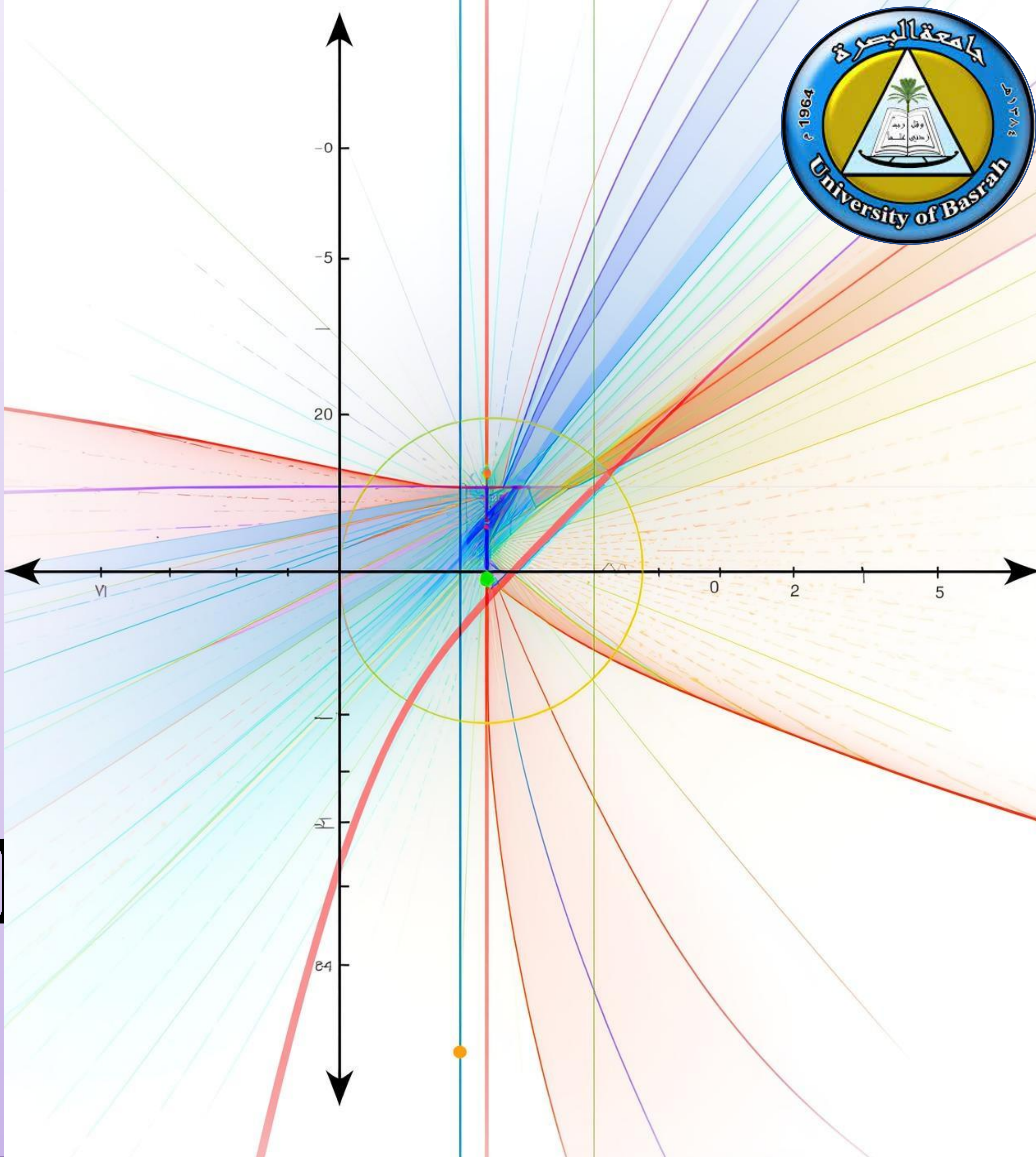
Complex Integrals: Foundations and Applicatio

Asset. Lect. Shereen M. Ibrahim



What Defines a Complex Integral?

Complex Integral

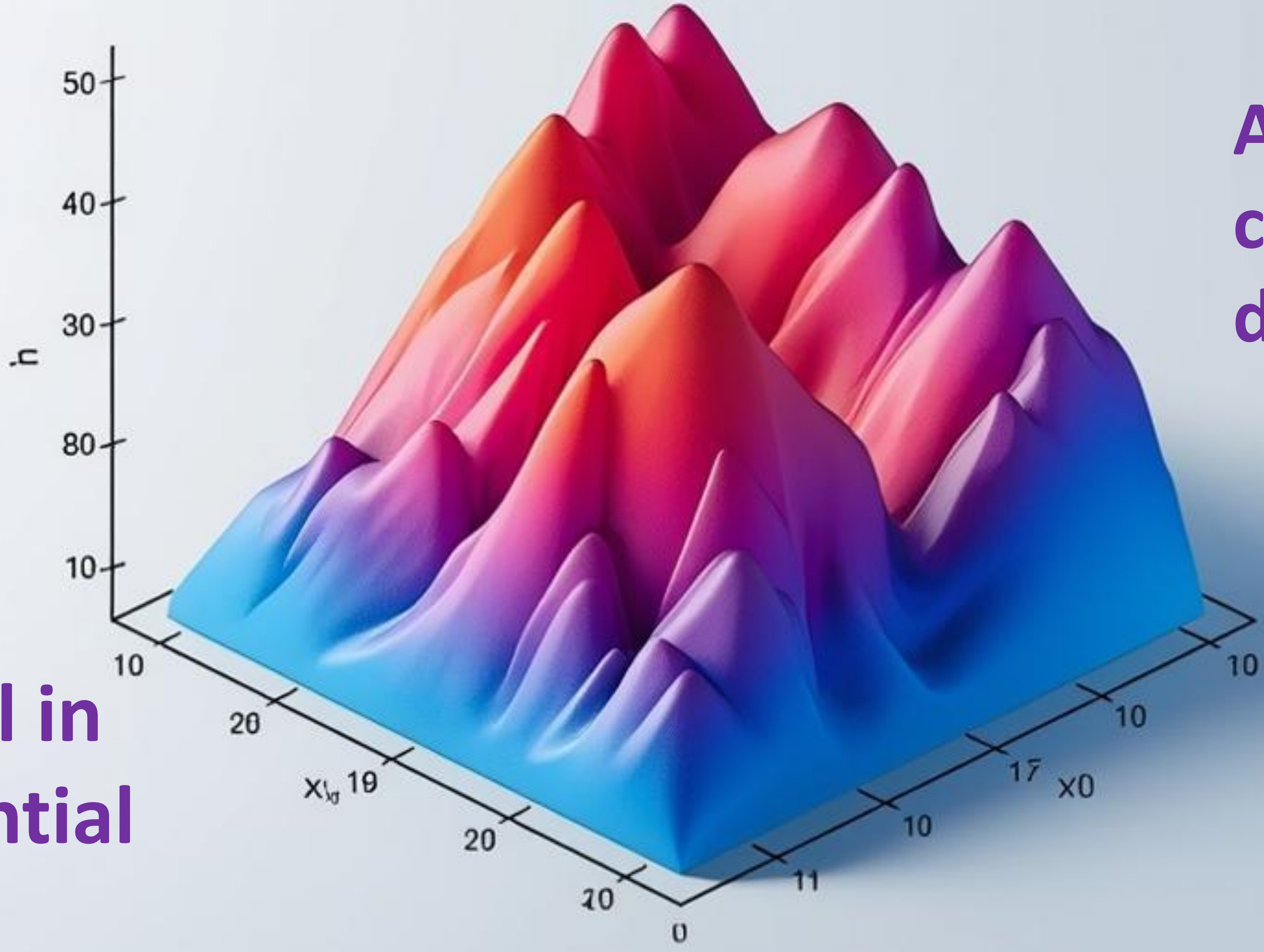
A complex integral is an integral where the integrand is a complex function, evaluated along a line or contour in the complex plane

Contour Integrals

Contour integrals are integrals taken over a specified path in the complex plane, differing from real integrals by considering complex functions' behavior and properties along these paths.

Key Distinction: All contour integrals are complex integrals, but the term "contour integral" emphasizes the path's properties and the application of powerful theorems like Cauchy's Residue Theorem

Applications of Complex Integrals



Analyzing electrical circuits, and in fluid dynamics.

In instrumental in solving differential equations

Complex Function

A complex function $f(z)$ can be written as

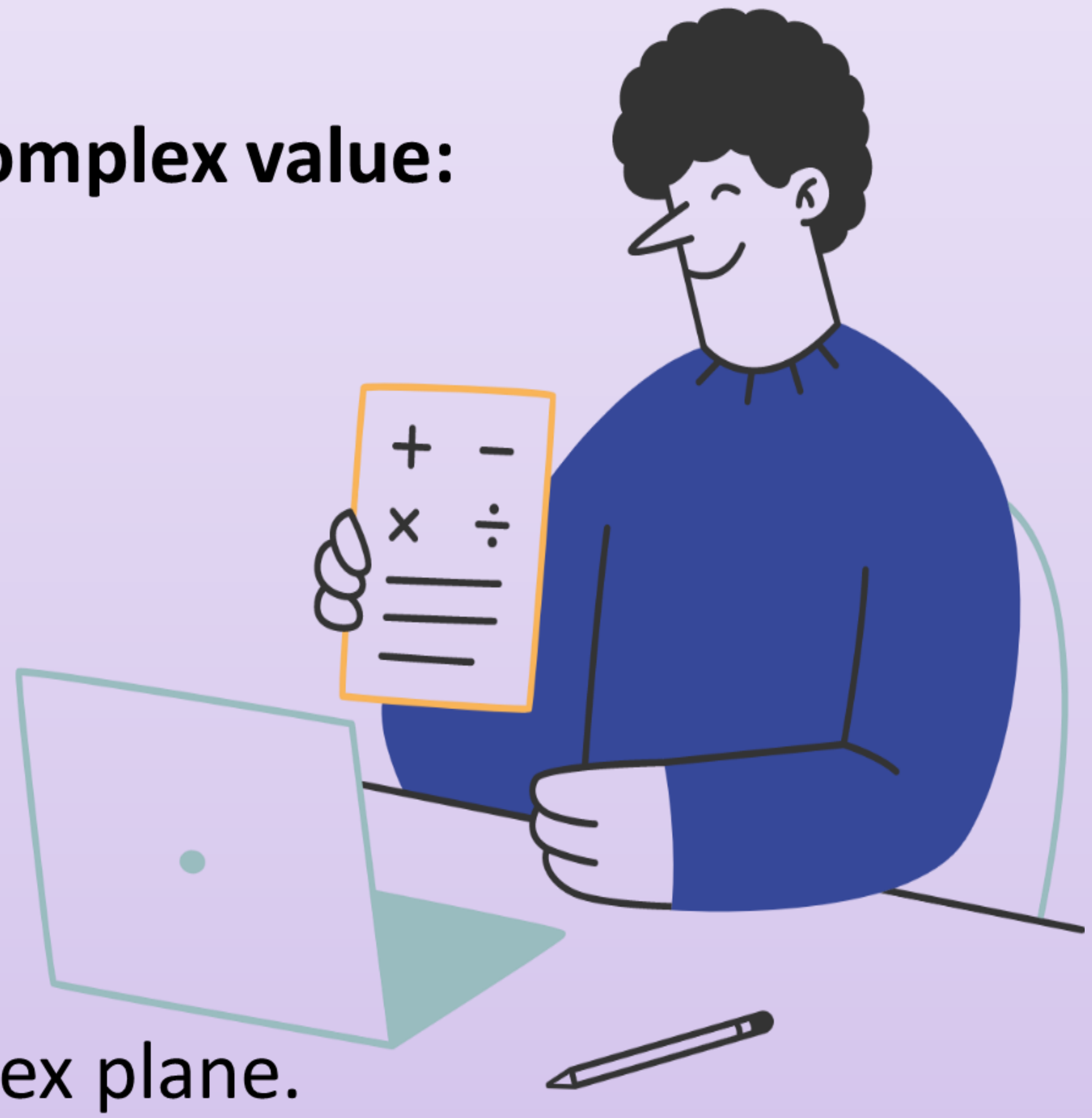
A complex function maps each complex number $z = x + iy$ to another complex value:

$$f(z) = u(x, y) + i\, v(x, y)$$

Where:

$u(x, y)$ =real part

$v(x, y)$ =imaginary part



The integral is often taken along a **contour** C, which is a curve in the complex plane.

Contour Integration: Integration along a path in the complex plane. The path can be a line, arc, or closed loop.

Types of Complex Functions

Types of Complex Functions

Polynomial Functions: $f(z)=z^2+3z+5$

Exponential Function: $f(z)=e^z$

Logarithmic Function: $f(z)=\log z=\ln|z|+i\arg(z)$

Complex Trigonometric Functions: $f(z)=\sin z, f(z)=\cos z$

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

Key Point

Analytic Functions

The concept of analytic functions emerged, highlighting their significance in complex analysis, leading to deeper understanding of singularities and the behavior of functions in the complex plane.

Cauchy's Integral Formula

The Cauchy's Integral Formula provided a powerful method for evaluating integrals of analytic functions, linking values of functions inside a contour to their integrals around that contour.

Cauchy's Integral Theorem

Cauchy's Integral Theorem established path independence for integrals of analytic functions over closed contours, a significant breakthrough in complex analysis, emphasizing the importance of contour choice.

Analytic Functions

Definition of an Analytic Function

- In mathematics, particularly in complex analysis:

An analytic function is a function that is complex differentiable at every point in some open set of the complex plane.

➢ This means it has a well-defined derivative that satisfies the Cauchy-Riemann equations in that region.

Equivalently, an analytic function can be represented by its Taylor power series expansion around any point in its domain.

Cauchy–Riemann Conditions

A function is analytic if it satisfies:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad , \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

EX: use Cauchy–Riemann proof to verify that the complex exponential function $f(z) = e^{z}$ is analytic everywhere.

$f(z) = e^{z}$

$e^{Z} = e^{X+iy} = e^{X} \, . \, \boldsymbol{e^{iy}}$

note: $\boldsymbol{e^{iy}}$ =cos(y)+i sin(y)

$f(z) = e^{X}$ (cosy+isiny)

Real part:

$$u_x = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$$

$$u_y = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$$

Imaginary part:

$$v_x = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

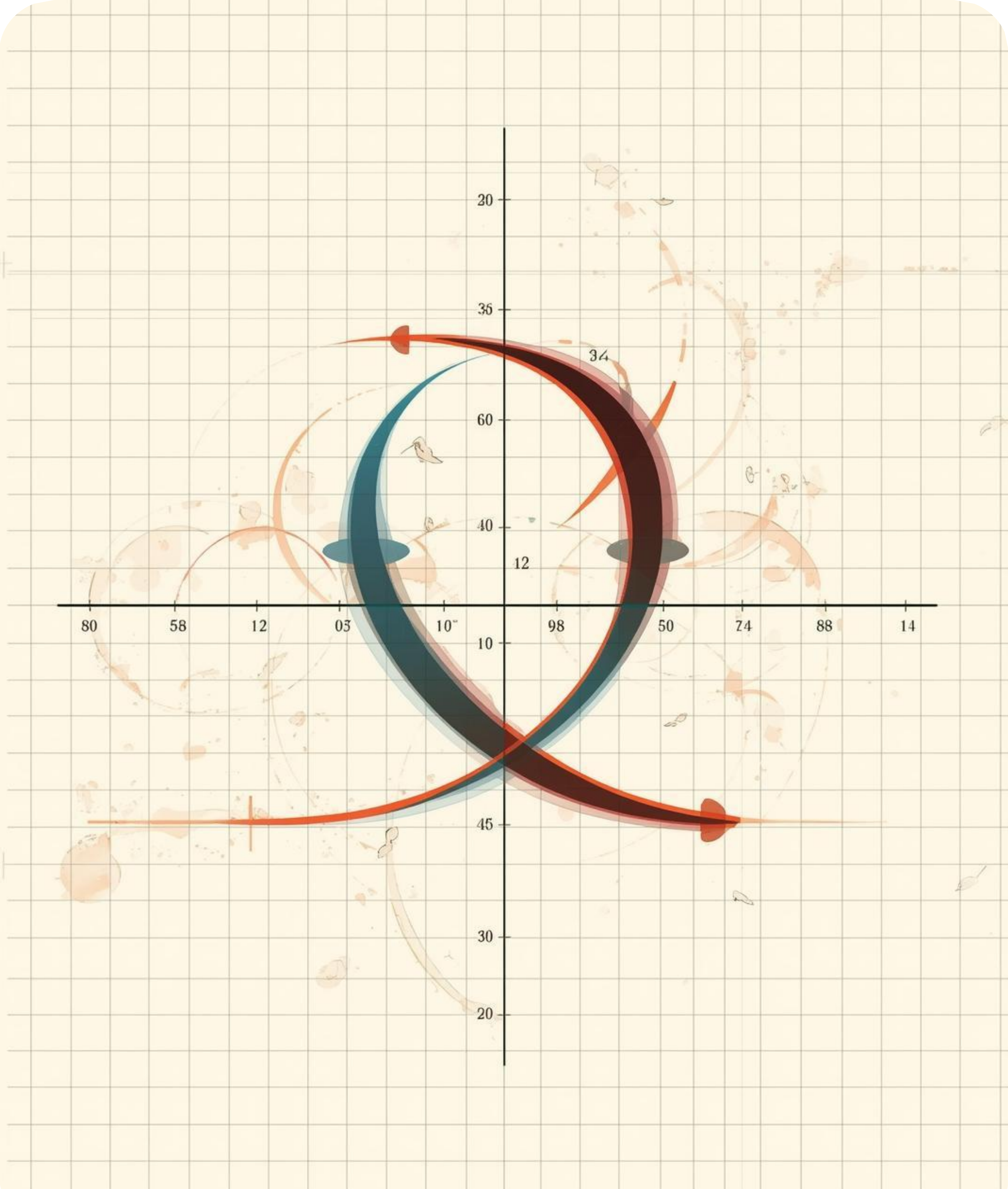
$$v_y = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

Class work Activity:

Each team will classify the following functions as **analytic** or **not analytic** why.

Functions:

1. $f(z) = z^2 + 3z + 5$
2. $f(z) = \overline{z}$

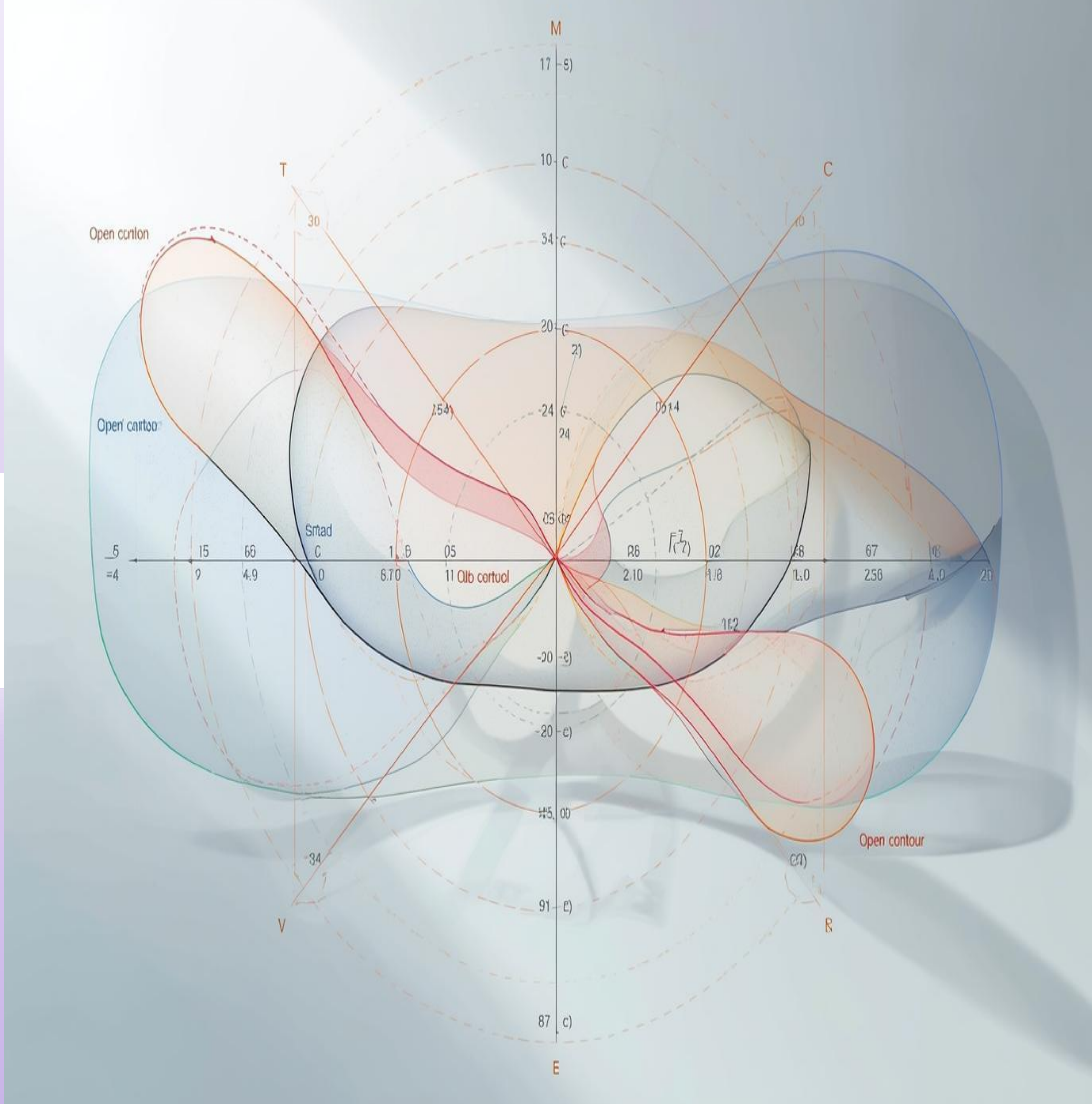


H.W 17

- classify the following functions as analytic or not analytic in the complex plane?

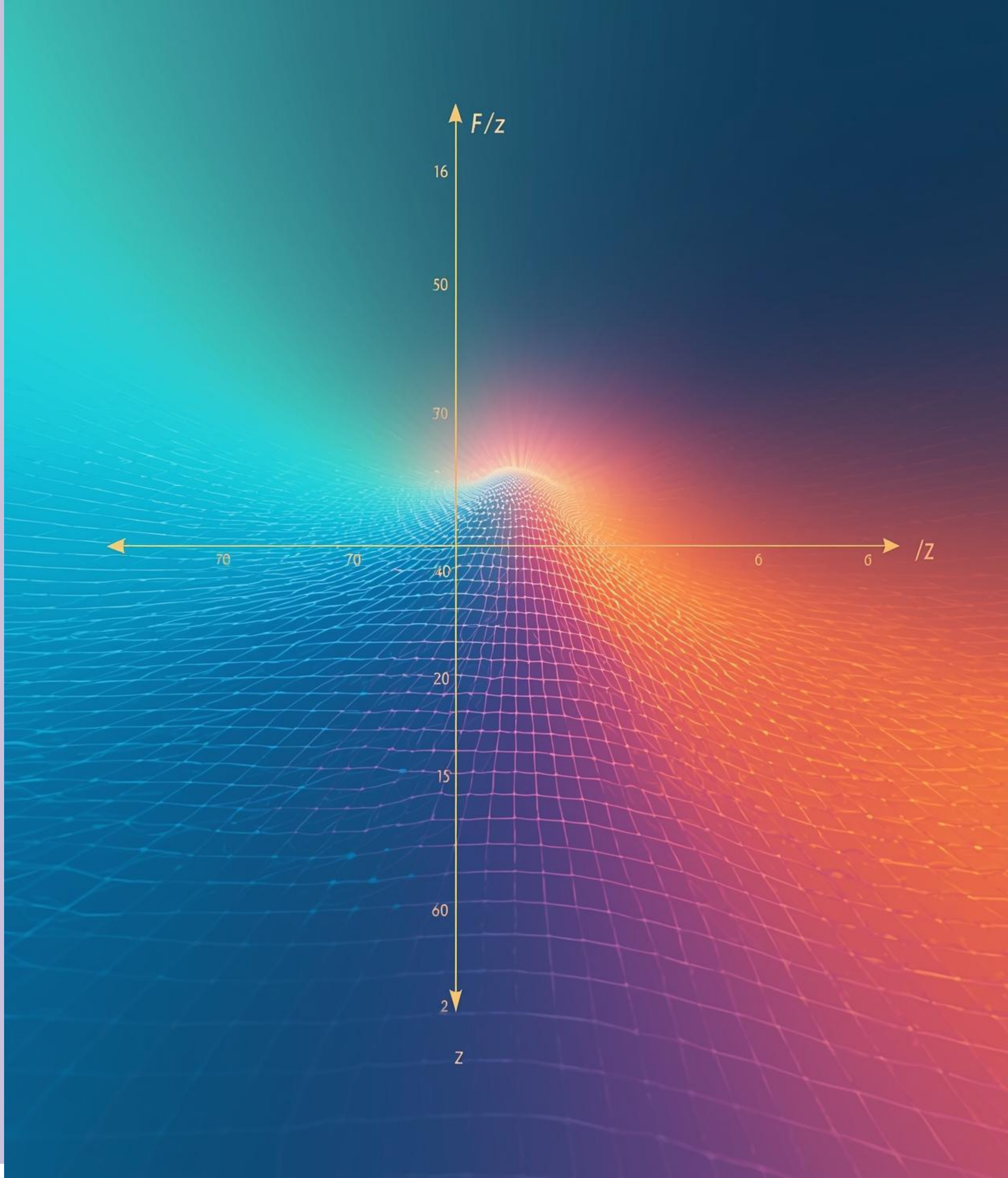
4. $f(z) = |z|$
5. $f(z) = \frac{1}{z}$ (analytic where?)
6. $f(z) = \sin z$

- Taylor power series expansion?

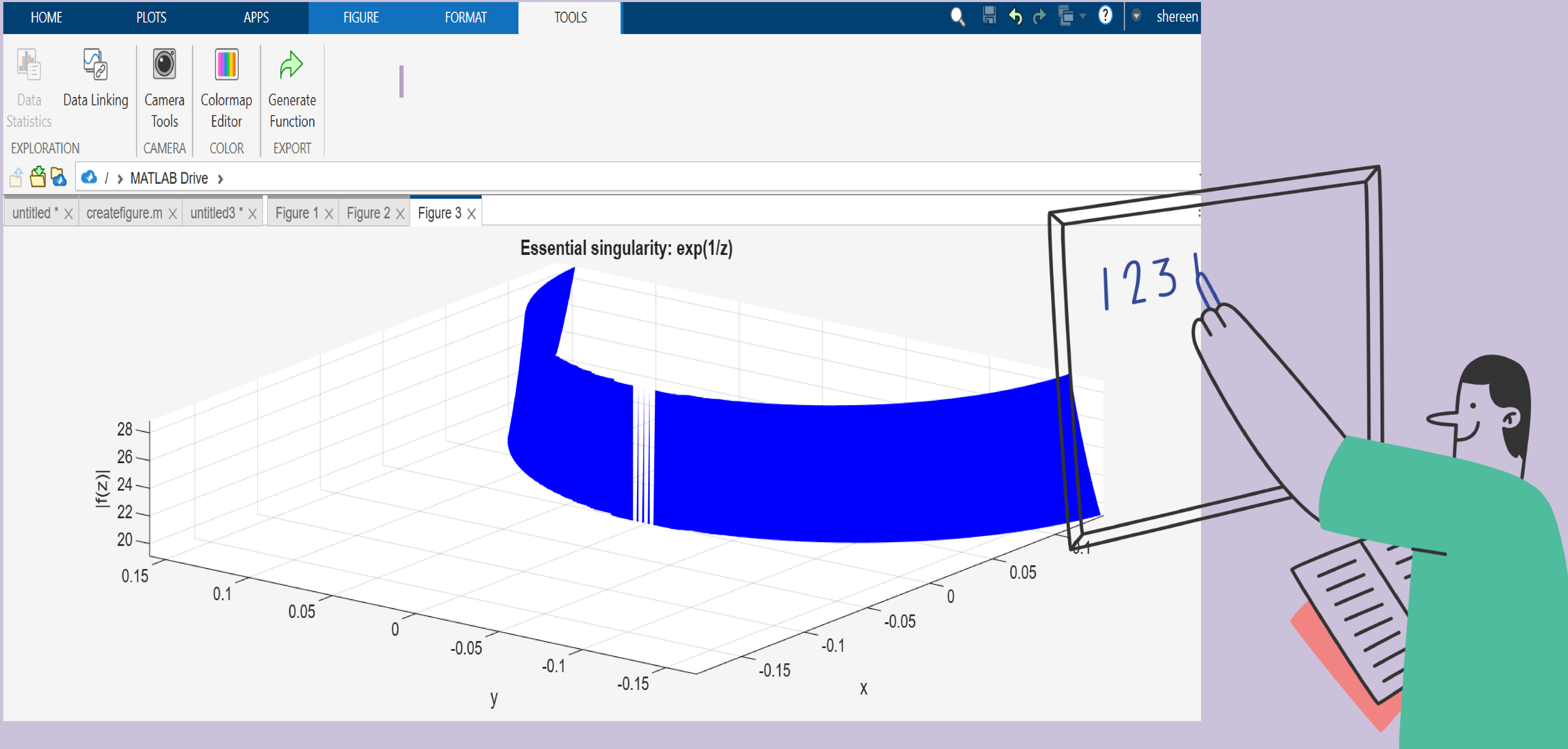


Analytic Functions and Singularities

The function f(z) = $\frac{1}{z}$ is analytic in the complex plane except at the **singularity** z=0. This point represents a **significant exception** where the function cannot be defined.



Singularities and Poles



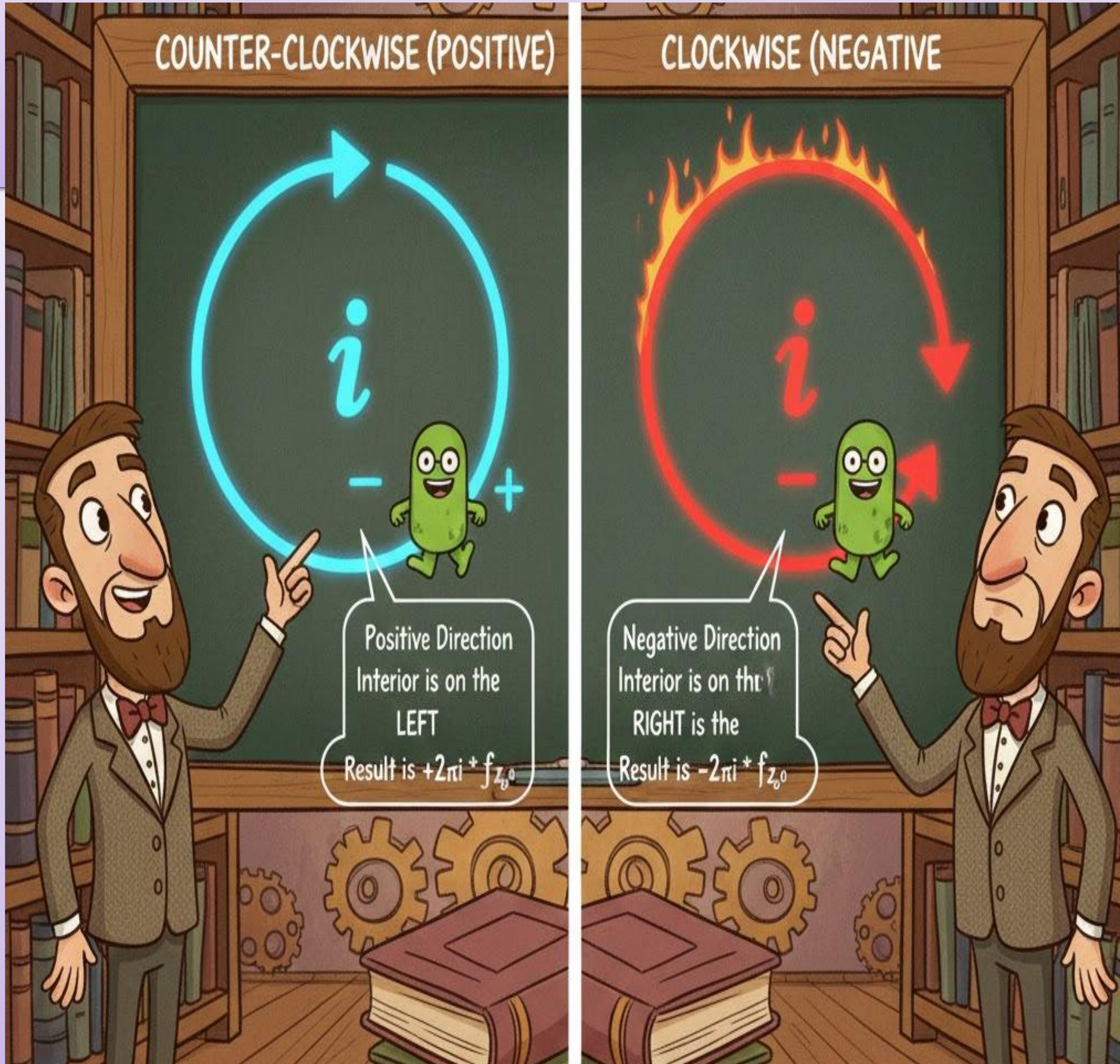
Singularities and Poles

Contour Integrals Pathe's

Examples of contours:

- Circle
- Rectangle
- Triangle

Any closed curve that starts and ends at the same point.



Cauchy's Integral

- **Cauchy's Integral Theorem (Cauchy-Goursat Theorem) :**

This is the foundational theorem. It tells us what happens when a function behaves "perfectly" (is analytic) everywhere inside a loop.

- **The Condition:** Let $f(z)$ be analytic in a simply connected domain D. Let C be a simple closed contour lying entirely within D.
- **The Result:** The integral of the function around the loop is exactly zero.

$$\oint_C f(z)\,dz = 0$$

✓ The key idea is that the function must have no singularities inside the contour

Example:

Evaluate the contour integral: $\oint_C z^2\, dz$

where C is the unit circle, $|z|=1$ traversed once counter-clockwise.

Step 1: Check for Analyticity

First, we substitute $z = x + iy$ into $f(z) = z^2$:

$$f(z) = (x + iy)^2$$

$$f(z) = x^2 + 2ixy + (iy)^2$$

$$f(z) = x^2 + 2ixy - y^2$$

$$f(z) = (x^2 - y^2) + i(2xy)$$

From this, we identify:

- Real Part: $u(x, y) = x^2 - y^2$
- Imaginary Part: $v(x, y) = 2xy$

Step 2: Apply the Theorem

Based on the Cauchy-Goursat Theorem:

$$\oint_C f(z)\, dz = 0$$

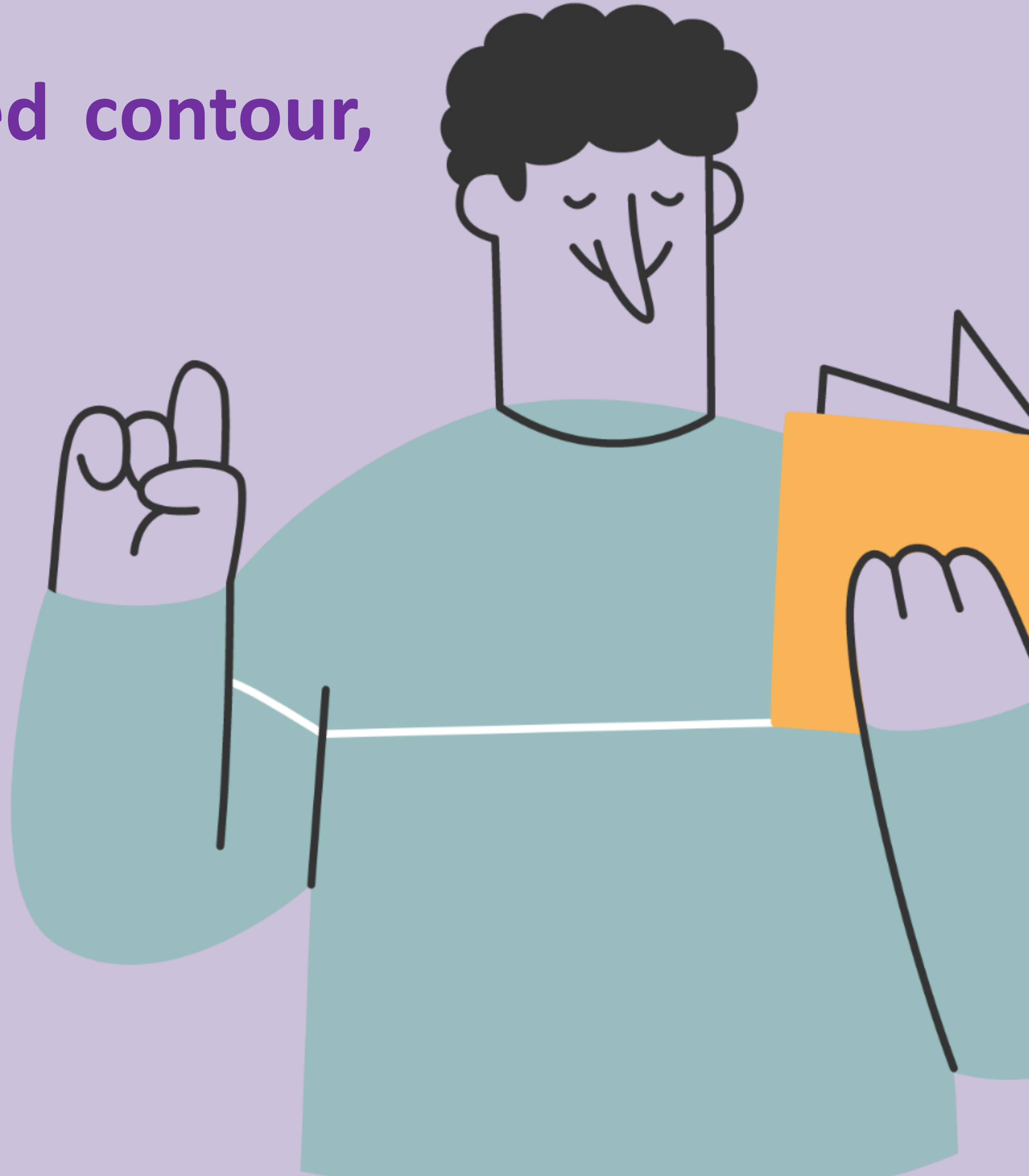
Therefore, the result is:

$$\oint_C z^2\, dz = 0$$

Cauchy's Integral Formula

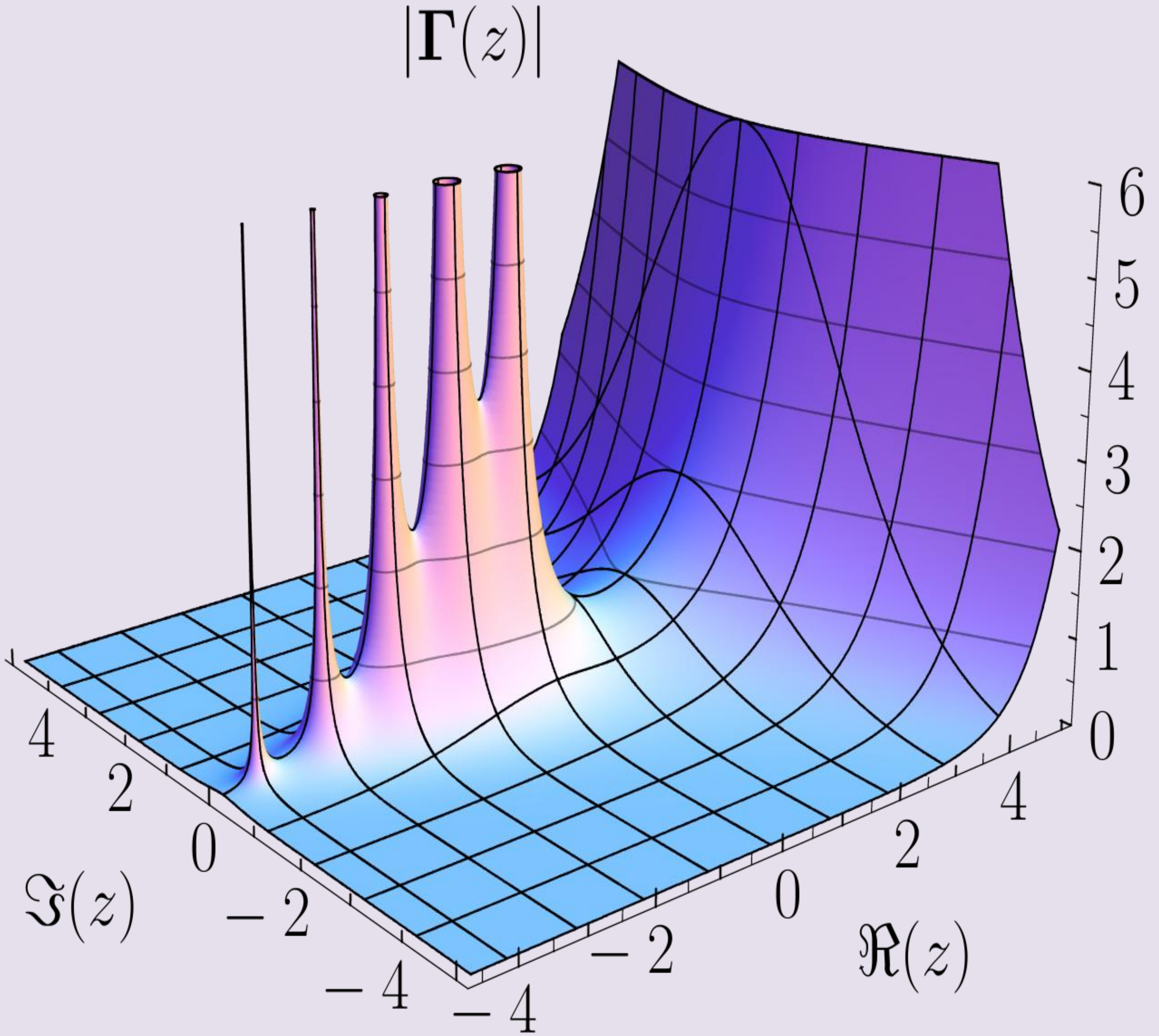
is a fundamental theorem in complex analysis that states that a holomorphic (analytic) function in a disk is entirely determined by its values on the boundary of that disk. It provides an integral formula to calculate the value of an analytic function at any point inside a simple closed contour, given the function's values on the contour itself.

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$



- This formula can also be extended to find <u>the derivatives of any order n</u> of the function $f(z)$ at point a:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz$$



Example:

Problem: Evaluate the integral $\oint_C \frac{e^z}{z-2}\, dz$ where C is the circle $|z| = 3.$

The contour integral taken in clockwise direction.

Step 1 : Identify $f(z)$ and a :

Step 2 : Check analyticity and contour:

- The contour C is a circle centered at the origin with a radius of 3.
  The point a=2 lies inside this contour since |2|<3

Step 3: Apply the Cauchy Integral Formula:

- According to the formula, the integral is equal to $2\pi i \cdot f(a)$

  $2\pi i \cdot f(2)$, $f(2)$= e^2

$$\oint_C \frac{e^z}{z-2}\, dz = \mathbf{2\pi i e^2}$$

Example:

Problem: Evaluate the integral $\oint_C \frac{\cos(z)}{(z-0)^2}\, dz$ where C is the circle $|z| = 1$.

contour integral taken in counterclockwise direction

Step 1 : Identify $f(z)$ and a :

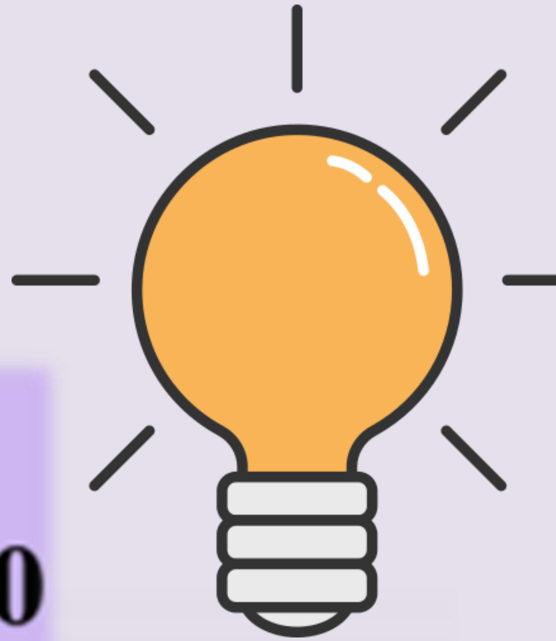
We compare the integral to the derivative formula $\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$.

Here, $f(z) = \cos(z)$, $a = 0$, and $n + 1 = 2$ (so $n = 1$).

Step 2 : Check analyticity and contour:

The function $f(z)$=cos(z) is entire (analytic everywhere).

cos(x+iy)=cosxcoshy−isinxsinhy

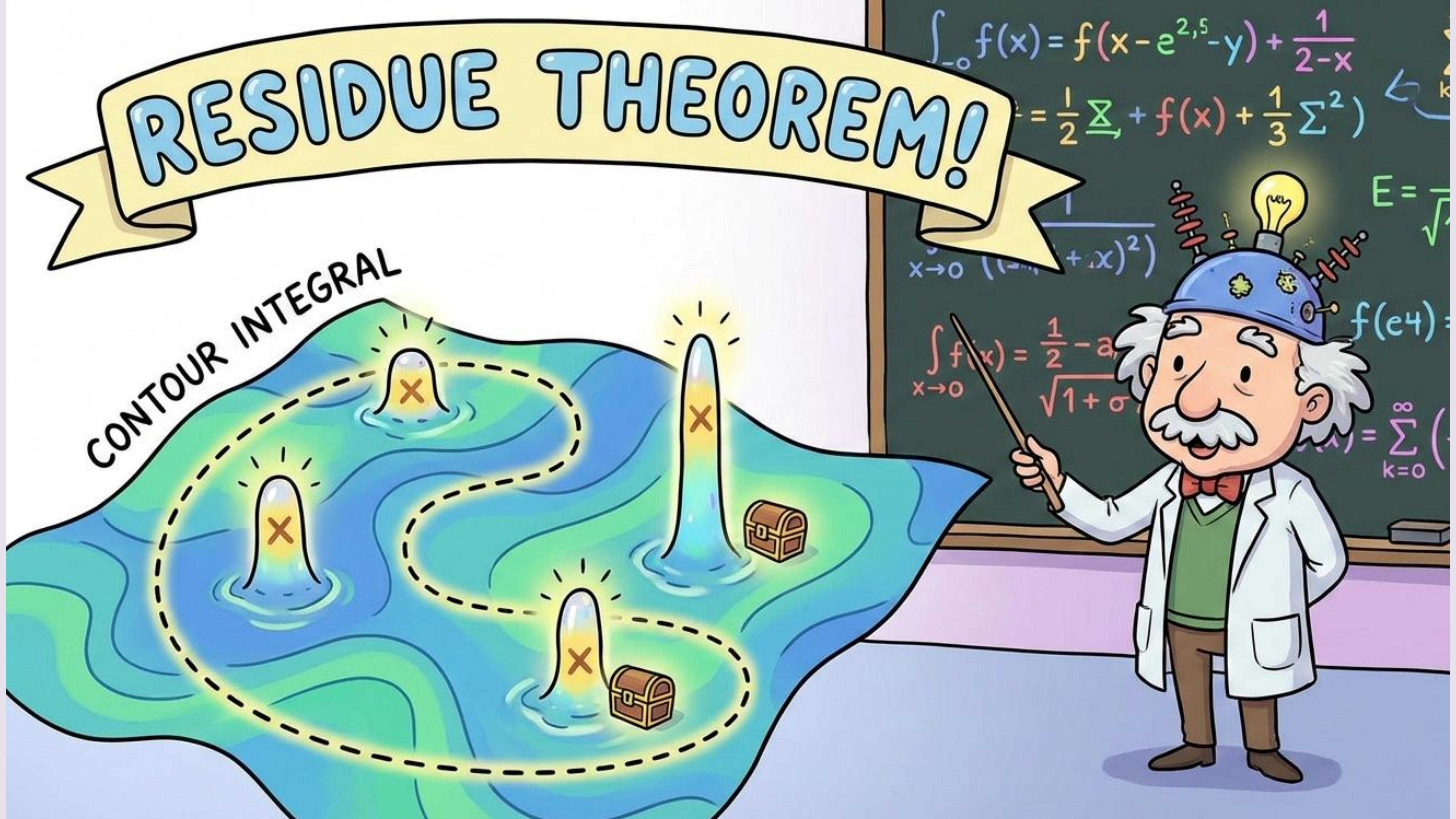


Step 3: Apply the Cauchy Integral Formula:

We need the first derivative of $f(z)$, which is $f'(z) = -\sin(z)$.
According to the formula for $n = 1$, the integral is $2\pi i \cdot f'(a)$.

$$\oint_C \frac{\cos(z)}{z^2}\, dz = 2\pi i \cdot f'(0)$$

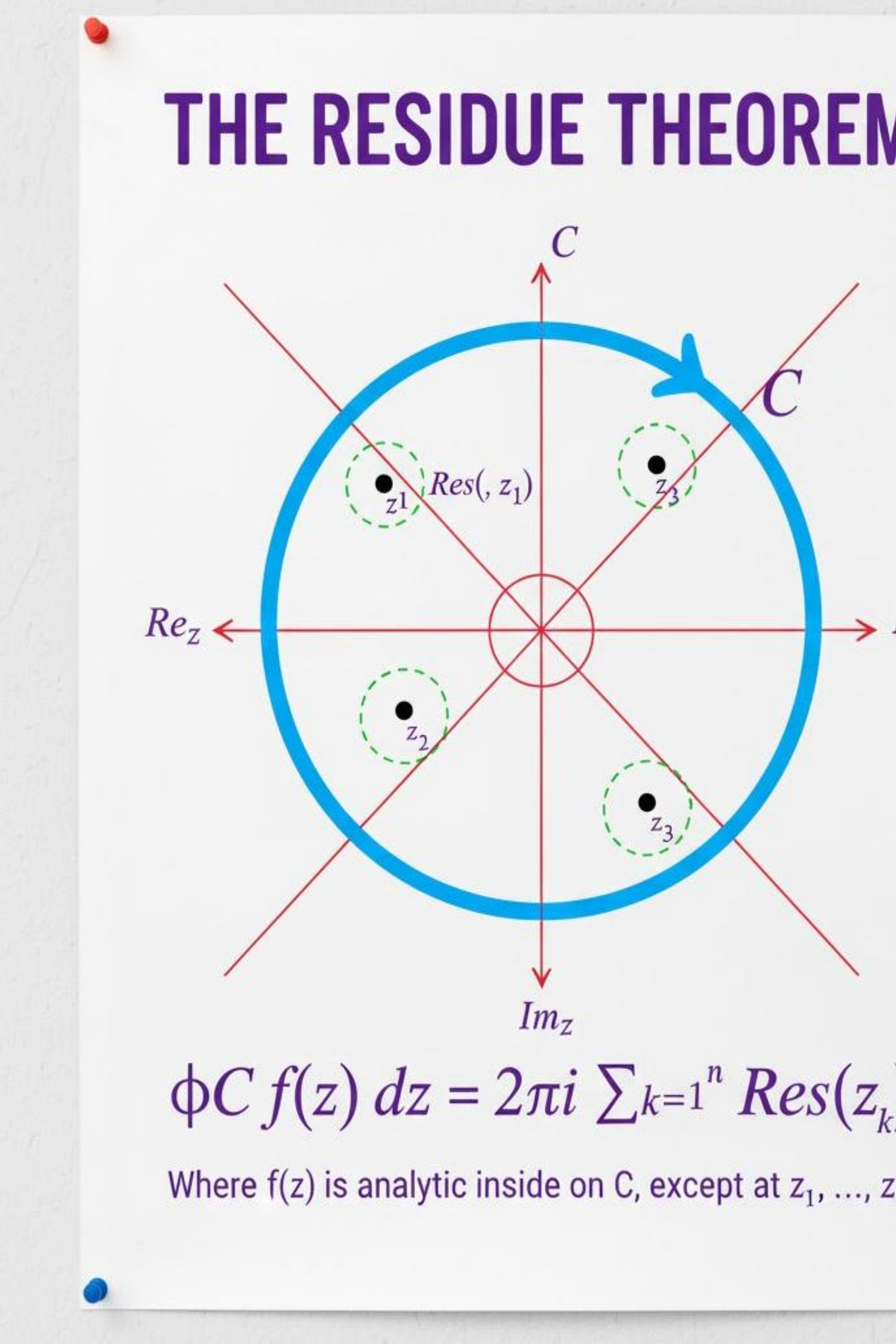
$$\oint_C \frac{\cos(z)}{z^2}\, dz = \mathbf{0}$$


RESIDUE THEOREM!
CONTOUR INTEGRAL

Residue Theorem

The **Residue Theorem** is a fundamental tool in complex integrals of analytic functions over closed curves. It relates around a closed path to the sum of "residues" at the isolated that path.

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$



Evaluate $\oint_C \frac{3z^3 + 4z^2 - 5z + 1}{z(z-1)(z+1)(z-2i)}\, dz$ where C is the circle $|z| = 3$.

Contour integral taken in counterclockwise

Step 1 : Identify $f(z)$ and a :

Step 2 : Check analyticity and contour:

Step 3: Apply the Residue Theorem :

According to the Residue Theorem:

$$\oint_C f(z)\, dz = 2\pi i \sum \text{Res}(f, z_k) = 2\pi i \cdot 3$$

$$\sum \text{Res} = \left(-\frac{i}{2}\right) + \left(\frac{3}{10} + \frac{6i}{10}\right) + \left(\frac{-7}{10} + \frac{14i}{10}\right) + \left(\frac{17}{5} - \frac{3i}{2}\right)$$

Group real and imaginary parts:

Real part:

$$0 + \frac{3}{10} - \frac{7}{10} + \frac{17}{5} = \frac{3 - 7 + 34}{10} = \frac{30}{10} = 3$$

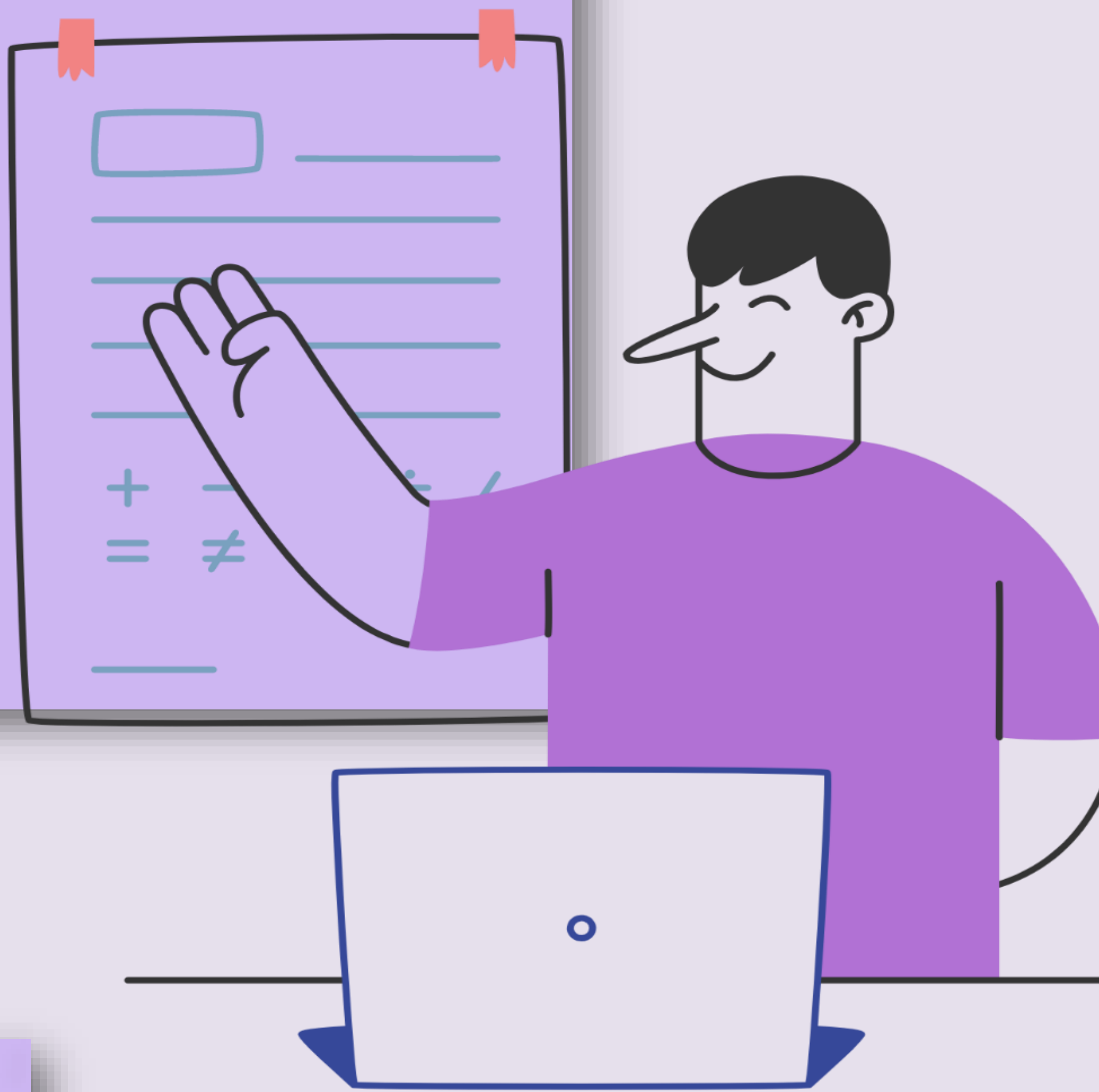
Imaginary part:

$$-\frac{1}{2} + \frac{6}{10} + \frac{14}{10} - \frac{3}{2} = -\frac{5}{10} + \frac{6}{10} + \frac{14}{10} - \frac{15}{10} = \frac{-5 + 6 + 14 - 15}{10} = \frac{0}{10} = 0$$

2. Residue Derivative Formula

For a pole of order m at $z = a$:

$$\operatorname{Res}(f, a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$



$$f(z) = \frac{g(z)}{(z-a)^m}, \quad m \geq 2$$

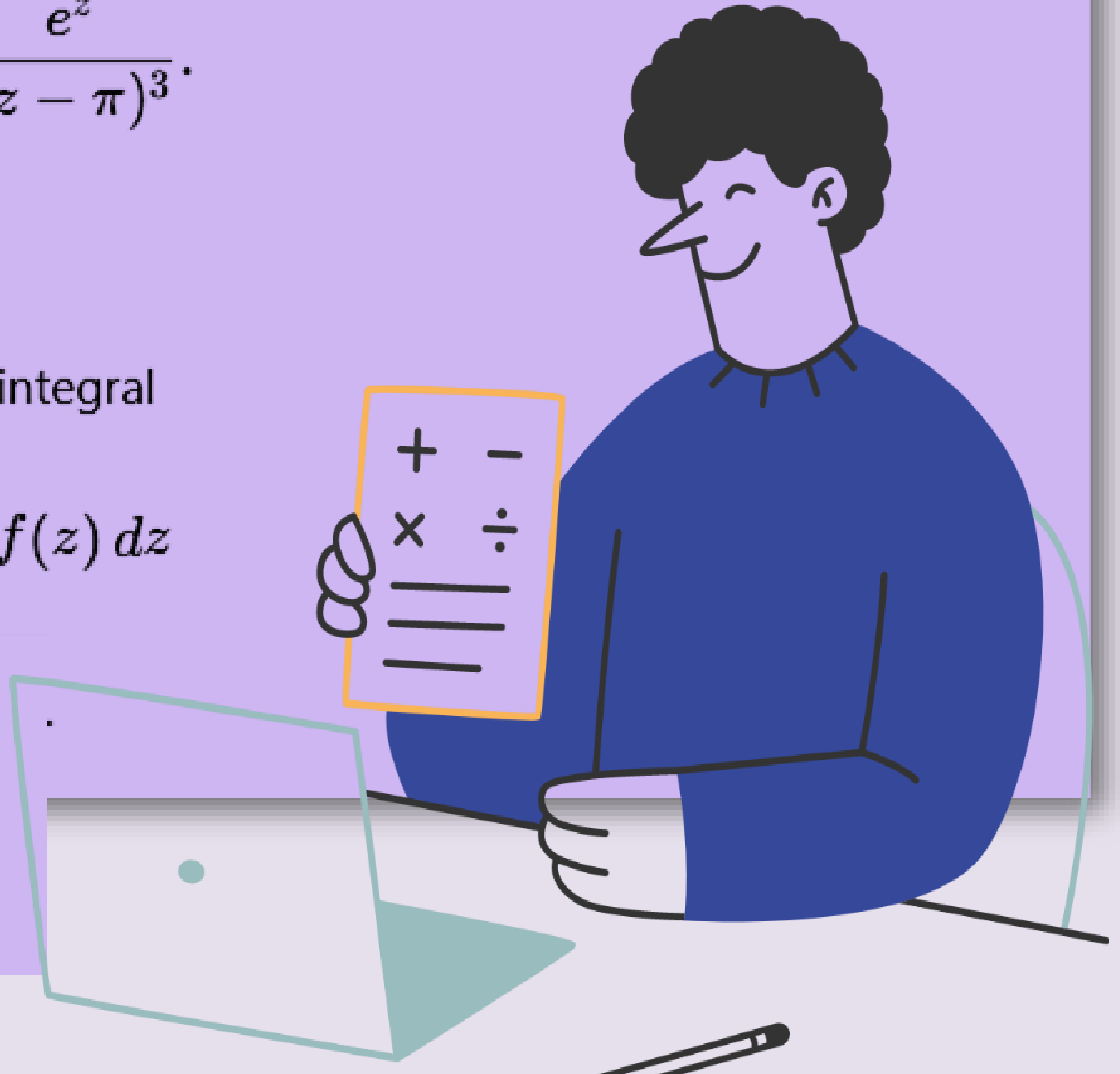
Discussion:

Let

$$f(z) = \frac{e^z}{(z-\pi)^3}.$$

1. Determine the **order of the pole** at $z = \pi$.
2. Compute the **residue** of $f(z)$ at $z = \pi$.
3. Using the **Residue Theorem**, evaluate the contour integral

$$\oint_{|z-\pi|=1} f(z)\, dz$$



Spot on! Since the order is $n = 3$, we need to differentiate **2 times** $(n-1)$.

So, our residue formula becomes:

$$\text{Res}(f,\pi) = \frac{1}{2!} \lim_{z\to\pi} \frac{d^2}{dz^2}\left[(z-\pi)^3 \cdot \frac{e^z}{(z-\pi)^3}\right]$$

THANK YOU

For your Attention !