at A and 356 kips at C. The dead load of the beam will be added in after the size of the beam is known.

(c) Design format: In design we will use $\phi V_n = V_u$ where, from ACI Code Section 9.3.2.6, $\phi = 0.75$.

2. Estimate the Size of the Beam. This will be done in two different ways.

(a) Limit ϕV_n to $\phi \times (0.7 \text{ to } 1.0) \times 10\sqrt{f'_c}\, b_w d$: ACI Code Section 11.7.3 limits the shear in a deep beam to $10\sqrt{f'_c}\, b_w d$. To allow a little leeway and to leave some capacity for the dead load, we will select the beam using maximum shear forces of $\phi 7\sqrt{f'_c}\, b_w d$ to $\phi 10\sqrt{f'_c}\, b_w d$. The maximum shear ignoring the dead load of the beam is 548 kips, so

$$b_w d = \frac{V_n}{\phi \times (7 \text{ to } 10)\sqrt{f'_c}} = \frac{548{,}000 \text{ lb}}{0.75(7 \text{ to } 10)\sqrt{4000}} = 1650 \text{ to } 1160 \text{ in.}^2$$

Try b_w equal to the width of the columns, $b_w = 20$ in. This gives d from 82.5 to 58.0 in. Assuming that $h \approx d/0.9$ gives h from 64.4 to 91.7 in. Try a 20-by-96-in. beam.

(b) Select height to keep the flattest strut at an angle of ≈40° from the tie. Note that a limit of 40° is more restrictive than the limit of 25° in ACI Code Section A.2.5. The strut-and-tie forces begin to grow rapidly for angles less than 40°. For the 120-in. shear span, the minimum height center to center of nodes is $120 \tan 40° = 101$ in. Assuming that the nodes at the bottom of this strut are located at midheight of a tie with an effective tie height of $0.1h$, this puts the lower nodes at $0.05h = 5.1$ in. from the bottom of the beam. We shall locate the node at B the same distance below the top, giving the required height of the beam as $h \approx 101/0.9 = 112$ in.

For a first trial, we shall assume $b_w = 20$ in. and $h = 108$ in. with the lower chord located at $0.05h = 5.4$ in. above the bottom of the beam and the center of the top node at $0.05h = 5.4$ in. below the top of the beam.

The weight of the beam is

$$[(20 \text{ in.}/12 \text{ in./ft.}) \times (108/12) \text{ ft} \times (16.5 + 1.67) \text{ ft} \times 0.150 \text{ kips/ft}^3)] = 40.9 \text{ kips}$$

The factored weight is $1.2 \times 40.9 = 49.1$ kips. To simplify the calculations we shall add this to the column load giving a total factored load of 954 kips. The corresponding reactions are 578 kips at A and 376 kips at C.

First trial design is based on a beam with $b_w = 20$ in. and $h = 108$ in., with $P_u = 954$ kips, and V_u equal to 578 kips in shear span A–B, and 376 kips in shear span B–C.

3. Compute Effective Compression Strengths, f_{ce}, for the Nodal Zones and Struts—Single-Span Deep Beam.

(a) Nodal zones (ACI Code Section A.5.2):

Nodal zone at A: This is a C–C–T node.

$$f_{ce} = 0.85\beta_n f'_c = 0.85 \times 0.80 \times 4000 = 2720 \text{ psi}$$

Nodal zone at *B*: This is a *C–C–C* node.

$$f_{ce} = 0.85\beta_n f'_c = 0.85 \times 1.0 \times 4000 = 3400 \text{ psi}$$

Nodal zone at *C*. This is a *C–C–T* node.

$$f_{ce} = 2720 \text{ psi.}$$

(b) Struts (ACI Code Section A.3.2):

Strut *A–B*: This strut has room for the width of the strut to be bottle-shaped. We will provide reinforcement satisfying ACI Code Section A.3.3.

$$f_{ce} = 0.85\beta_s f'_c = 0.85 \times 0.75 \times 4000 = 2550 \text{ psi}$$

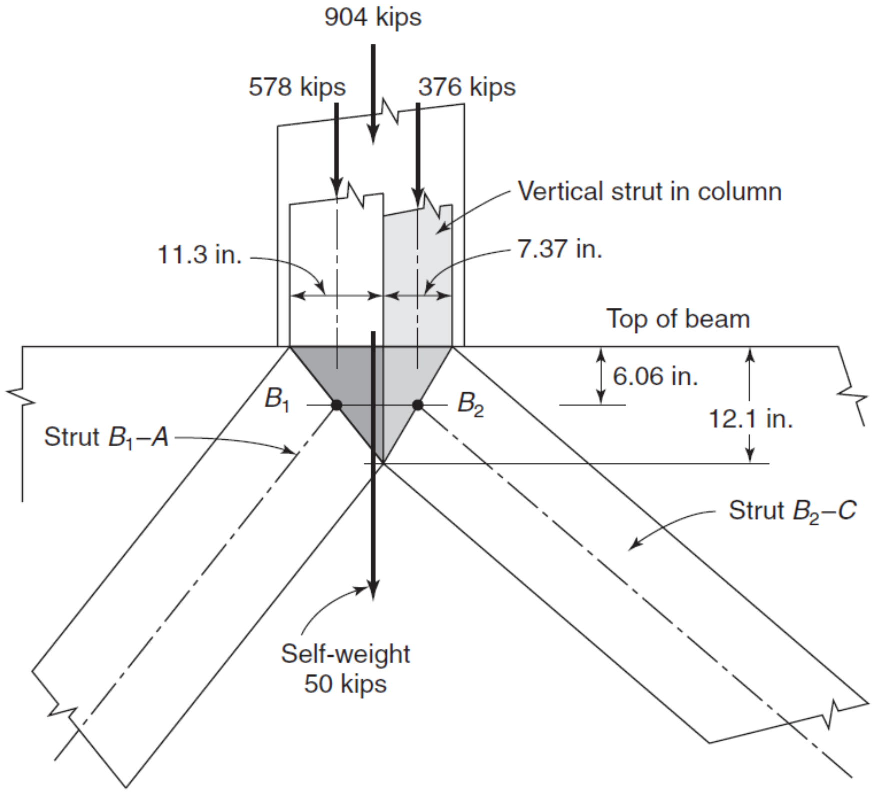
Strut *B–C*: This also is a bottle-shaped strut, and $f_{ce} = 2550$ psi

The effective compression strengths of the struts govern. **Use $f_{ce} = 2550$ psi** throughout.

4. Locate Nodes—First Trial—Single Span Deep Beam.

Nodal zones at *A* and *C*: In step 2(a), we assumed that the nodes at *A* and *C* are located at $0.05h = 5.4$ in. above the bottom of the beam. Use this location in the first trial strut-and-tie model.

Nodal zone at B: Assume node B is located at $0.05h = 5.4$ in. below the top of the beam. The node at B is subdivided into two subnodes, as shown in Fig. 17-30, one assumed to transmit a factored load of 578 kips to the left reaction and the other transmitting 376 kips to the right reaction. Each of the subnodes is assumed to receive load from a vertical strut in the column over B. Both of the subnodes are C–C–C nodes. The effective strength of the nodal zone at B is $f_{ce} = 3400$ psi. For struts $A-B_1$ and B_2-C, $f_{ce} = 2550$ psi. We shall use $f_{ce} = 2550$ psi for the faces of the nodal zone loaded by bottle-shaped struts, and $f_{ce} = 3400$ psi for the face of the nodal zone that



is loaded by compression stresses in the column and for the vertical struts in the column over point B.

$$w_{B1} = \frac{578{,}000}{0.75 \times 3400 \times 20} = 11.3 \text{ in.}$$

$$w_{B2} = 7.37 \text{ in.}$$

The total width of the vertical struts in the column at B is $\Sigma w_{BS} = 18.7$ in.

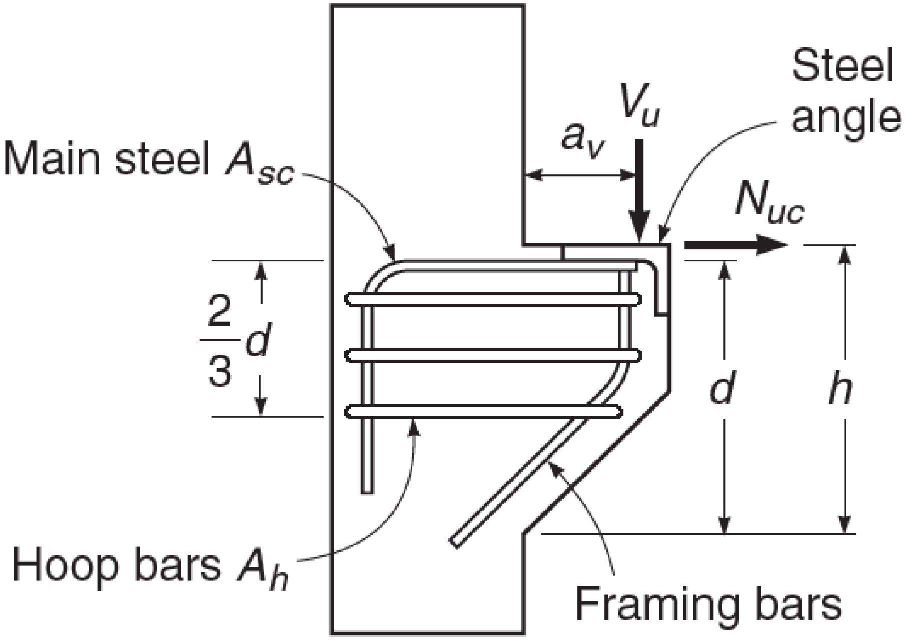
This will fit in the 20-in.-square column. If it did not, ACI Code Section A.3.5 allows us to include the capacity of the column bars, $\phi A_s f_y$, when computing the capacity of the strut.

Because the vertical load components acting at B_1 and B_2 are different, the vertical division of node B is not located at the center of the column. Assuming equal widths for the unloaded portions of the column outside the two vertical struts in Fig. 17-30, the resultant force in the left-hand strut acts at point B_1, which is 3.7 in. to the left of the center of the column, and the right-hand strut acts at point B_2, located 5.7 in. to the right of the center.

The horizontal distance from A to B is 78 in.; that from B to C is 120 in.

BRACKETS AND CORBELS:

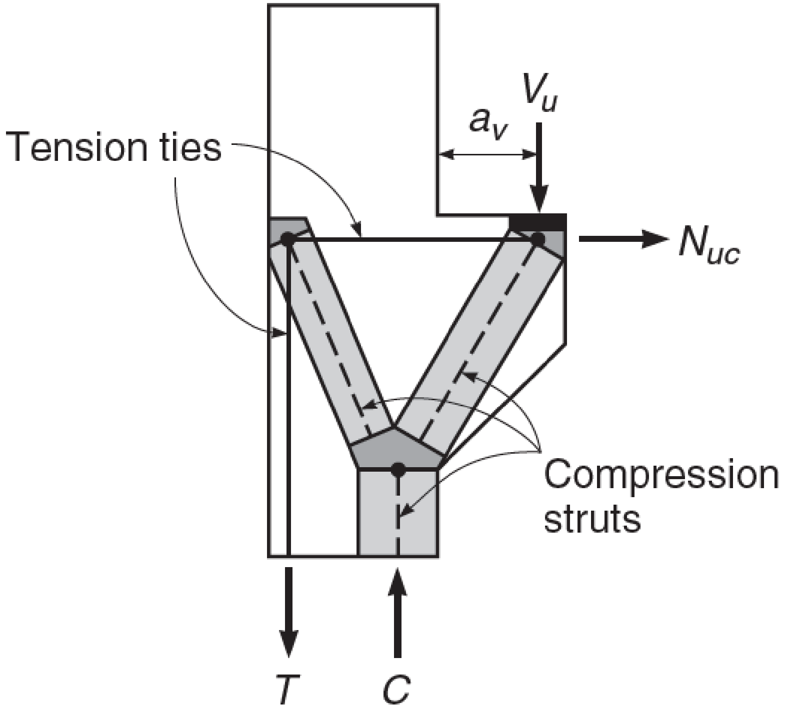
- Brackets are widely used in precast construction for supporting precast beams at the columns.
- When they project from a wall, rather than from a column, they are properly called corbels, although the two terms are often used interchangeably.
- Brackets are designed mainly to provide for the vertical reaction V_u at the end of the supported beam, but they must also resist a horizontal force N_{uc}.



- Steel bearing plates or angles are generally used on the top surface of the brackets to provide a uniform contact surface and to distribute the reaction.
- A corresponding steel bearing plate or angle is usually provided at the lower corner of the supported member.
- If the two plates are welded together, horizontal forces clearly must be allowed for in the design.
- Even with Teflon or elastomeric bearing pads, frictional forces will develop due to volumetric change.

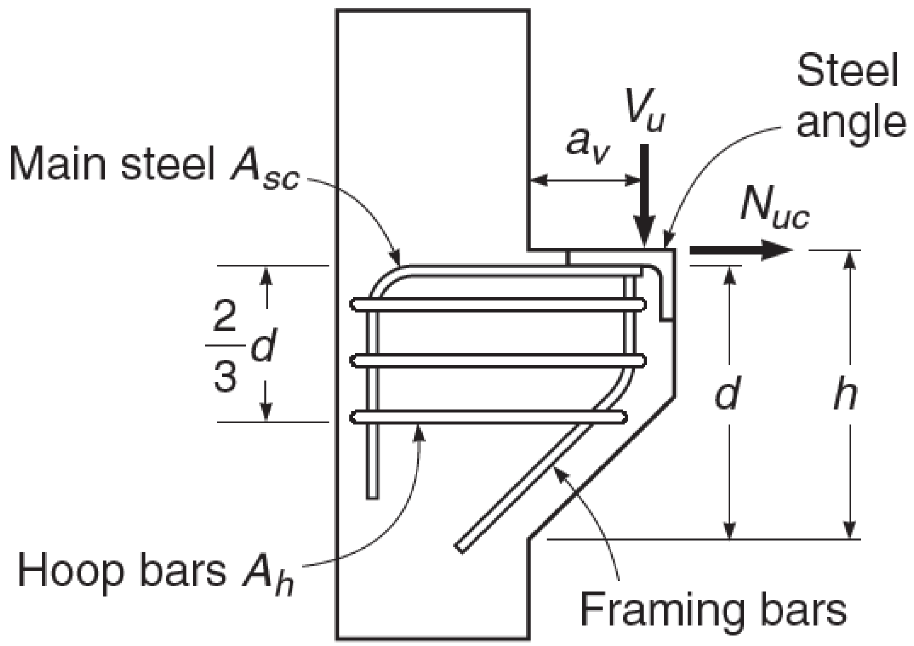
Analysis of R. C. Brackets:

- The structural performance of a bracket can be visualized easily by means of the strut-and-tie model.
- The downward thrust of the load Vu is equilibrated by the vertical component of the reaction from the diagonal compression strut that carries the load down into the column.
- The outward thrust at the top of the strut is balanced by the tension in the horizontal tie bars across the top of the bracket; these also take the tension, if any, imparted by the horizontal force Nuc .
- .



- At the left end of the horizontal tie, the tension is equilibrated by the horizontal component of thrust from the second compression strut shown
- The vertical component of this thrust requires the tensile force shown acting downward at the left side of the supporting column.
- The steel required, according to the strut-and-tie model is as shown in figure.

- The main bars Asc must be carefully anchored because they need to develop their full yield strength fy directly under the load Vu.
- For this reason they are usually welded to the underside of the bearing angle and a 90° hook is provided for anchorage at the left side.



- Closed hoop bars with area A_h confine the concrete in the two compression struts and resist a tendency for splitting in a direction parallel to the thrust.
- The framing bars shown are usually of about the same diameter as the stirrups and serve mainly to improve the stirrup anchorage at the outer face of the bracket.
- The bracket may also be considered as a very short cantilevered beam, with flexural tension at the column face resisted by the top bars Asc .
- Either concept will result in about the same area of main reinforcement.

- A second possible mode of failure is by direct shear along a plane more or less flush with the vertical face of the main part of the column.
- Shear-friction reinforcement crossing such a crack would include the area Asc previously placed in the top tie and the area Ah from the hoops below it.
- Other failure modes include:

1) flexural tension failure, with yielding of the top bars followed by crushing of the concrete at the bottom of the bracket;
2) crushing of the concrete under the bearing angle (particularly if end rotation of the supported beam causes the force Vu to be applied too close to the outer corner of the bracket); and
3) direct tension failure, if the horizontal force Nuc is larger than anticipated.

Provisions of ACI code:

- The provisions of ACI Code for the design of brackets and corbels have been developed mainly based on tests and relate to the flexural model of bracket behavior.
- They apply to brackets and corbels with a shear span ratio a_v/d of 1.0 or less.
- Brackets and corbels with a_v/d less than 2 may be designed using strut-and-tie models.
- The distance **d is measured at the column face**, and the depth at the outside edge of the bearing area **must not be less than 0.5 d**.
- The usual design basis is employed, that is,

$$\phi M_n \geq M_u \text{ and } \phi V_n \geq V_u ,$$

- and for brackets and corbels (for which shear dominates the design), ϕ is to be taken equal to 0.75 for all strength calculations, including flexure and direct tension as well as shear.

- The section at the face of the supporting column must simultaneously resist:

a) the shear Vu ,

b) the moment [$Mu = Vu\ a_v + Nuc\ (h - d)$],and

c) the horizontal tension Nuc .

- Unless special precautions are taken, a horizontal tension not less than 20 percent of the vertical reaction must be assumed to act.
- This tensile force is to be regarded as live load, and a load factor of 1.6 should be applied.

- An amount of steel A_f to resist the moment Mu can be found by the usual methods for flexural design.
- Thus,

$$A_f = \frac{M_u}{\phi f_y (d - a/2)}$$

where $a = A_f f_y / 0.85 f_c' b$. An additional area of steel A_n must be provided to resist the tensile component of force:

$$A_n = \frac{N_{uc}}{\phi f_y} \tag{11.7}$$

The total area required *for flexure and direct tension* at the top of the bracket is thus

$$A_{sc} \geq A_f + A_n \tag{11.8}$$

- Design for shear is based on the shear-friction method and the total shear-friction reinforcement A_{vf} is found by

$$A_{vf} = \frac{V_u}{\phi \mu f_y}$$

where the friction factor μ for monolithic construction is 1.4λ, where $\lambda = 1.0$ for normalweight concrete and 0.75 for both sand-lightweight and all-lightweight concrete, in accordance with ACI Code 11.6. The value of $V_n = V_u/\phi$ must not exceed the smallest of $0.2f'_c b_w d$, $(480 + 0.08f'_c)b_w d$, and $1600b_w d$ at the support face for normalweight concrete or the smaller of $(0.2 - 0.07a_v/d)f'_c b_w d$ and $(800 - 280a_v/d)b_w d$ for lightweight concrete. Then, according to ACI Code 11.8, the total area required *for shear plus direct tension* at the top of the bracket is

$$A_{sc} \geq \frac{2}{3} A_{vf} + A_n \qquad (11.10)$$

Table 22.9.4.4—Maximum V_n across the assumed shear plane

Condition	Maximum V_n		
Normalweight concrete placed monolithically or placed against hardened concrete intentionally roughened to a full amplitude of approximately 6 mm	Least of (a), (b), and (c)	$0.2f_c'A_c$	(a)
		$(3.3 + 0.08f_c')A_c$	(b)
		$11A_c$	(c)
Other cases	Lesser of (d) and (e)	$0.2f_c'A_c$	(d)
		$5.5A_c$	(e)

with the remaining part of A_{vf} placed in form of closed hoops having area A_h in the lower part of the bracket, as shown in Fig. 11.23*a*.

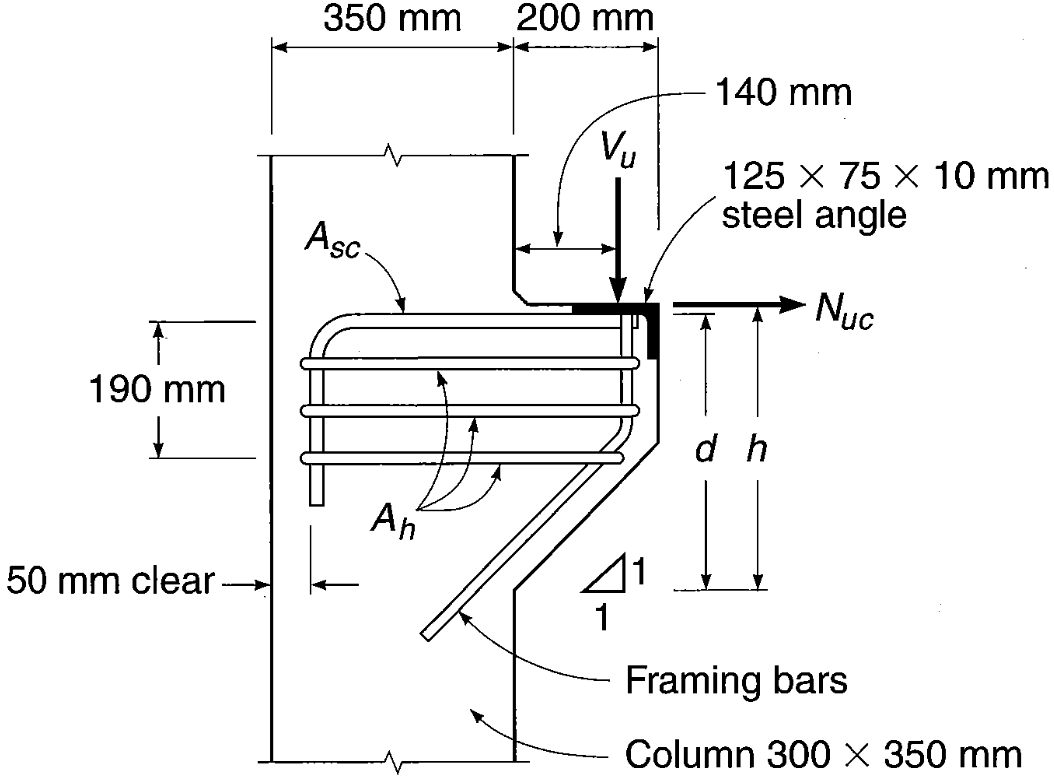
Thus, the total steel area A_{sc} required at the top of the bracket is equal to the larger of the values given by Eq. (11.8) or (11.10). An additional restriction, that A_{sc} not be less than $0.04(f'_c/f_y)bd$, is intended to avoid the possibility of sudden failure upon formation of a flexural tensile crack at the top of the bracket.

According to the ACI Code, closed hoop stirrups having area A_h (see Fig. 11.23*a*) not less than $0.5(A_{sc} - A_n)$ must be provided and be uniformly distributed within two-thirds of the effective depth adjacent to and parallel to A_{sc}. This requirement is more clearly stated as follows:

According to the ACI Code, closed hoop stirrups having area A_h (see Fig. 11.23*a*) not less than $0.5(A_{sc} - A_n)$ must be provided and be uniformly distributed within two-thirds of the effective depth adjacent to and parallel to A_{sc}. This requirement is more clearly stated as follows:

$$A_h \geq 0.5A_f \qquad \text{and} \qquad \geq \frac{1}{3} A_{vf} \tag{11.11}$$

EXAMPLE: A column bracket having the general features shown is to be designed to carry the end reaction from a long-span precast girder. Vertical reactions from service dead and live loads are 110 and 225 kN, respectively, applied at a_v = 140 mm from the column face. A steel bearing plate will be provided for the girder, which will rest directly on a 125 × 75 × 10 mm steel angle anchored at the outer corner of the bracket. Bracket reinforcement will include main steel A_{sc} welded to the underside of the steel angle, closed hoop stirrups having total area A_h distributed appropriately through the bracket depth, and framing bars in a vertical plane near the outer face. Select appropriate concrete dimensions, and design and detail all reinforcement. Material strengths are fc = 35 MPa and fy = 420 MPa.



SOLUTION. The vertical factored load to be carried is

$$V_u = 1.2 \times 110 + 1.6 \times 225 = 492 \text{ kN}$$

In the absence of a roller or low-friction support pad, a horizontal tensile force of

$$N_{uc} = 0.20 \times 492 = 98 \text{ kN}$$

will be included. According to the shear friction provisions of the ACI Code, the nominal shear strength V_n must not exceed $0.2f'_c bd$, $(3.3 + 0.08f'_c)bd$, or $11bd$. With $f'_c = 35$ MPa, the second limit controls. Then, with $V_u = \phi V_n$ and with the column width $b = 300$ mm,

$V_{u1} = 0.2 \times 35 \times bd = 7\ bd$

$V_{u2} = (3.3 + 0.08 \times 35)bd = 6.1\ bd$

$V_{u3} = 11bd$

The second governs, thus

$\phi \times 6.1\ bd = 492 \times 10^3$

b = 300 mm, $\phi = 0.75$

d= 358.5 mm

h=358.5+25= 383.5 mm

Use h = 400 mm, thus d= 375mm

375 mm, the exact value depending on the bar diameter chosen for A_{sc}. If a 45° slope is used, as indicated in Fig. 11.24, the bracket depth at the outside of the bearing area will be 200 mm. This is not less than $0.5d = 187$ mm, as required. For the bracket geometry selected, $a_v/d = 140/375 = 0.37$. This does not exceed the 1.0 limit imposed by the ACI Code.

The total shear friction steel is found from Eq. (11.9):

$$A_{vf} = \frac{V_u}{\phi \mu f_y} = \frac{492 \times 10^3}{0.75 \times 1.4 \times 420} = 1116 \text{ mm}^2$$

The bending moment to be resisted is

$$M_u = V_u a_v + N_{uc}(h - d)$$

$$= 492 \times 0.14 + 98 \times 0.025 = 71 \text{ kN-m}$$

The depth of the flexural compression stress block will be estimated to be 50 mm, so, from Eq. (11.6),

$$A_f = \frac{M_u}{\phi f_y (d - a/2)} = \frac{71 \times 10^6}{0.75 \times 420(375 - 50/2)} = 644 \text{ mm}^2$$

Checking the stress block depth gives

$$a = \frac{A_f f_y}{0.85 f_c' b} = \frac{644 \times 420}{0.85 \times 35 \times 300} = 30 \text{ mm}$$

so the revised steel area is

$$A_f = \frac{71 \times 10^6}{0.75 \times 420(375 - 30/2)} = 626 \text{ mm}^2$$

The tensile force of 98 kN requires an additional steel area, from Eq. (11.7), of

$$A_n = \frac{N_{uc}}{\phi f_y} = \frac{98 \times 10^3}{0.75 \times 420} = 311 \text{ mm}^2$$

Thus, from Eqs. (11.8) and (11.10), respectively, the total steel area at the top of the bracket must not be less than

$$A_{sc} \geq A_f + A_n = 626 + 311 = 937 mm^2$$

or less than

$$A_{sc} \geq \tfrac{2}{3} A_{vf} + A_n = \tfrac{2}{3} \times 1116 + 311 = 1055 \text{ mm}^2$$

The second requirement controls here. The minimum steel requirement of

$$A_{sc,\min} = 0.04 \frac{f'_c}{f_y} bd = 0.04 \times \frac{35}{420} \times 300 \times 400 = 400 \text{ mm}^2$$

is seen not to control. A total of three No. 22 (No. 7) bars, providing $A_{sc} = 1161 \text{ mm}^2$, will be used.

Closed hoop steel having a total area A_h not less than $0.5(A_{sc} - A_n)$ must be provided. Thus,

$$A_h \geq 0.5A_f = 0.5 \times 644 = 322 \text{ mm}^2$$

and

$$A_h \geq 0.5 \times \tfrac{2}{3} A_{vf} = \tfrac{1}{3} \times 1116 = 372 \text{ mm}^2$$

The second requirement controls. Three No. 10 (No. 3) closed hoops will be provided, giving total area $A_h = 426$ mm^2. These must be placed within $\frac{2}{3}$ of the effective depth of the main steel. A spacing of 63 mm will be satisfactory, as indicated in Fig. 11.24. A pair of No. 10 (No. 3) framing bars will be added at the inside corner of the hoops to improve anchorage, as shown.

Anchorage of the No. 22 (No. 7) bars will be provided at the right end by welding to the underside of the steel angle and at the left end by a standard 90° bend (see Fig. 5.10). The basic development length for hooked bars (Table 5.3) is

$$l_{dh} = \left(\frac{0.24\psi_e f_y}{\lambda\sqrt{f_c'}}\right) d_b = \left(\frac{0.24 \times 1 \times 420}{1 \times \sqrt{35}}\right) 22.2 = 378 \text{ mm}$$

Closed hoop steel having a total area A_h not less than $0.5(A_{sc} - A_n)$ must be provided. Thus,

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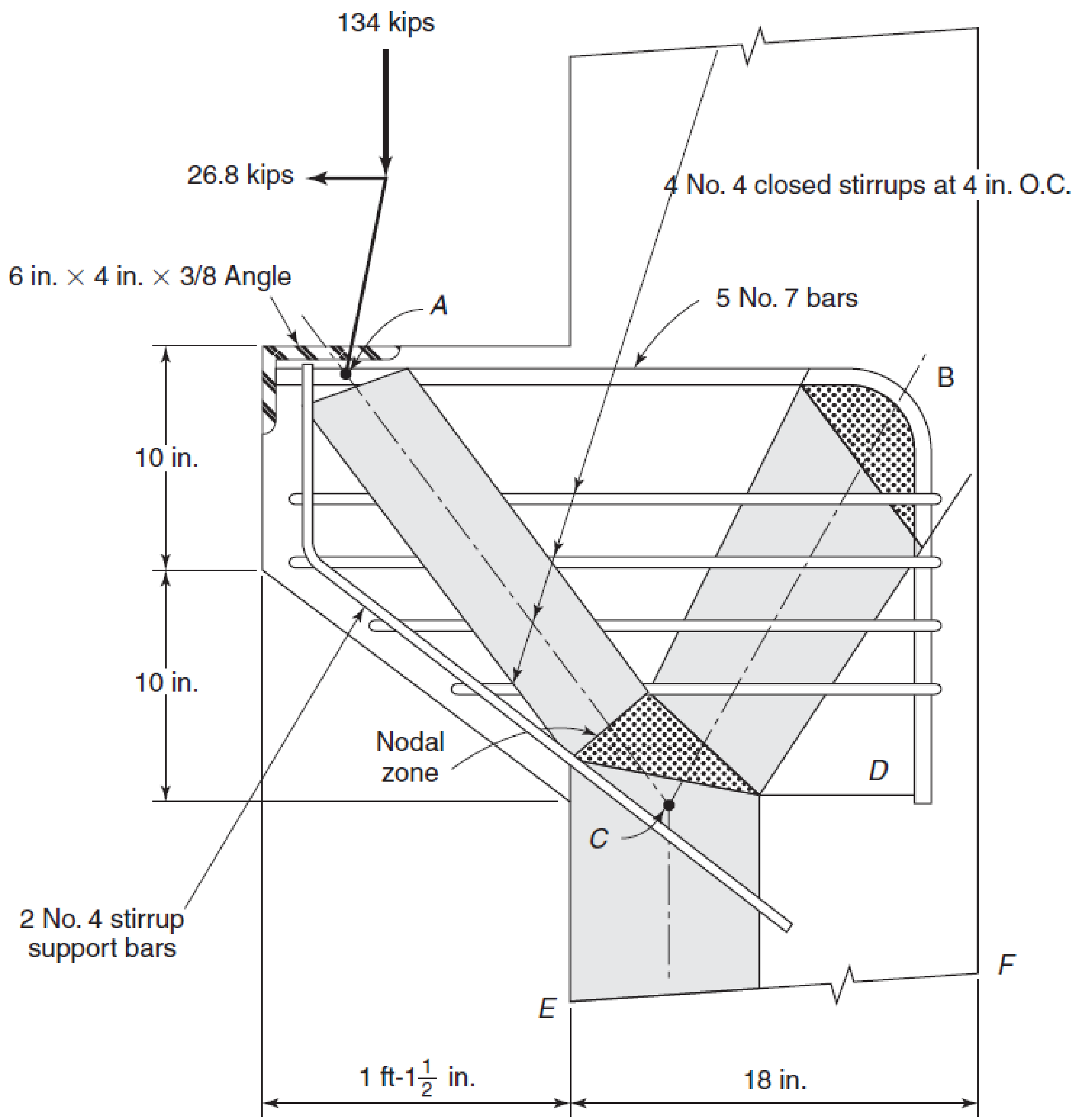
Two modification factors apply here. The first is 0.7, provided at least 50 mm cover is maintained at the end of the hook, and the second is (required A_s)/(provided A_s) = 1055/1161 = 0.91. Thus, the required development length past the face of the column is

$$l_{dh} = 378 \times 0.7 \times 0.91 = 241 \text{ mm}$$

This requirement is easily met. The hook extension will be $12d_b = 12 \times 22.2 = 266$ mm. For the hoop bars, a standard 35° hook, as shown in Fig. 5.9*b*, will be used.

EXAMPLE: Design of a Corbel via a Strut-and-Tie Model -Design a corbel to support the reaction from a precast beam. The end of the beam is 12 in. (305mm) wide. The column is 16 in. (406mm) square. The unfactored beam reaction is 60 kips (267 kN) dead load and 39 kips (173.5 kN) live load. The beam is partially or fully restrained against longitudinal shrinkage. Use $f_c' = 5000\,psi = 34.5\,MPa$ normal-weight concrete and $f_y = 60{,}000\,psi = 414\,MPa$.Use the load factors and strength-reduction factors from ACI Code.



1. Compute the Factored Load—Corbel.

$$U = 1.4D = 1.4 \times 60 = 84 \text{ kips}$$

$$U = 1.2D + 1.6L = 1.2 \times 60 + 1.6 \times 39 = 134 \text{ kips}$$

Thus, the factored vertical load on the corbel is 134 kips.

2. Compute the Distance From the Face of the Column to the Beam Reaction—Corbel.

Assume a 12-in.-wide bearing plate. From ACI Code Section 10.14.1, the maximum bearing stress is $0.85f'_c = 0.85 \times 5000 = 4250$ psi. Using $\phi = 0.65$ as the strength-reduction factor for bearing, the required width of the bearing plate is

$$\frac{134{,}000 \text{ lb}}{0.65 \times 4250 \text{ psi} \times 12} = 4.04 \text{ in.}$$

ACI Code Section 7.7.1 requires 1.5-in. cover to stirrups or main reinforcement. Try a bearing plate 6 in. by 12 in. by 1.5 in. thick, with the top surface flush with the top of the corbel. This will provide 1.5-in. cover to the principal reinforcement at the top of the corbel.

Assume that the beam extends 7.5 in. past the center of the bearing plate. The design gap between the end of the beam and the face of the column is 1 in. Erection tolerances could make this as large as 2 in. or as small as 0 in. Thus, the beam reaction could act as far as 9.5 in. from the face of the column.

3. Establish the Depth of the Corbel. ACI Code Section 11.8 does not give guidance in choosing the size of a corbel. ACI Code Section 11.8.3.2.1 limits the interface shear transfer stress to the smallest of $0.2f'_c$, $480 + 0.08f'_c$, and 1600 psi. For 5000-psi concrete, the second limit governs. We shall use this as guidance, but use a corbel somewhat larger than the minimum. To simplify forming, the corbel will have the same width as the column, $b = 16$ in. Then

$$d = \frac{134{,}000 \text{ lb}}{0.75 \times 880 \text{ psi} \times 16 \text{ in.}} = 12.7 \text{ in.}$$

where the strength-reduction factor for the strut-and-tie model, $\phi = 0.75$, is used.

The smallest corbel that satisfies ACI Code Section 11.8.3.2.1 would have $b = 16$ in., $h = 15$ in., and $d = 15.0 - 1.5 - d_b/2$—say, 13 in. For conservatism, try $b = 16$ in. and $h = 20$ in.; then, assuming No. 8 bars in the tie A–B, $d = 20$ in. $-$ (1.5-in. cover) $-$ (1/2 bar diameter) $= 18$ in.

Try $b = 16$ in., $h = 20$ in., $d = 18$ in.

4. Select Design Method—Corbel. ACI Code Section 11.8.1 allows corbels with a/d between 1.0 and 2.0 to be designed by using Appendix A. Corbels with a/d less than 1.0 can be designed by using Appendix A or by using the traditional ACI corbel design method in 11.8. Here, $a/d = 9.5/18 < 1.0$, so either design method may be used. We shall use Appendix A.

5. Select a First Trial Strut-and-Tie Model—Corbel. ACI Code Section 11.8.3.4 requires corbels to be designed for a shear, V_u, equal to the factored beam reaction and for a factored tensile force, N_{uc}, equal to the actual tension force acting on the corbel, but not less than 0.2 V_u. This force represents the forces induced by restrained shrinkage of the structure supported by the corbel [17-24].

Design the corbel for $V_u = 134$ kips and $N_{uc} = 26.8$ kips: (See Fig. 17-38.) The forces V_u and N_{uc} can be resolved into an inclined force that intercepts the centroid of the tie reinforcement at node A, which is located at $9.5 + 0.2 \times (1.5 + 0.5)$ in. =9.9 in., from the face of the column.

Figure 17-38 shows the final strut-and-tie model. Assuming (a) that the column reinforcement is No. 9 bars with No. 3 ties and (b) that the nodes B and D are in the plane of the column reinforcement located at 1.5-in. cover + 0.375 in. ties + 1.270/2 = 2.5 in. from the right-hand face of the column, gives the distance from A to B as 9.9 in. + 16 in. $-$ 2.5 in. = 23.4 in.

Locate node *C*: Node *C* is located at a distance $a/2$ from the left side of the column, where a is the depth of the stress block resisting the force in the strut *C–E*. ACI Code Section A.5.2 gives the effective compression strength of the nodal zone as

$$f_{ce} = 0.85\beta_n f'_c \tag{17-7b}$$

(ACI Eq. A-8)

The node at *C* anchors three compression struts and a tension tie, to be discussed later. From ACI Code Section A.5.2, a nodal zone anchoring one tie has $\beta_n = 0.80$, so that

$$f_{ce} = 0.85 \times 0.80 \times 5000 \text{ psi} = 3400 \text{ psi}$$

and

$$a = \frac{P_{u,C-E} \text{ kips} \times 1000 \text{ lb/kip}}{0.75 \times 3400 \text{ psi} \times 16 \text{ in.}} = 0.0245 P_{u,C-E} \text{ in.}$$

where $P_{u,C-E}$ is in kips and the 0.75 is the ϕ factor for strut-and-tie models from ACI Code Section 9.3.2.6. Summing moments about *D* gives

$$\Sigma M_D = 0 = 134 \text{ kips} \times 23.4 \text{ in.} + 26.8 \text{ kips} \times 18 \text{ in.} - P_{u,C-E} \times (13.5 \text{ in.} - a/2)$$

For $a = 0.0245P_{u,C-E}$, this becomes

$$0 = 295{,}000 - 1100P_{u,C-E} + (P_{u,C-E})^2$$

$$P_{u,C-E} = \frac{1100 \pm \sqrt{1100^2 - 4 \times 1 \times 295{,}000}}{2} = 637 \text{ or } 463 \text{ kips}$$

Thus, $P_{u,C-E} = 463$ kips and $a = 0.0245 \times 463 = 11.3$ in. Node *C* is $11.3/2 = 5.65$ in. from the left edge of the column adjacent to *C*, and *C–D* is 13.5 in. − 5.65 in. = 7.85 in.

The nodal zone at *C* takes up 11.35 in. out of the 13.5 in. width between the compression face of the column and node *D*. For comparison, the rectangular stress block in a beam would have a depth of about $0.3d$. This indicates that the column is too small. We shall increase the width of the column to 18 in. The distance from *A* to *B* increases to 25.4 in.

6. Recompute *a* for the New Depth of Column—Second Model—Corbel. Summing moments about *D* gives

$$\Sigma M_D = 0 = 134 \times 25.4 + 26.8 \times 18 - P_{u,C-E} \times (15.5 - a/2)$$

$$0 = 317{,}000 - 1265P_{u,C-E} + (P_{u,C-E})^2$$

which gives $P_{u,C-E} = 345$ kips. Thus, $P_{u,C-E}$ drops from 464 kips to 345 kips, corresponding to $a = 8.45$ in., and $a/2 = 4.23$ in. The distance *C–D* is 15.5 in. − 4.23 in. = 11.3 in.

7. Solve for Forces in the Struts and Ties—Second Model—Corbel.

Node *A*: Figure 17-40a shows the forces acting at node *A*.

Strut A—C has a horizontal projection of $25.4 - 11.3 = 14.1$ in. and a vertical projection of 18 in. It supports a factored vertical force component of 134 kips, a horizontal force component of

$$\frac{14.1}{18} \times 134 \text{ kips} = 105 \text{ kips}$$

and an axial force of $\sqrt{134^2 + 105^2} = 170$ kips compression. The angle between *A–C* and the tie *A–B* is $\arctan(18/14.1) = 51.9°$.

Tie A–B: Summing horizontal forces at node *A* gives $P_{u,A-B} = 105 + 26.8 = 132$ kips tension.

Node *B*: Figure 17-40b shows the forces acting at node *B*.

Strut B–C: Member *B–C* has a vertical projection of 18 in. and a horizontal projection of 15.5 in. − $a/2$, where $a/2 = 4.23$ in., making the horizontal projection of *B–C* 11.3 in.

Columns

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Introduction:

- Columns are members that carry loads mainly in compression.
- Usually columns carry bending moments as well, about one or both axes of the cross section, and the bending action may produce tensile forces over a part of the cross section.
- Even in such cases, columns are generally referred to as compression members, because the compression forces dominate their behavior.
- Concrete columns can be roughly divided into the following three categories:

A. Short compression blocks or pedestals.
B. *Short reinforced concrete columns.*
C. Long or slender reinforced concrete columns.

Types of columns:

- Columns may be classified based on the following different categories:

I. The length.

II. The shape of the cross section.

III. Type of lateral reinforcement.

IV. The frame bracing

V. The materials

Based on loading, columns may be classified as follows:

a) Axially loaded columns, where loads are assumed acting at the center of the column section.

b) Eccentrically loaded columns, where loads are acting at a distance (e) from the center of the column section.

- The distance *e* could be along the *x* or *y* axis, causing moments either about the *x* or *y* axis.

c) Bi-axially loaded columns, where the load is applied at any point on the column section, causing moments about both the *x* and *y* axes simultaneously.

<u>Based on length, columns may be classified as follows:</u>

a) Short columns, where the column’s failure is due to the crushing of concrete or the yielding of the steel bars under the full load capacity of the column.

b) Long columns, where buckling effect and slenderness ratio must be taken into consideration in the design, thus reducing the load capacity of the column relative to that of a short column.

Based on the shape of the cross section:

Column sections may be:

- square, rectangular, round, L-shaped, octagonal, or any desired shape with an adequate side width or dimensions.

Based on type of lateral reinforcement:

Columns may be classified as follows:

a) Tied columns containing steel ties to confine the main longitudinal bars in the columns.

b) Spiral columns containing spirals (spring-type reinforcement) to hold the main longitudinal reinforcement and to help increase the column ductility before failure.

Based on frame bracing:

Columns may be part of a frame that is braced against sidesway or unbraced against sidesway.

- Bracing may be achieved by using shear walls or bracings in the building frame.
- In braced frames, columns resist mainly gravity loads, and shear walls resist lateral loads and wind loads.
- In unbraced frames, columns resist both gravity and lateral loads, which reduces the load capacity of the columns.

Based on materials:

Columns may be reinforced, prestressed, composite (containing rolled steel sections such as I-sections), or a combination of rolled steel sections and reinforcing bars.

- Concrete columns reinforced with longitudinal reinforcing bars are the most common type used in concrete buildings.

Short compression blocks or pedestals:

- If the height of an upright compression member is less than three times its least lateral dimensions, it may be considered to be a pedestal.
- The ACI states that a pedestal may be designed with unreinforced or plain concrete with a maximum design compressive stress equal to $0.85\phi f'c$ where ϕ is 0.65.
- Should the total load applied to the member be larger than $0.85\phi f'c\ Ag$, it will be necessary either:

1. to enlarge the cross-sectional area of the pedestal, or
2. to design it as a reinforced concrete column.

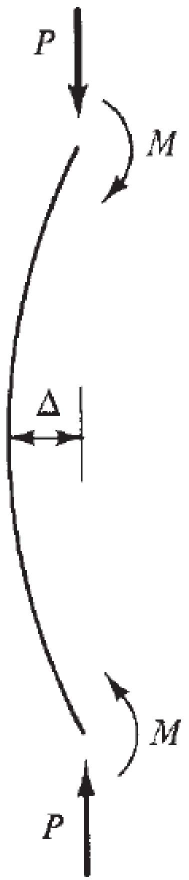
Short reinforced concrete columns:

- Should a reinforced concrete column fail due to initial material failure, it is classified as a short column.
- The load that it can support is controlled by the dimensions of the cross section and the strength of the materials of which it is constructed.

Long or slender reinforced concrete columns:

- As columns become more slender, bending deformations will increase, as will the resulting secondary moments.
- If these moments are of such magnitude as to significantly reduce the axial load capacities of columns, those columns are referred to as being *long* or *slender*.
- When a column is subjected to *primary moments* (those moments caused by applied loads, joint rotations, etc.), the axis of the member will deflect laterally.
- Then additional moments equal to the column load times the lateral deflection will be applied to the column.
- These latter moments are called *secondary moments* or ($P\Delta$) *moments*.

- A column that has large secondary moments is said to be a slender column, and it is necessary to size its cross section for the sum of both the primary and secondary moments.
- The ACI's intent is to permit columns to be designed as short columns if the secondary or (*PΔ*) effect does not reduce their strength by more than 5%.
- Effective slenderness ratios are used to classify columns as being short or slender.
- When the ratios are larger than certain values (depending on whether the columns are braced or unbraced laterally), they are classified as slender columns.



secondary moment = $P\Delta$

Axial Load Capacity of short Columns:

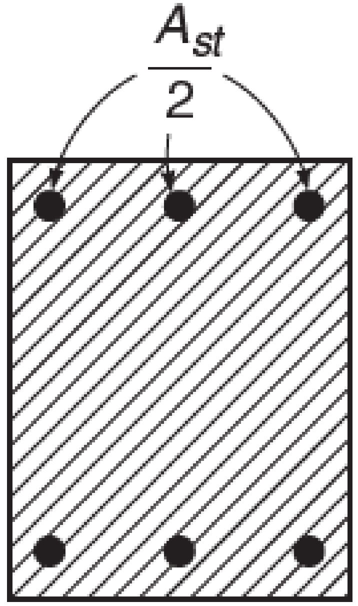
- At failure, the theoretical ultimate strength or nominal strength of a short axially loaded column is quite accurately determined by the expression:

$$P_n = 0.85f'_c(A_g - A_{st}) + f_y A_{st}$$

in which:

A_g is the gross concrete area and

A_{st} is the total cross-sectional area of longitudinal reinforcement, including bars and steel shapes.



Actual section

- According to ACI Code 22.4.2, the *design strength* of an axially loaded column is to be found with the introduction of certain strength reduction factors.
- The ACI factors are lower for columns than for beams, reflecting their greater importance in a structure.
- A beam failure would normally affect only a local region, whereas a column failure could result in the collapse of the entire structure.
- In addition, these factors reflect differences in the behavior of tied columns and spirally reinforced columns.
- A basic ϕ factor of 0.75 is used for spirally reinforced columns and 0.65 for tied columns, vs. $\phi = 0.90$ for most beams.

- A further limitation on column strength is imposed by ACI Code 22.4.2 to allow for accidental eccentricities of loading not considered in the analysis.
- This is done by imposing an upper limit on the axial load that is less than the calculated design strength.
- This upper limit is taken as 0.85 times the design strength for spirally reinforced columns and 0.80 times the calculated strength for tied columns.
- Thus, according to ACI Code 22.4.2, for spirally reinforced columns

$$\phi P_{n,\max} = 0.85\phi[0.85f_c'(A_g - A_{st}) + f_y A_{st}]$$

with $\phi = 0.75$. For tied columns

$$\phi P_{n,\max} = 0.80\phi[0.85f_c'(A_g - A_{st}) + f_y A_{st}]$$

with $\phi = 0.65$.

Example 9.3

Design an axially loaded short square tied column for $P_u = 2600$ kN if $f'_c = 28$ MPa and $f_y = 350$ MPa. Initially assume $\rho = 0.02$.

SOLUTION

Selecting Column Dimensions

$$\phi P_n = \phi 0.80[0.85 f'_c(A_g - A_{st}) + f_y A_{st}] \qquad \text{(ACI Equation 10-2)}$$

$$2600 \text{ kN} = (0.65)(0.80)[(0.85)(28 \text{ MPa})(A_g - 0.02A_g) + (350 \text{ MPa})(0.02A_g)]$$

$$A_g = 164\ 886 \text{ mm}^2$$

Use 400 mm × 400 mm ($A_g = 160\ 000$ mm^2)

Selecting Longitudinal Bars

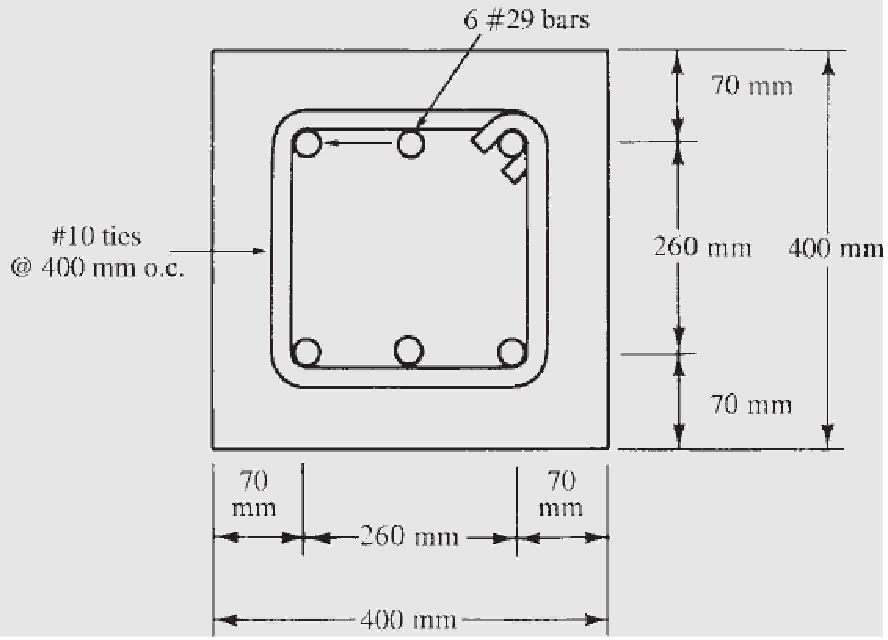
$$2600 \text{ kN} = (0.65)(0.80)[(0.85)(28 \text{ MPa})(160\ 000 \text{ mm}^2 - A_{st}) + (350 \text{ MPa})A_{st}]$$

$$A_{st} = 3654 \text{ mm}^2$$

Use 6 #29 (3870 mm^2)

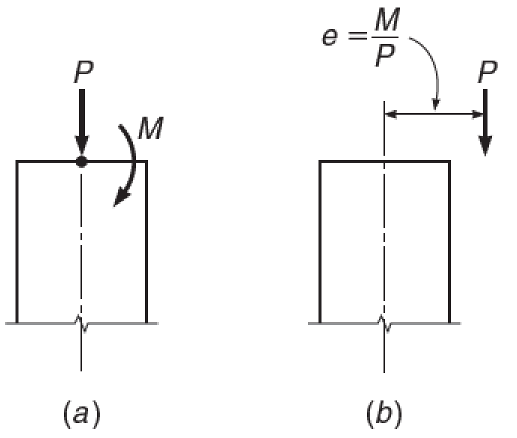
Design of Ties (Assuming #10 SI Ties)

(a) 16 mm × 28.7 mm = 459.2 mm
(b) 48 mm × 9.5 mm = 456 mm
(c) Least col. dim. = 400 mm ← Use #10 ties @ 400 mm



COMPRESSION PLUS BENDING OF RECTANGULAR COLUMNS:

- When a member is subjected to combined axial compression *P* and moment *M*, it is usually convenient to replace the axial load and moment with an equal load *P* applied at eccentricity $e = M/P$.
- All columns may then be classified in terms of the equivalent eccentricity.



a) Those having relatively small *eccentricity:*

- are generally characterized by compression over the entire concrete section,
- and if overloaded, will fail by crushing of the concrete accompanied by yielding of the steel in compression on the more heavily loaded side.

b) Columns with large eccentricity:

- are subject to tension over at least a part of the section,
- and if overloaded, may fail due to tensile yielding of the steel on the side farthest from the load.

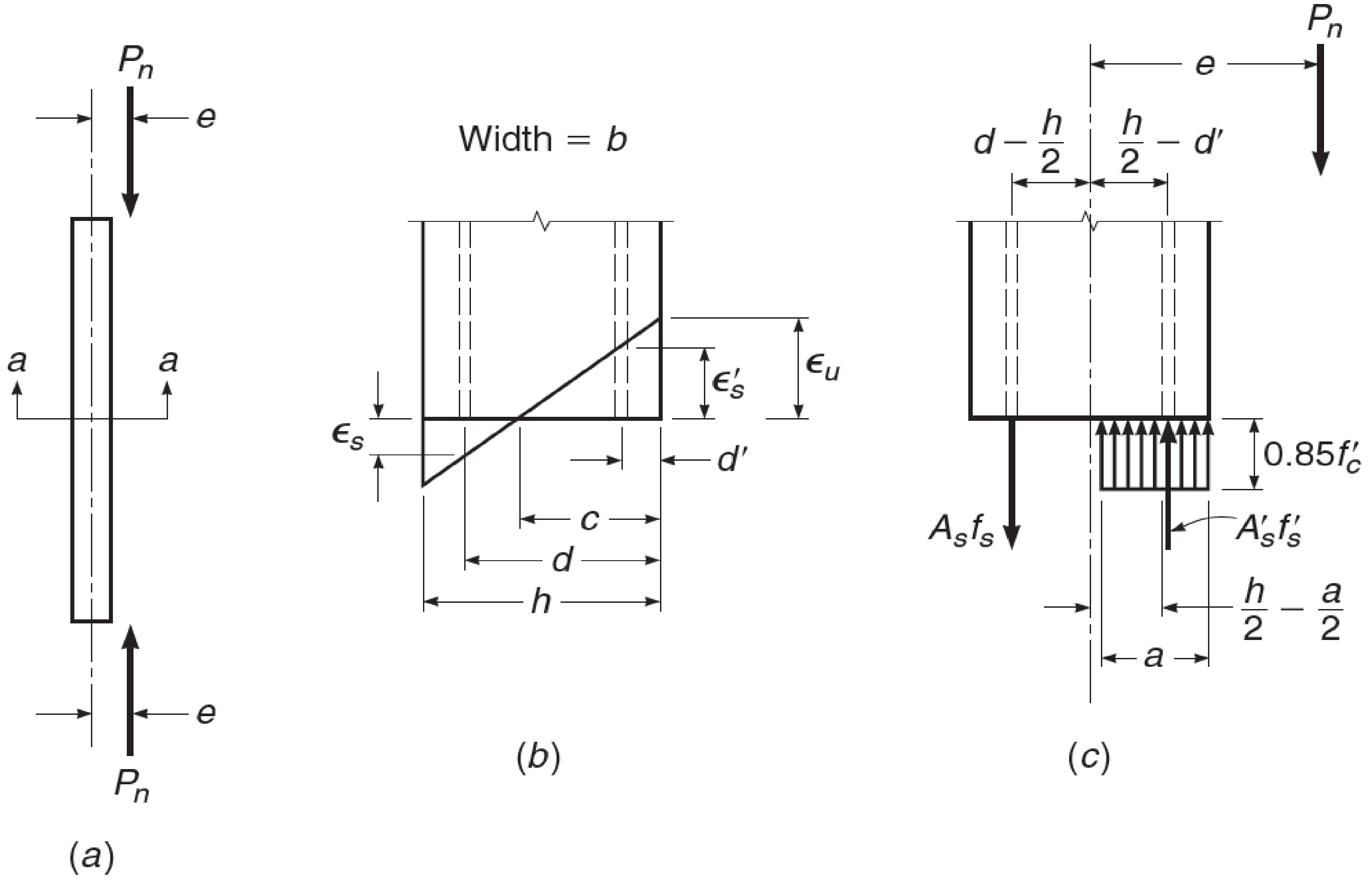
- For columns, load stages below the ultimate are generally not important.
- Cracking of concrete, even for columns with large eccentricity, is usually not a serious problem, and lateral deflections at service load levels are seldom, if ever, a factor.
- Design of columns is therefore based on the factored load, which must not exceed the design strength, that is,

$$\phi M_n \geq M_u$$

$$\phi P_n \geq P_u$$

STRAIN COMPATIBILITY ANALYSIS AND INTERACTION DIAGRAMS:

FIGURE 9.9
Column subject to eccentric compression: (*a*) loaded column; (*b*) strain distribution at section *a*-*a*; and (*c*) stresses and forces at nominal strength.



- Equilibrium between external and internal axial forces shown requires that:

$$P_n = 0.85 f_c' \, ab + f_s' A_s' - f_s A_s \tag{9.7}$$

Also, the moment about the centerline of the section of the internal stresses and forces must be equal and opposite to the moment of the external force P_n, so that

$$M_n = P_n e = 0.85 f_c' \, ab \left(\frac{h}{2} - \frac{a}{2}\right) + f_s' A_s' \left(\frac{h}{2} - d'\right) + f_s A_s \left(d - \frac{h}{2}\right) \tag{9.8}$$

- These are the two basic equilibrium relations for rectangular eccentrically compressed members.

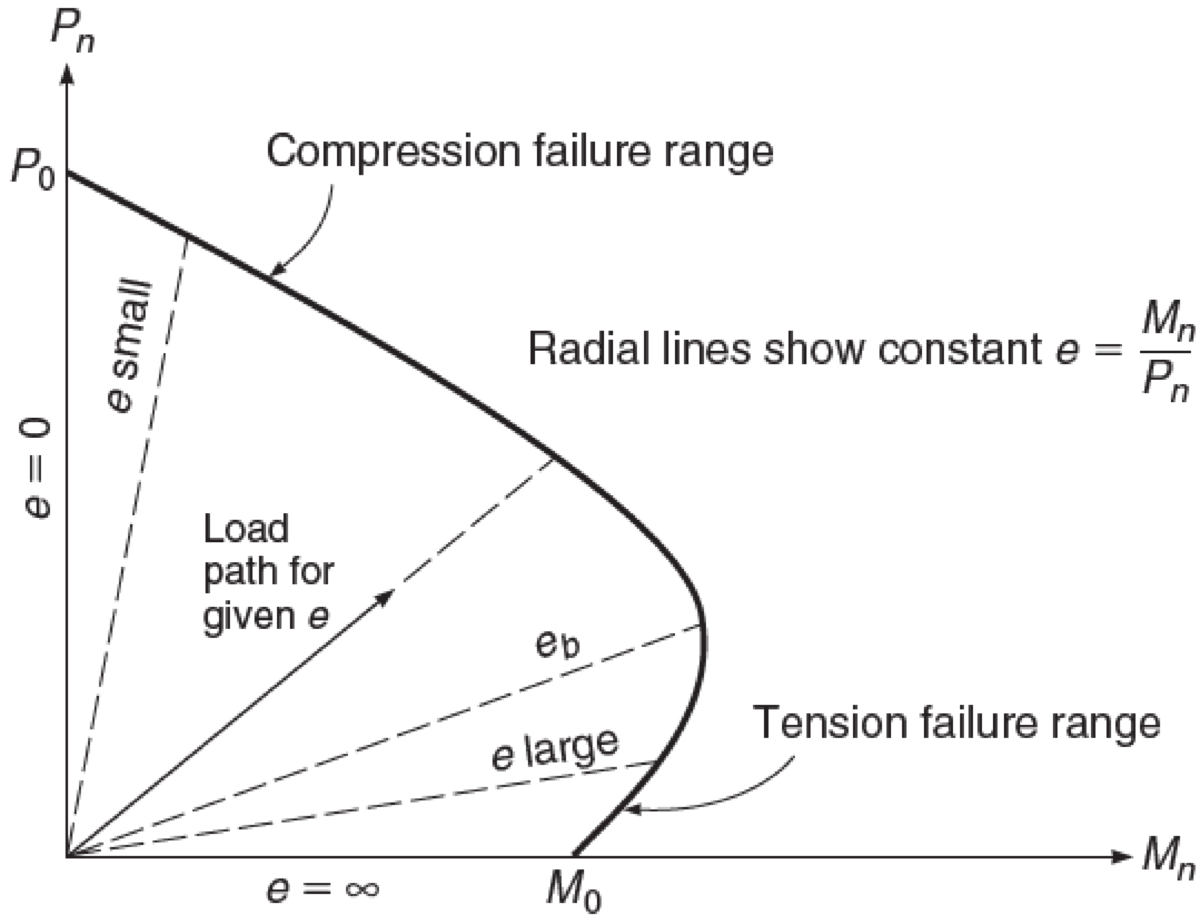
The fact that the presence of the compression reinforcement A'_s has displaced a corresponding amount of concrete of area A'_s is neglected in writing these equations. If necessary, particularly for large reinforcement ratios, one can account for this very simply. Evidently, in the above equations, a nonexistent concrete compression force of amount A'_s $(0.85f'_c)$ has been included as acting in the displaced concrete at the level of the compression steel. This excess force can be removed in both equations by multiplying A'_s by $f'_s - 0.85f'_c$ rather than by f'_s.

For large eccentricities, failure is initiated by yielding of the tension steel A_s. Hence, for this case, $f_s = f_y$. When the concrete reaches its ultimate strain ϵ_u, the compression steel may or may not have yielded; this must be determined based on compatibility of strains. For small eccentricities, the concrete will reach its limit strain ϵ_u before the tension steel starts yielding; in fact, the bars on the side of the column farther from the load may be in compression, not tension. For small eccentricities, too, the analysis must be based on compatibility of strains between the steel and the adjacent concrete.

For a given eccentricity determined from the frame analysis (that is, $e = M_u/P_u$) it is possible to solve Eqs. (9.7) and (9.8) for the load P_n and moment M_n that would result in failure as follows. In both equations, f'_s, f_s, and a can be expressed in terms of a single unknown c, the distance to the neutral axis. This is easily done based on the geometry of the strain diagram, with ϵ_u taken equal to 0.003 as usual, and using the stress-strain curve of the reinforcement. The result is that the two equations contain only two unknowns, P_n and c, and can be solved for those values simultaneously. However, to do so in practice would be complicated algebraically, particularly because of the need to incorporate the limit f_y on both f'_s and f_s.

- A better approach, providing the basis for practical design, is to construct a *strength interaction diagram* defining the failure load and failure moment for a given column for the full range of eccentricities from zero to infinity.
- For any eccentricity, there is a unique pair of values of P_n and M_n that will produce the state of incipient failure.

- That pair of values can be plotted as a point on a graph relating Pn and Mn.
- A series of such calculations, each corresponding to a different eccentricity, will result in a curve.



Interaction diagram for nominal column strength in combined bending and axial load.

- On such a diagram, any radial line represents a particular eccentricity $e = M/P$.
- For that eccentricity, gradually increasing the load will define a load path as shown, and when that load path reaches the limit curve, failure will result.
- Note that the vertical axis corresponds to $e = 0$, and P_0 is the capacity of the column if concentrically loaded.
- The horizontal axis corresponds to an infinite value of e, that is, pure bending at moment capacity M_0 .
- Small eccentricities will produce failure governed by concrete compression, while large eccentricities give a failure triggered by yielding of the tension steel.

For a given column, selected for trial, the interaction diagram is most easily constructed by selecting successive choices of neutral axis distance c, from infinity (axial load with eccentricity 0) to a very small value found by trial to give $P_n = 0$ (pure bending). For each selected value of c, the steel strains and stresses and the concrete force are easily calculated as follows. For the tension steel,

$$\epsilon_s = \epsilon_u \frac{d - c}{c} \tag{9.9}$$

$$f_s = \epsilon_u E_s \frac{d - c}{c} \le f_y \tag{9.10}$$

while for the compression steel,

$$\epsilon_s' = \epsilon_u \frac{c - d'}{c} \tag{9.11}$$

$$f_s' = \epsilon_u E_s \frac{c - d'}{c} \le f_y \tag{9.12}$$

The concrete stress block has depth

$$a = \beta_1 c \leq h \tag{9.13}$$

and consequently the concrete compressive resultant is

$$C = 0.85 f_c' \, ab \tag{9.14}$$

- The nominal axial force Pn and nominal moment Mn corresponding to the selected neutral axis location can then be calculated from Eqs. (9.7) and (9.8), respectively, and thus a single point on the strength interaction diagram is established.
- The calculations are then repeated for successive choices of neutral axis to establish the curve defining the strength limits.

BALANCED FAILURE:

As already noted, the interaction curve is divided into a compression failure range and a tension failure range.† It is useful to define what is termed a *balanced failure mode* and corresponding eccentricity e_b with the load P_b and moment M_b acting in combination to produce failure, with the concrete reaching its limiting strain ϵ_u at precisely the same instant that the tensile steel on the far side of the column reaches yield strain. This point on the interaction diagram is the dividing point between compression failure (small eccentricities) and tension failure (large eccentricities).

The values of P_b and M_b are easily computed with reference to Fig. 9.9. For balanced failure,

$$c = c_b = d \frac{\epsilon_u}{\epsilon_u + \epsilon_y} \tag{9.15}$$

and

$$a = a_b = \beta_1 c_b \tag{9.16}$$

Equations (9.9) through (9.14) are then used to obtain the steel stresses and the compressive resultant, after which P_b and M_b are found from Eqs. (9.7) and (9.8).

- Note that, in contrast to beam design, one cannot restrict column designs such that yielding failure rather than crushing failure would always be the result of overloading.
- The type of failure for a column depends on the value of eccentricity e, which in turn is defined by the load analysis of the structure.
- It is important to observe that in the region of compression failure the larger the axial load P_n , the smaller the moment M_n that the section is able to sustain before failing.
- However, in the region of tension failure, the reverse is true; the larger the axial load, the larger the simultaneous moment capacity.
- In the compression failure region, failure occurs through overstraining of the concrete.

EXAMPLE: Column strength interaction diagram. A 300 × 500 mm column is reinforced with four No. 9 (No. 29mm) bars of area 645 mm^2 each, one in each corner as shown. The concrete cylinder strength f_c'=28 MPa and the steel yield strength is 420 MPa . Determine:

a) the load P_b , moment M_b , and corresponding eccentricity e_b for balanced failure;

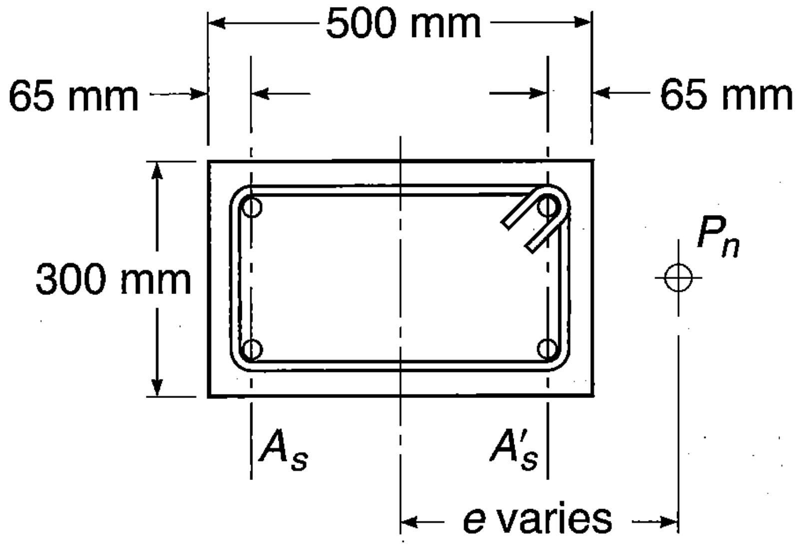
b) the load and moment for a representative point in the tension failure region of the interaction curve;

c) The load and moment for a representative point in the compression failure region;

d) the axial load strength for zero eccentricity.

Then (e) sketch the strength interaction diagram for this column.

Finally, (f) design the transverse reinforcement, based on ACI Code provisions.



Solution.

(*a*) The neutral axis for the balanced failure condition is easily found from Eq. (8.15) with $\epsilon_u = 0.003$ and $\epsilon_y = 420/200{,}000 = 0.0021$

$$c_b = 435 \times \frac{0.003}{0.0051} = 256 \text{ mm}$$

giving a stress-block depth $a = 0.85 \times 256 = 218$ mm. For the balanced failure condition, by definition, $f_s = f_y$. The compressive steel stress is found from Eq. (8.12):

$$f_s' = 0.003 \times 200{,}000 \frac{256 - 65}{256} = 448 \text{ MPa} \qquad \text{but} \qquad \leq 420 \text{ MPa}$$

confirming that the compression steel, too, is at the yield. The concrete compressive resultant is

$$C = 0.85 \times 28 \times 218 \times 300 \times 10^{-3} = 1557 \text{ kN}$$

The balanced load P_b is then found from Eq. (8.7) to be

$$P_b = 1557 + 1290 \times 420 \times 10^{-3} - 1290 \times 420 \times 10^{-3} = 1557 \text{ kN}$$

and the balanced moment from Eq. (8.8) is

$$M_b = 1557(250 - 109) + 1290 \times 420(250 - 65) + 1290 \times 420(435 - 250)$$
$$= 420 \times 10^3 \text{ kN-mm} = 420 \text{ kN-m}$$

The corresponding eccentricity of load is $e_b = 241$ mm.

(*b*) Any choice of c *smaller than* $c_b = 256$ mm will give a point in the tension failure region of the interaction curve, with eccentricity larger than e_b. For example, choose $c = 125$ mm. By definition, $f_s = f_y$. The compressive steel stress is found to be

$$f_s' = 0.003 \times 200{,}000 \frac{125 - 65}{125} = 288 \text{ MPa}$$

With the stress-block depth $a = 0.85 \times 125 = 106$, the compressive resultant is $C = 0.85 \times 28 \times 106 \times 300 \times 10^{-3} = 757$ kN. Then from Eq. (8.7), the thrust is

$$P_n = 757 + 1290 \times 288 \times 10^{-3} - 1290 \times 420 \times 10^{-3}$$
$$= 587 \text{ kN}$$

and the moment capacity from Eq. (8.8) is

$$M_n = 757(250 - 53) + 1290 \times 288(250 - 65) + 1290 \times 420(435 - 250)$$
$$= 318 \times 10^3 \text{ kN-mm} = 318 \text{ kN-m}$$

giving eccentricity $e = 318{,}000/587 = 542$ mm, well above the balanced value.

(*c*) Now selecting a c value *larger than* c_b to demonstrate a compression failure point on the interaction curve, choose $c = 460$ mm, for which $a = 0.85 \times 460 = 391$ mm. The compressive concrete resultant is $C = 0.85 \times 28 \times 391 \times 300 \times 10^{-3} = 2792$ kN. From Eq. (8.10) the stress in the steel at the left side of the column is

$$f_s = 0.003 \times 200{,}000 \frac{435 - 460}{460} = -33 \text{ MPa}$$

Note that the negative value of f_s indicates correctly that A_s is in compression if c is greater than d, as in the present case. The compressive steel stress is found from Eq. (8.12) to be

$$f_s' = 0.003 \times 200{,}000 \frac{460 - 65}{460} = 515 \text{ MPa} \quad \text{but} \quad \leq 420 \text{ MPa}$$

Then the column capacity is

$$P_n = 2792 + 1290 \times 420 \times 10^{-3} + 1290 \times 33 \times 10^{-3} = 3376 \text{ kN}$$
$$M_n = 2792(250 - 196) + 1290 \times 420(250 - 65) - 1290 \times 33(435 - 250)$$

(*d*) The axial strength of the column if concentrically loaded corresponds to $c = \infty$ and $e = 0$. For this case,

$$P_n = 0.85 \times 28 \times 300 \times 500 \times 10^{-3} + 2580 \times 420 \times 10^{-3} = 4654 \text{ kN}$$

Note that, for this as well as the preceding calculations, subtraction of the concrete displaced by the steel has been neglected. For comparison, if the deduction were made in the last calculation,

$$P_n = 0.85 \times 28(300 \times 500 - 2580) \times 10^{-3} + (2580 \times 420) \times 10^{-3} = 4592 \text{ kN}$$

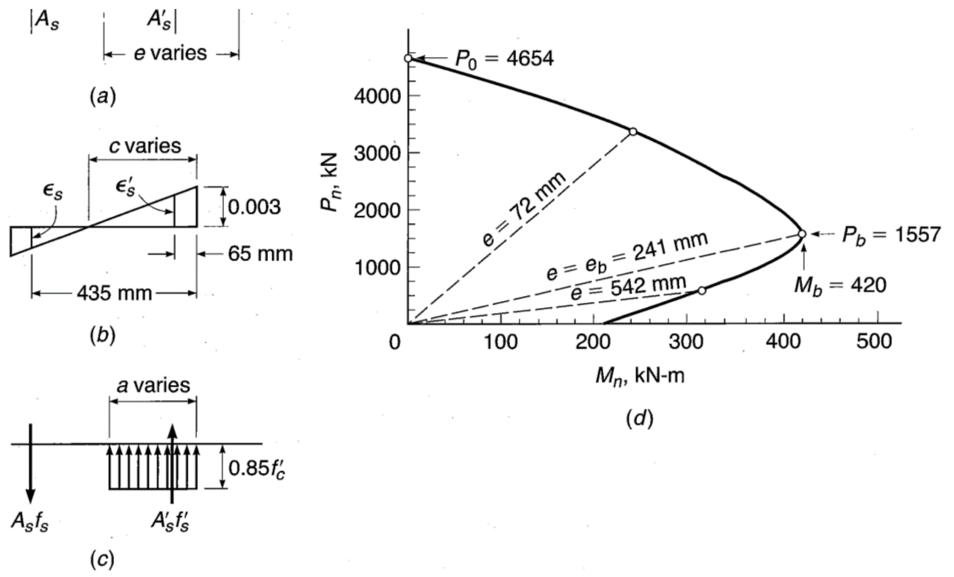
The error in neglecting this deduction is only 1 percent in this case; the difference generally can be neglected, except perhaps for columns with reinforcement ratios close to the maximum of 8 percent. In the case of design aids, however, such as those presented in Refs. 8.2 and 8.7 and discussed in Section 8.10, the deduction is usually included for all reinforcement ratios.

(*e*) From the calculations just completed, plus similar repetitive calculations that will not be given here, the strength interaction curve of Fig. 8.11*d* is constructed. Note the characteristic shape, described earlier, the location of the balanced failure point as well as the "small eccentricity" and "large eccentricity" points just found, and the axial load capacity. In the process of developing a strength interaction curve, it is possible to select the values of steel strain ϵ_s, as done in step *a*, for use in steps *b* and *c*. Selecting ϵ_s uniquely establishes the neutral axis depth *c*, as shown by Eqs. (8.9) and (8.15), and is useful in determining M_n and P_n for values of steel strain that correspond to changes in the strength reduction factor ϕ, as will be discussed in Section 8.9.

(*f*) The design of the column ties will be carried out following the ACI Code restrictions. For the minimum permitted tie, a No. 10 (No. 3) bar having a diameter of 9.5 mm, used with No. 29 (No. 9) longitudinal bars having a diameter of 28.7 mm in a column the least dimension of which is 300 mm, the tie spacing is not to exceed

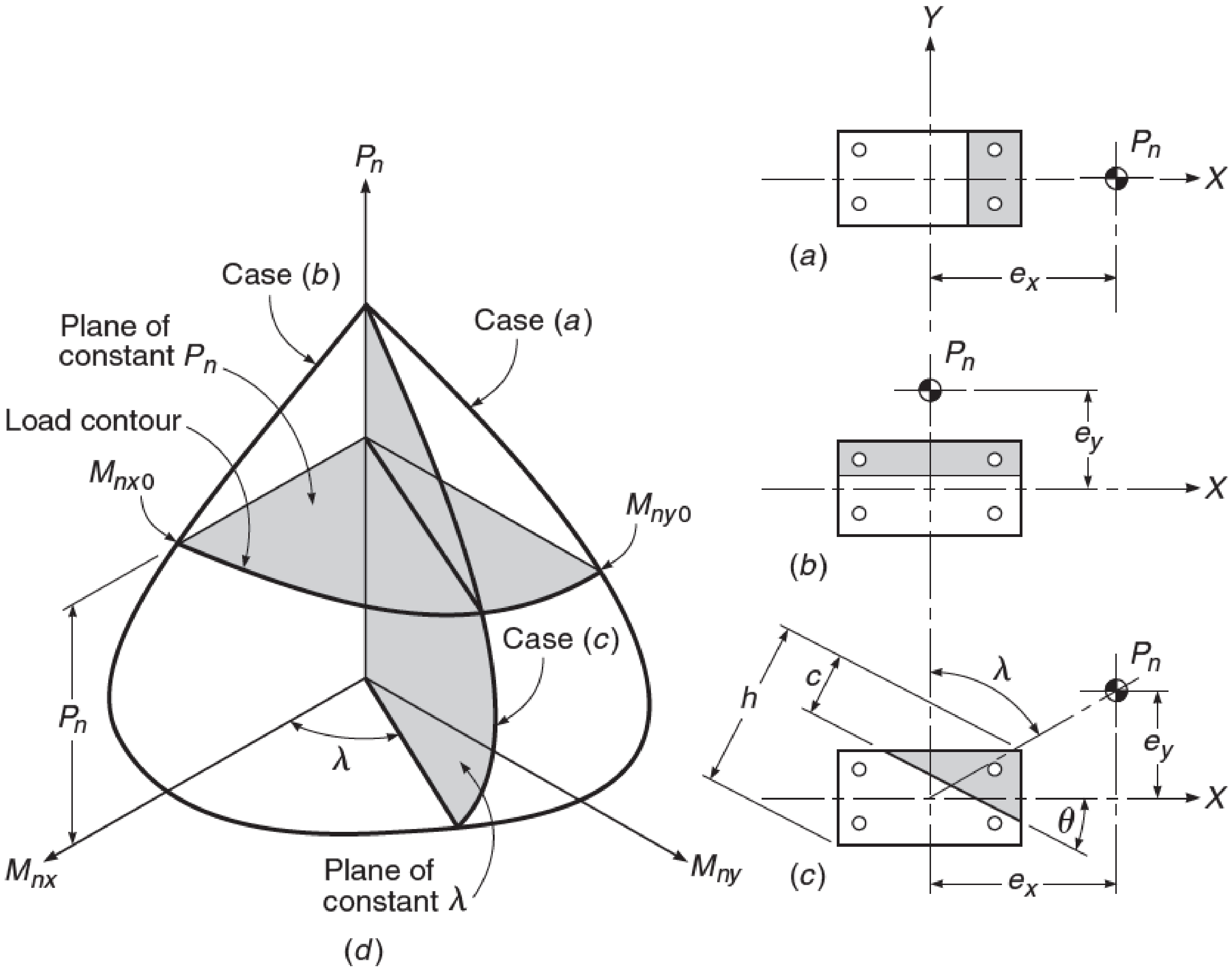
$$48 \times 9.5 = 456 \text{ mm}$$
$$16 \times 28.7 = 459 \text{ mm}$$
$$b = 300 \text{ mm}$$

The last restriction controls in this case, and No. 10 (No. 3) ties will be used at 300 mm spacing, detailed as shown in Fig. 8.11*a*. Note that the permitted spacing as controlled by the first and second criteria, 456 mm, must be reduced because of the 300 mm column dimension.



BIAXIAL BENDING:

FIGURE 9.16
Interaction diagram for compression plus biaxial bending: (*a*) uniaxial bending about *Y* axis; (*b*) uniaxial bending about *X* axis; (*c*) biaxial bending about diagonal axis; and (*d*) interaction surface.



Slender Columns:

- When a column bends or deflects laterally an amount Δ, its axial load will cause an increased column moment equal to $P\,\Delta$.
- This moment will be superimposed onto any moments already in the column.
- Should this $P\,\Delta$ moment be of such magnitude as to reduce the axial load capacity of the column significantly, the column will be referred to as a *slender column*.

A column is said to be *slender* if its cross-sectional dimensions are small compared with its length. The degree of slenderness is generally expressed in terms of the slenderness ratio ℓ/r, where ℓ is the unsupported length of the member and r is the radius of gyration of its cross section, equal to $\sqrt{I/A}$. For square or circular members, the value of r is the same about either axis; for other shapes, r is smallest about the minor principal axis, and it is generally this value that must be used in determining the slenderness ratio of a freestanding column.

- It has long been known that a member of great slenderness will collapse under a smaller compression load than a stocky member with the same cross-sectional dimensions.
- When a stocky member, say with $\ell/r = 10$ (such as, a square column of length equal to about 3 times its cross-sectional dimension h), is loaded in axial compression, it will fail at the load given by Eq. (9.3), because at that load both concrete and steel are stressed to their maximum carrying capacity and give way, respectively, by crushing and by yielding.
- If a member with the same cross section has a slenderness ratio $\ell/r = 100$ (such as, a square column hinged at both ends and of length equal to about 30 times its section dimension), it may fail under an axial load equal to one-half or less of that given by Eq. (9.3).
- In this case, collapse is caused by buckling, that is, by sudden lateral displacement of the member between its ends, with consequent overstressing of steel and concrete by the bending stresses that are superimposed on the axial compressive stresses.

- Most columns in practice are subjected to bending moments as well as axial loads.
- These moments produce:

a) lateral deflection of a member between its ends, and

b) may also result in relative lateral displacement of joints.

- Associated with these lateral displacements are *secondary moments* that add to the primary moments and that may become very large for slender columns, leading to failure.
- A practical definition of a slender column is one for which there is a significant reduction in axial load capacity because of these secondary moments.
- In the development of ACI Code column provisions, for example, any reduction greater than about 5 percent is considered significant, requiring consideration of slenderness effects.

- The ACI Code and Commentary contain detailed provisions governing the design of slender columns.
- ACI Code presents approximate methods for accounting for slenderness through the use of ***moment magnification factors***.
- The provisions are quite similar to those used for many years for steel columns designed under the American Institute of Steel Construction (AISC) Specification.
- Alternatively, in ACI Code 6.7 and 6.8, a more fundamental approach is endorsed, in which the effect of lateral displacements is accounted for directly in the frame analysis.
- The latter approach, known as second-order analysis, is often incorporated as a feature in commercially available structural analysis software.

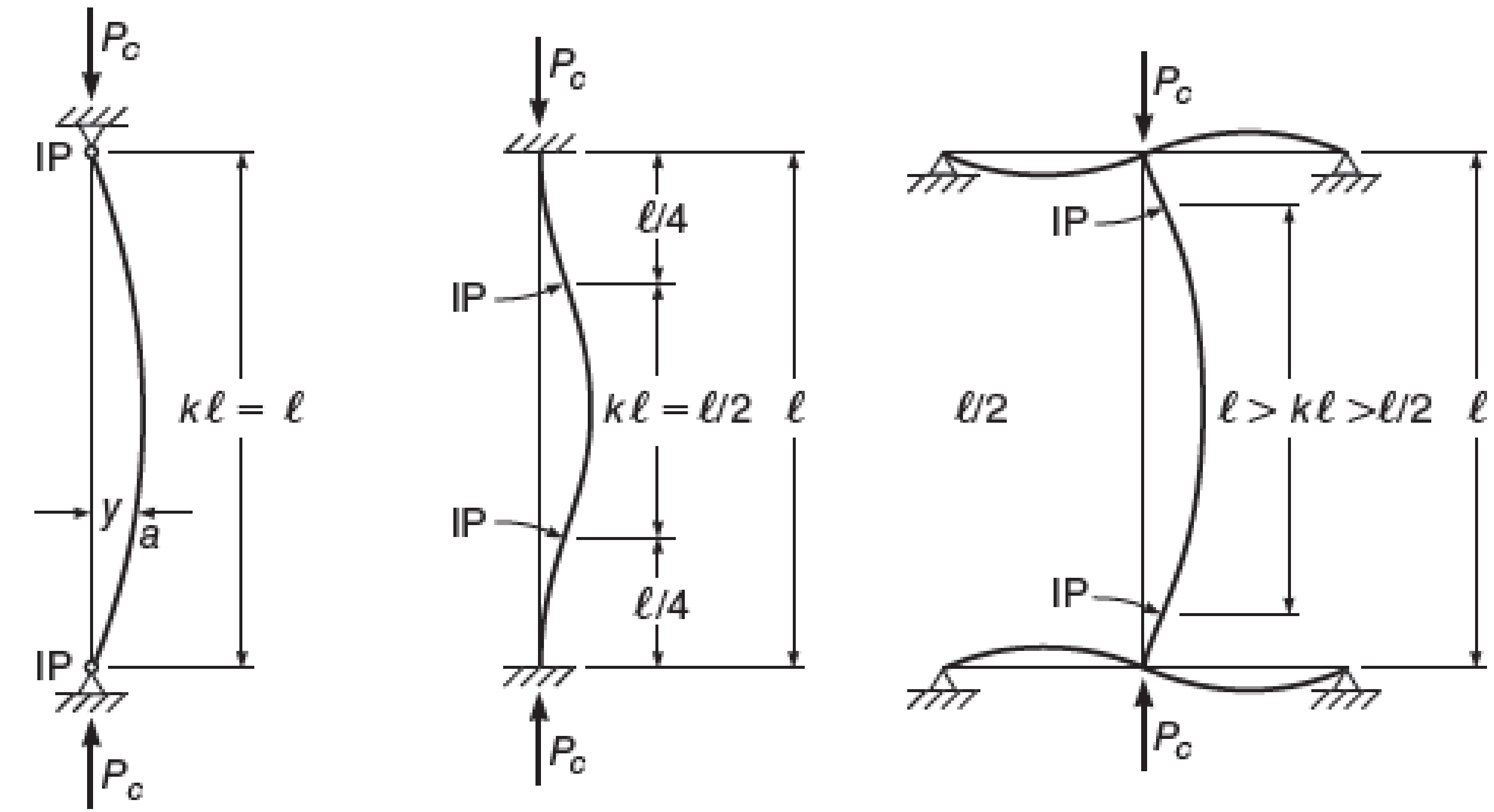
CONCENTRICALLY LOADED COLUMNS:

- The basic information on the behavior of straight, concentrically loaded slender columns was developed by Euler more than 250 years ago.
- In generalized form, it states that such a member will fail by buckling at the critical load

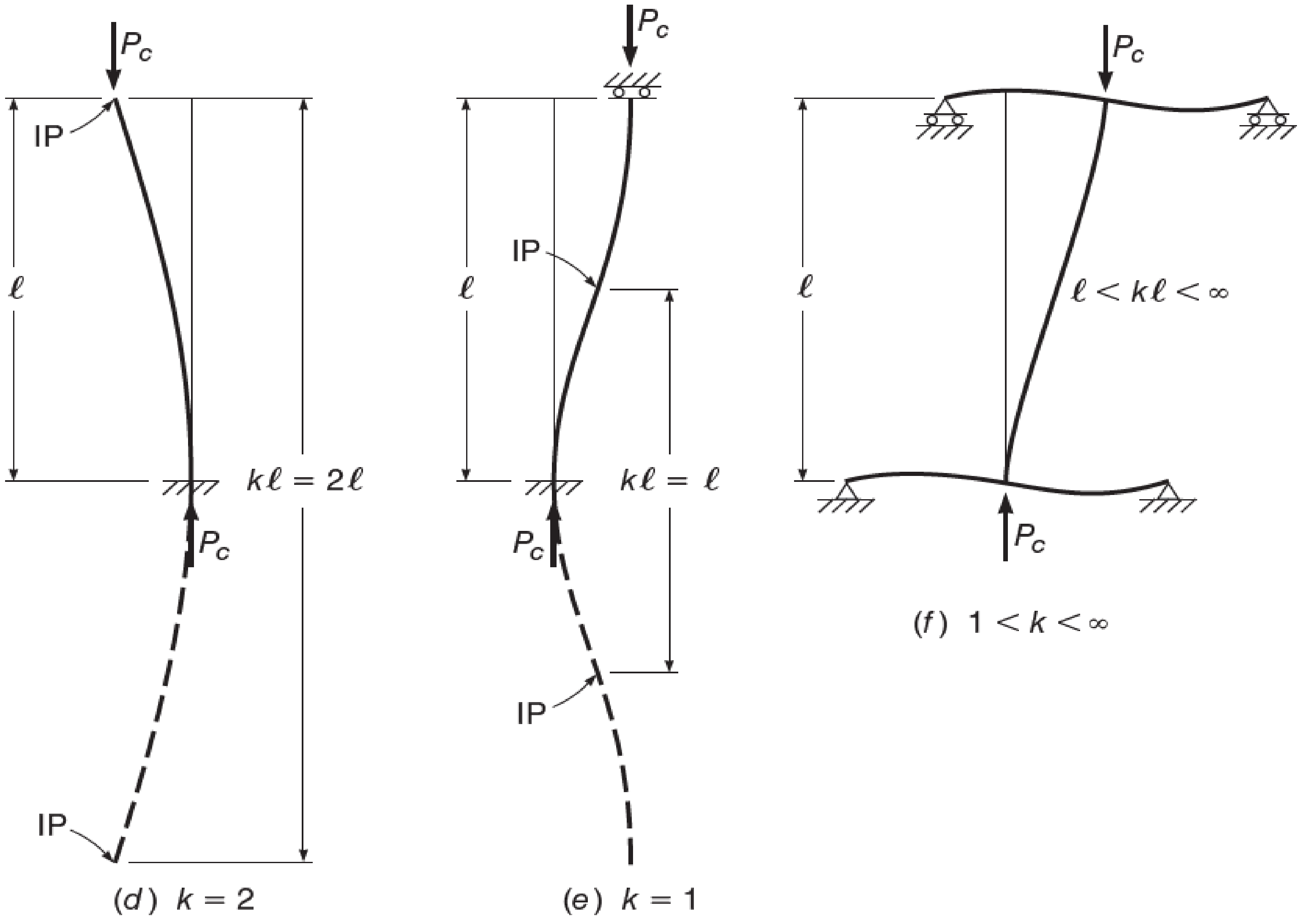
$$P_c = \frac{\pi^2 E_t I}{(k\ell)^2} \qquad (10.1)$$

- It is seen that the buckling load decreases rapidly with increasing slenderness ratio $k\ell/r$.
- At the load given by Eq. (10.1), the originally straight member buckles into a half sine wave, as shown in Fig. 10.1 a.
- In this bent configuration, bending moments *P.y* act at any section such as a; y is the deflection at that section.
- These deflections continue to increase until the bending stress caused by the increasing moment, together with the original compression stress, overstresses and fails the member.

FIGURE 10.1
Buckling and effective length of axially loaded columns.



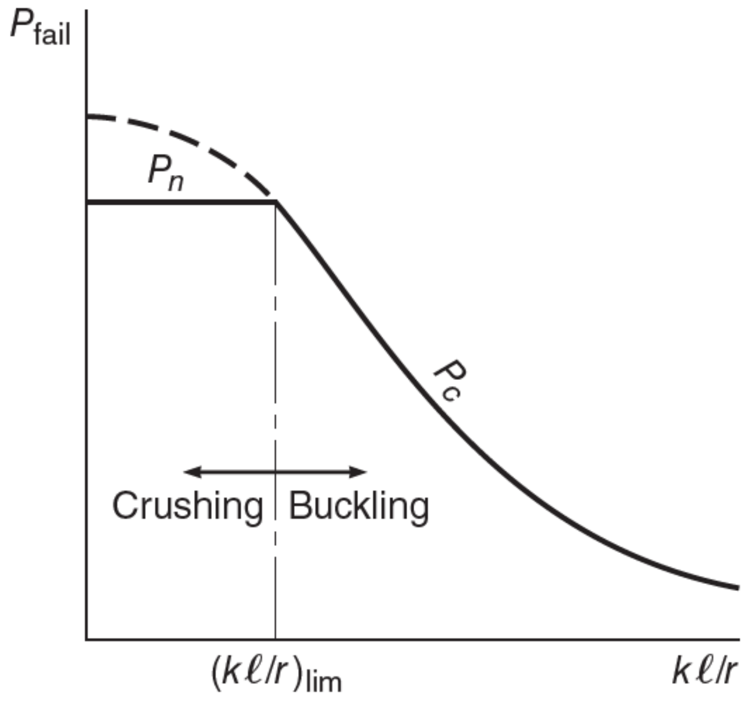
(a) $k = 1$ (b) $k = 1/2$ (c) $1/2 < k < 1$



P_c
IP
ℓ
$k\ell = 2\ell$
P_c
IP
(d) $k = 2$
P_c
ℓ
IP
$k\ell = \ell$
P_c
IP
(e) $k = 1$
P_c
ℓ
$\ell < k\ell < \infty$
P_c
(f) $1 < k < \infty$

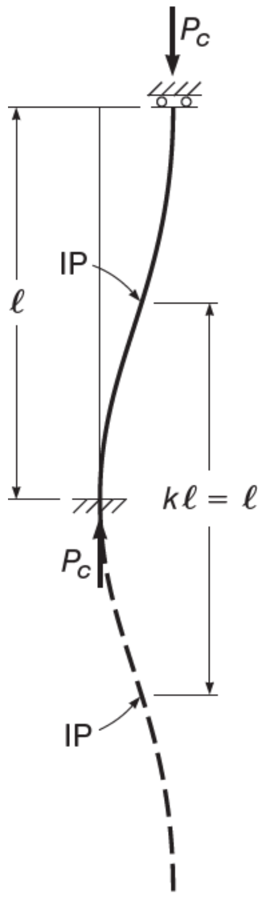
Effect of slenderness on failure load:

- A plot of the buckling load vs. the slenderness ratio, the so-called column curve, therefore has the shape shown.
- The curve shows the reduction in buckling strength with increasing slenderness.
- For very stocky columns, the value of the buckling load, calculated from Eq. (10.1), exceeds the direct crushing strength of the stocky column Pn given by Eq. (9.3).
- Correspondingly, there is a limiting slenderness ratio $(k\ell/r)_{lim}$.
- For values smaller than this, failure occurs by simple crushing, regardless of $k\ell/r$; for values larger than $(k\ell/r)_{lim}$, failure occurs by buckling, the buckling load or stress decreasing for greater slenderness.



Effect of sidesway on buckling load:

- If the column is rotationally fixed at both ends but one end can move laterally with respect to the other, it buckles as shown, with an effective length $k\ell = \ell$.
- If comparing this column, fixed at both ends but free to sidesway, with a fixed-fixed column that is braced against sidesway, it can be noted that the effective length of the former is twice that of the latter.
- From Eq. (10.1), the buckling strength of an elastic fixed-fixed column that is free to sidesway is only one-quarter that of the same column when braced against sidesway.
- ***This is an illustration of the general fact that compression members free to buckle in a sidesway mode are always considerably weaker than when braced against sidesway.***



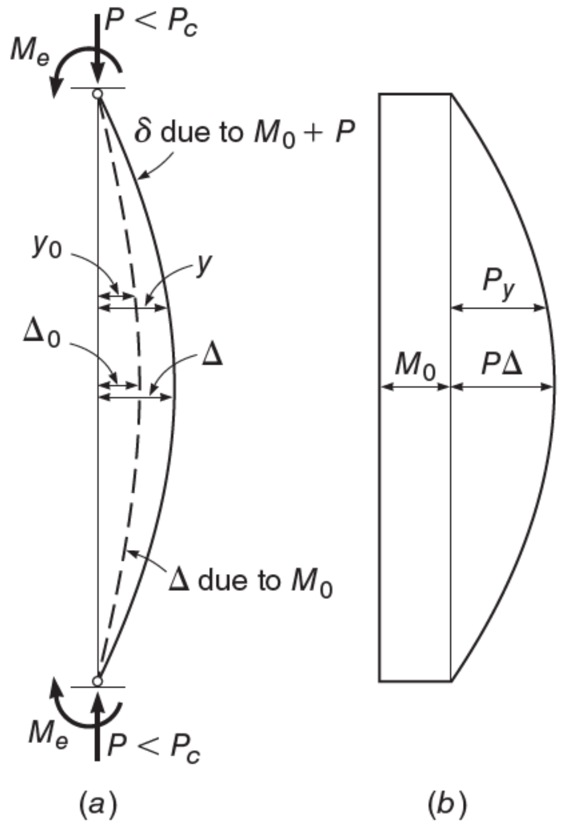
(e) $k = 1$

Notes on effect of sidesway:

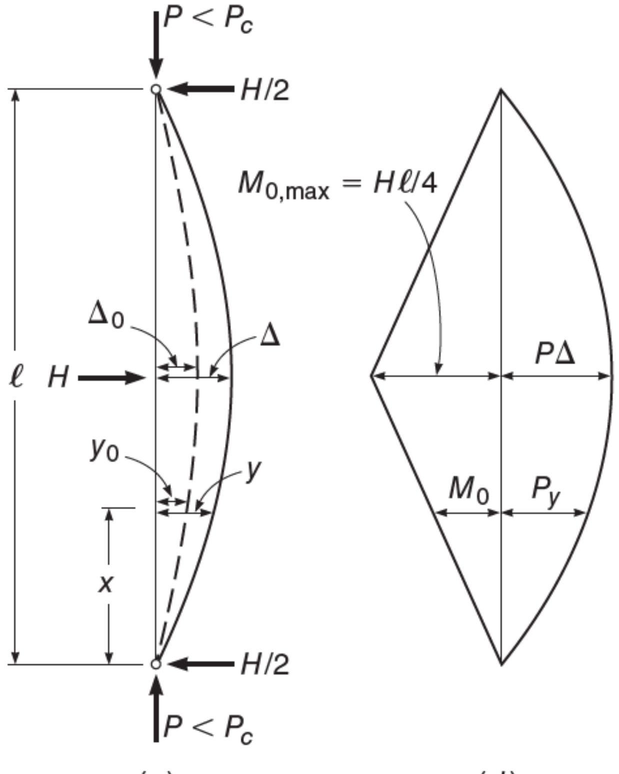
- The strength of concentrically loaded columns decreases with increasing slenderness ratio $k\ell/r$.
- In columns that are braced against sidesway or that are parts of frames braced against sidesway, the effective length $k\ell$, that is, the distance between inflection points, falls between $\ell/2$ and ℓ, depending on the degree of end restraint.
- The effective lengths of columns that are not braced against sidesway or that are parts of frames not so braced are always larger than ℓ, the more so the smaller the end restraint. In consequence, the buckling load of a frame not braced against sidesway is always substantially smaller than that of the same frame when braced.

COMPRESSION PLUS BENDING:

- Most reinforced concrete compression members are also subject to simultaneous flexure, caused by transverse loads or by end moments owing to continuity.
- The behavior of members subject to such combined loading also depends greatly on their slenderness.
- The member shown is axially loaded by P and bent by equal end moments Me.
- If no axial load were present, the moment M_0 in the member would be constant throughout and equal to the end moments Me . This is shown in Fig. b.
- In this situation, that is, in simple bending without axial compression, the member deflects as shown by the dashed curve of Fig. a, where y_0 represents the deflection at any point caused by bending only.



- When P is applied, the moment at any point increases by an amount equal to P times its lever arm.
- The increased moments cause additional deflections, so that the deflection curve under the simultaneous action of P and M_0 is the solid curve of Fig. a.
- At any point, then, the total moment is now: $M = M_0 + Py$ (10.2)
- That is, the total moment consists of the moment M_0 that acts in the presence of P and the additional moment caused by P, equal to P times the deflection.
- This is one illustration of the so-called P -Δ effect.
- A similar situation is shown in Fig. 10.4 c, where bending is caused by the transverse load H.
- When P is absent, the moment at any point x is $M_0 = \frac{Hx}{2}$ with a maximum value at midspan equal to $Hl/4$.
- The corresponding M_0 diagram is shown in Figure.



(c) (d)

- When P is applied, additional moments Py are caused again, distributed as shown, and the total moment at any point in the member consists of the same two parts as in Eq. (10.2).
- The deflections y of elastic columns of the type shown in previous figures can be calculated from the deflections y_0, that is, from the deflections of the corresponding beam without axial load, using the following expression.

$$y = y_0 \frac{1}{1 - P/P_c} \tag{10.3}$$

If Δ is the deflection at the point of maximum moment M_{max}, as shown in Fig. 10.4, M_{max} can be calculated using Eqs. (10.2) and (10.3).

$$M_{max} = M_0 + P\Delta = M_0 + P\Delta_0 \frac{1}{1 - P/P_c} \tag{10.4}$$

It can be shown (Ref. 10.2) that Eq. (10.4) can be written as

$$M_{max} = M_0 \frac{1 + \psi P/P_c}{1 - P/P_c} \tag{10.5}$$

where ψ is a coefficient that depends on the type of loading and varies between about ± 0.20 for most practical cases. Because P/P_c is always significantly smaller than 1, the second term in the numerator of Eq. (10.5) is small enough to be neglected. Doing so, one obtains the simplified design equation

$$M_{\max} = M_0 \frac{1}{1 - P/P_c} \tag{10.6}$$

where $1/(1 - P/P_c)$ is known as the *moment magnification factor*, which reflects the amount by which the moment M_0 is magnified by the presence of a simultaneous axial force P.

Since P_c decreases with increasing slenderness ratio, it is seen from Eq. (10.6) that the moment M in the member increases with the slenderness ratio $k\ell/r$. The situation is shown schematically in Fig. 10.5. It indicates that, for a given transverse loading (that is, a given value of M_0), an axial force P causes a larger additional moment in a slender member than in a stocky member.

In the two members in Fig. 10.4, the largest moment caused by P, namely $P\Delta$, adds directly to the maximum value of M_0; for example,

$$M_0 = \frac{H\ell}{4}$$

in Fig. 10.4d. As P increases, the maximum moment at midspan increases at a rate faster than that of P in the manner given by Eqs. (10.2) and (10.6) and shown in Fig. 10.6. The member will fail when the simultaneous values of P and M become equal to P_n and M_n, the nominal strength of the cross section at the location of maximum moment.

FIGURE 10.5
Effect of slenderness on column moments.

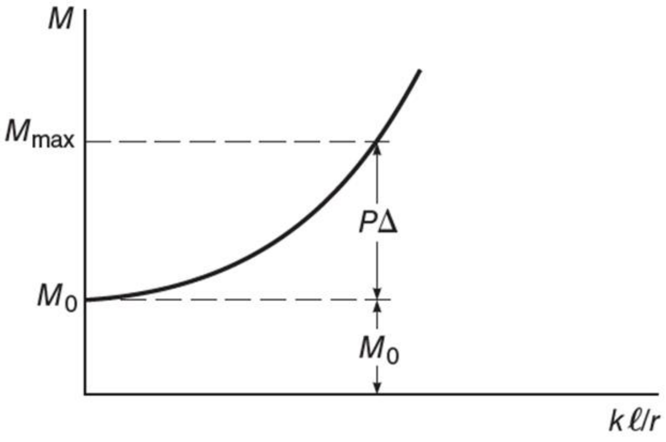
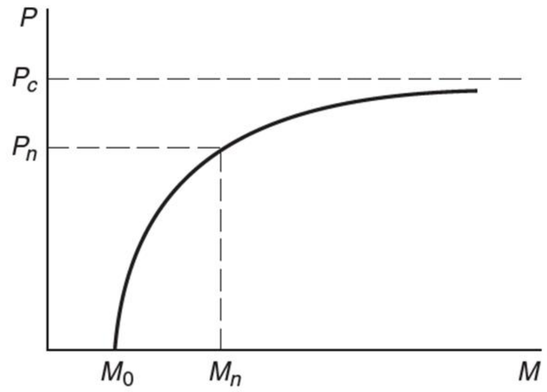
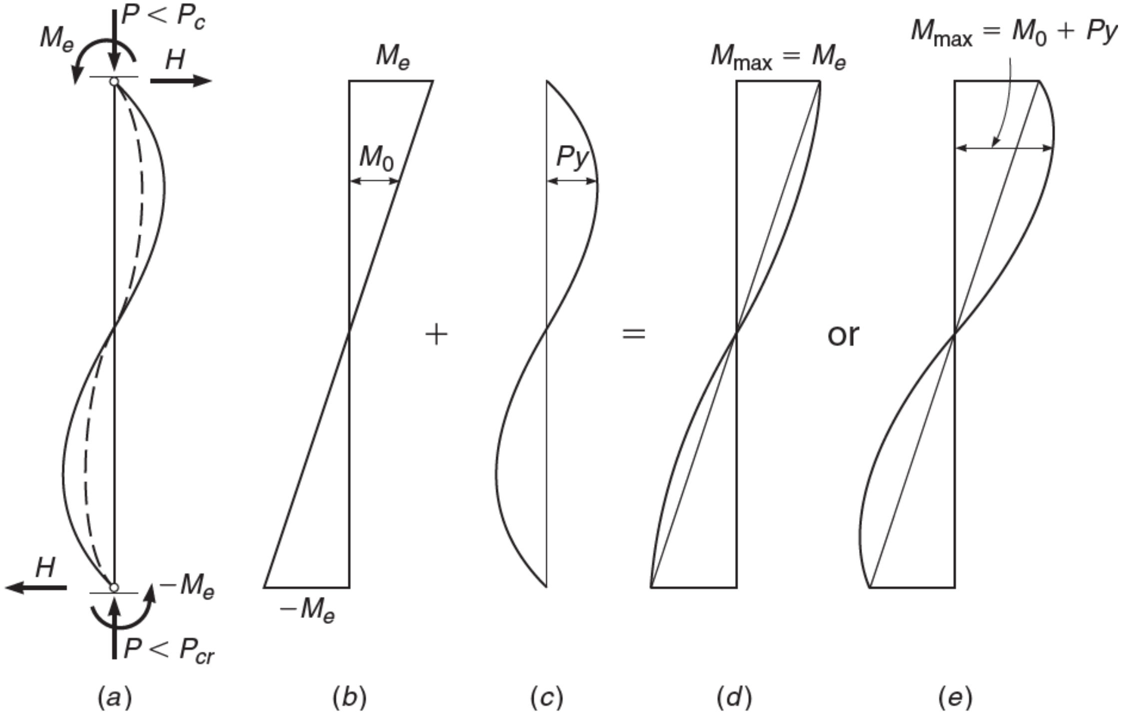


FIGURE 10.6
Effect of axial load on column moments.



- This direct addition of the maximum moment caused by *P* to the maximum moment caused by the transverse load, clearly the most unfavorable situation, does not result for all types of deformations.
- For instance, the member in Fig. 10.7 a, with equal and opposite end moments, has the M_0 diagram shown in Fig. 10.7 b.
- The deflections caused by M_0 alone are again magnified when an axial load P is applied.

FIGURE 10.7
Moments in slender members with compression plus bending, bent in double curvature.



- In this case, these deflections under simultaneous bending and compression can be approximated by:

$$y = y_0 \frac{1}{1 - P/4P_c} \tag{10.7}$$

- By comparison with Eq. (10.3), it is seen that the deflection magnification here is much smaller.

It can be shown (Ref. 10.2) that the way in which moment magnification depends on the relative magnitude of the two end moments (as in Figs. 10.4*a* and 10.7*a*) can be expressed by a modification of Eq. (10.6):

$$M_{\max} = M_0 \frac{C_m}{1 - P/P_c} \tag{10.8}$$

where

$$C_m = 0.6 - 0.4\,\frac{M_1}{M_2} \geq 0.4 \tag{10.9}$$

Here M_1 is the numerically smaller and M_2 the numerically larger of the two end moments; hence, by definition, $M_0 = M_2$. The fraction M_1/M_2 is defined as negative if the end moments produce single curvature and positive if they produce double curvature. It is seen that when $M_1 = M_2$, as in Fig. 10.4a, $C_m = 1$, so that Eq. (10.8) becomes Eq. (10.6), as it should. Note that Eq. (10.9) *applies only to members braced against sidesway*. As will become apparent from the discussion that follows, for members not braced against sidesway, maximum moment magnification usually occurs, that is, $C_m = 1$.

- Members that are braced against sidesway include columns that are parts of structures in which sidesway is prevented in one of various ways:

i. by walls sufficiently strong and rigid in their own planes to effectively prevent horizontal displacement,

ii. By special bracing in vertical planes, in buildings by designing the utility core to resist horizontal loads and furnish bracing to the frames, or

iii. by bracing the frame against some other essentially immovable support.

ACI CRITERIA FOR SLENDERNESS EFFECTS IN COLUMNS:

- The Code provisions are as follows:

  1. For compression members braced against sidesway (that is, in nonsway structures), the effects of slenderness may be neglected when $k\ell_u/r \leq 34 + 12M_1/M_2$ and $k\ell_u/r \leq 40$.
  2. For compression members not braced against sidesway (that is, in sway structures), the effects of slenderness may be neglected when $k\ell_u/r$ is less than 22.

In these provisions, k is the effective length factor (see Section 10.2); ℓ_u is the unsupported length, taken as the clear distance between floor slabs, beams, or other members providing lateral support; M_1 is the smaller factored end moment on the compression member; M_2 is the larger factored end moment on the compression member (if transverse loading occurs between supports, M_2 is the largest moment in member); and M_1/M_2 is negative if the member is bent in single curvature and positive if bent in double curvature.

ACI CRITERIA FOR NONSWAY VS. SWAY STRUCTURES:

- In practice, a structure is seldom either completely braced or completely unbraced.
- It is necessary, therefore, to determine in advance if bracing provided by shear walls, elevator and utility shafts, stairwells, or other elements is adequate to restrain the structure against significant sway effects.
- As suggested in ACI Commentary 6.6.4.1, a compression member can be assumed braced if it is located in a story in which the bracing elements (shear walls, etc.) have a stiffness substantial enough to limit lateral deflection to the extent that the column strength is not substantially affected.
- Such a determination can often be made by inspection.
- If not, ACI Code 6.6.4.3 provides two alternate criteria for determining if columns and stories are treated as nonsway or sway.
- To be considered as a nonsway or braced column, the first criterion requires that the increase in column end moments due to second-order effects not exceed 5 percent of the first-order end moments.

ACI MOMENT MAGNIFIER METHOD FOR NONSWAY FRAMES:

- A slender reinforced concrete column reaches the limit of its strength when the combination of P and M at the most highly stressed section causes that section to fail.
- In general, P is essentially constant along the length of the member.
- This means that the column approaches failure when, at the most highly stressed section, the axial force P combines with a moment $M = M_{max}$, as given by Eq. (10.8), so that this combination becomes equal to Pn and Mn , which will cause the section to fail.

- For slender column design, the axial load and end moments in a column are first determined using conventional frame analysis.
- The member is then designed for that axial load and a simultaneous magnified column moment.

For a nonsway frame, the ACI Code equation for magnified moment, acting with the factored axial load P_u, is written as

$$M_c = \delta M_2 \tag{10.13}$$

where the moment magnification factor is

$$\delta = \frac{C_m}{1 - P_u/0.75P_c} \geq 1 \tag{10.14}$$

The 0.75 term in Eq. (10.14) is a *stiffness reduction factor*, designed to provide a conservative estimate of P_c. The critical load P_c, in accordance with Eq. (10.1), is given as

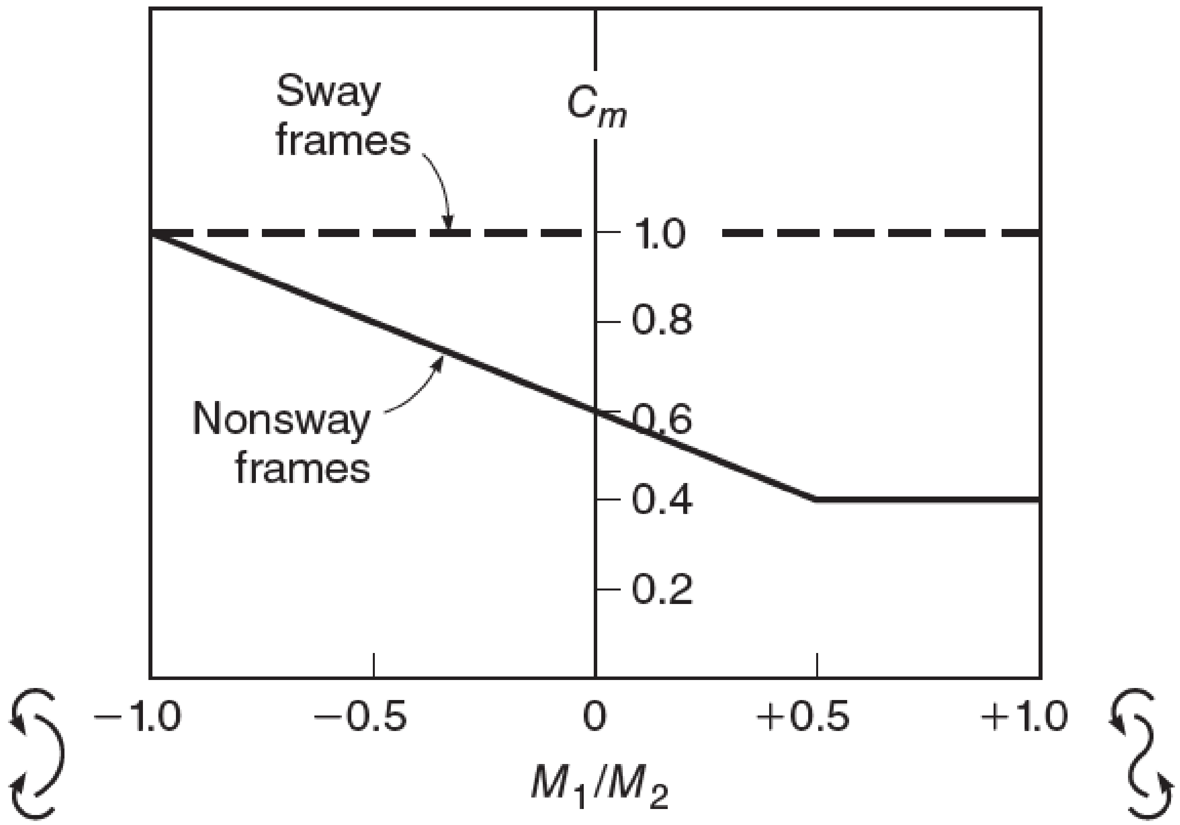
$$P_c = \frac{\pi^2 (EI)_{eff}}{(k\ell_u)^2} \tag{10.15}$$

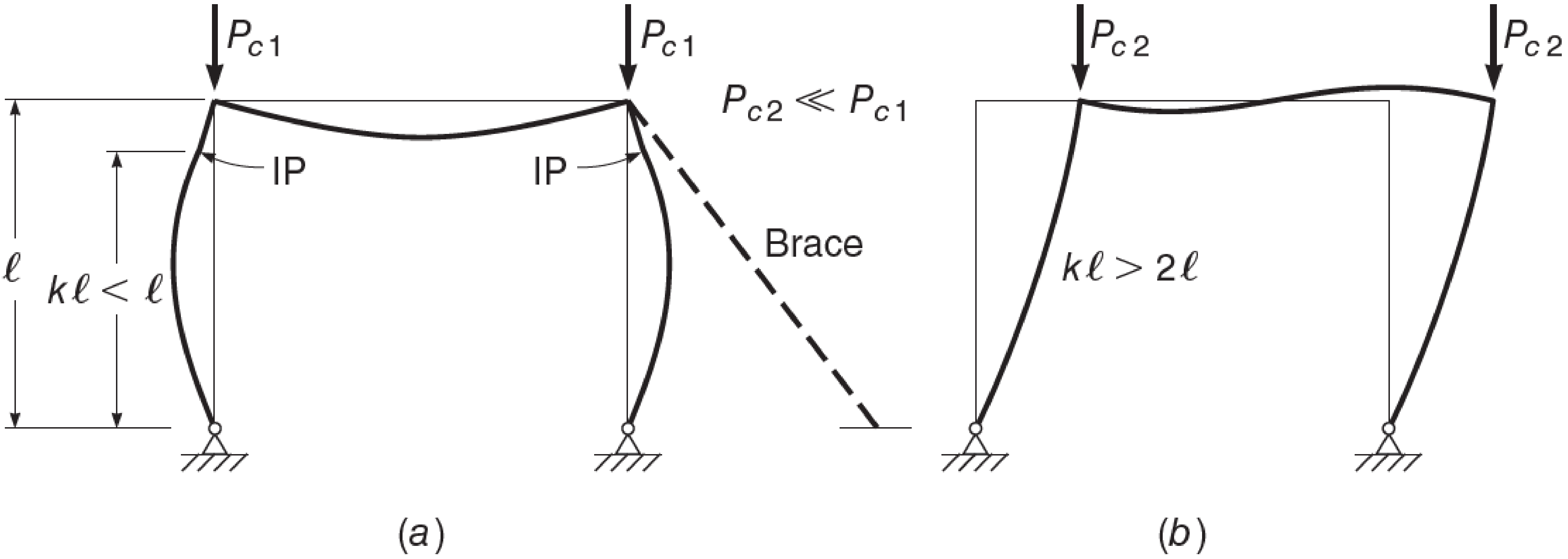
where $(EI)_{eff}$ is defined later in this section and ℓ_u is defined as the unsupported length of the compression member. The value of k in Eq. (10.15) should be set equal to 1.0, unless calculated using the values of E_c and I given in Section 10.5 and procedures described later in this section.

In Eq. (10.14), the value of C_m is as previously given in Eq. (10.9):

$$C_m = 0.6 - 0.4\,\frac{M_1}{M_2} \geq 0.4 \tag{10.9}$$

FIGURE 10.11
Values of C_m for slender columns in sway and nonsway frames.





Rigid-frame buckling:
(*a*) laterally braced and
(*b*) unbraced.

- The ACI Code provides for the capacity-reducing effects of slenderness in nonsway frames by means of the moment magnification factor δ .
- However, it is well known that for columns with no or very small applied moments, that is, axially or nearly axially loaded columns, increasing slenderness also reduces the column strength.
- For this situation, ACI Code 6.6.4.5 provides that the factored moment M_2 in Eq. (10.13) not be taken less than

$$M_{2,\min} = P_u(15 + 0.03h)$$

about each axis separately, where 0.6 and h are in mm.

For members in which $M_{2,min}$ exceeds M_2 , the value of C_m in Eq. (10.9) is taken equal to 1.0 or is based on the ratio of the computed end moments M1 and M2 .

EXAMPLE 9.1 **Design of a slender column in a nonsway frame.** Figure 9.14 shows an elevation view of a multistory concrete frame building, with 1200 mm wide × 300 mm deep beams on all column lines, carrying two-way slab floors and roof. The clear height of the columns is 3.95 m. Interior columns are tentatively dimensioned at 450 × 450 mm, and exterior columns at 400 × 400 mm. The frame is effectively braced against sway by stair and elevator shafts having concrete walls that are monolithic with the floors, located in the building corners (not shown in the figure). The structure will be subjected to vertical dead and live loads. Trial calculations by first-order analysis indicate that the pattern of live loading shown in Fig. 9.14, with full load distribution on roof and upper floors and a checkerboard pattern adjacent to column $C3$, produces maximum moments with single curvature in that column, at nearly maximum axial load. Dead loads act on all spans. Service load values of dead and live load axial force and moments for the typical interior column $C3$ are as follows:

Dead load	*Live load*
$P = 990$ kN	$P = 745$ kN
$M_2 = 3$ kN-m	$M_2 = 146$ kN-m
$M_1 = -3$ kN-m	$M_1 = 135$ kN-m

The column is subjected to double curvature under dead load alone and single curvature under live load.