Z -Transform — Theory, Properties, and Petroleum Engineering Applications

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Course: Engineering Analysis & Numerical
Level: Undergraduate 📈 💼

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1. Introduction 🌍 📃

For **discrete-time** signals and systems, we use the **Z-Transform**. It is especially useful when the Fourier Transform **fails to converge** or when we want to analyze system behavior **outside the unit circle**. **Z-Transform** gives you more power (convergence, system behavior).

❖ Z-Transform Definition

We define the **Z-transform** of a signal x[n] as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

It's the **most general** — it works in the **complex plane**.

The **Z-transform** is a tool used in **digital signal processing (DSP)** and **control systems** to analyze **discrete-time signals and systems**

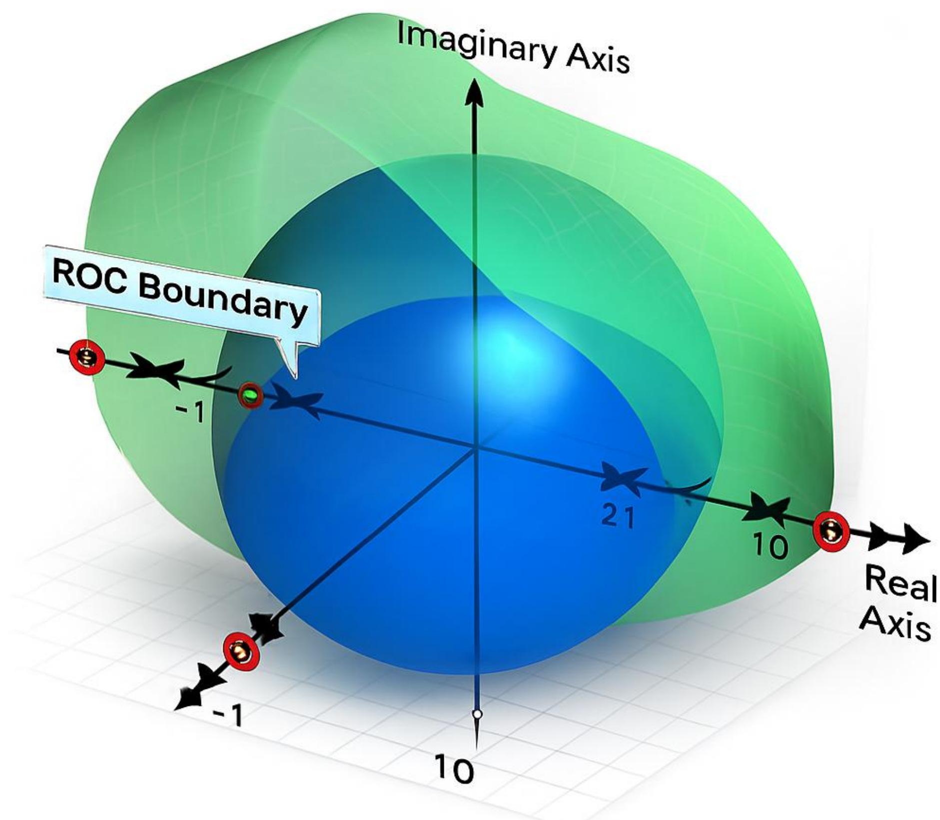
- ✓ x[n]: signal in time domain
- ✓ X(z): signal in **Z-domain**
- ✓ z: complex number z=r$e^{i\omega}$

The Z-transform exists only for the values of z where the **sum converges** — this set is called the **Region of Convergence (ROC)**.

So, The Z-transform is a powerful tool used in digital signal processing to analyze discrete-time signals.

It transforms a time-domain signal $x[n]$ into a complex frequency-domain representation $X(z)$.

A key concept in this transformation is the Region of Convergence (ROC) — the set of values of z for which the Z-transform converges.



So now let's summary, what is the Z-transform?

The Z-transform converts a discrete-time signal (like a sequence of numbers) into a complex function. It's defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Here:

- $x[n]$**is your signal.**
- z **is a complex variable.**
- **The sum may or may not converge depending on** z**.**

2- The Region of Convergence ROC

The **Region of Convergence (ROC)** is the set of all values of z for which the Z-transform sum **converges** (i.e., does not go to infinity). It tells us **where in the complex plane** the transform is valid.

Why is the ROC important?

- It tells us **whether** the Z-transform exists.
- It helps determine **stability** and **causality**.
- Different signals can have **the same Z-transform expression**, but **different ROCs**, leading to different meanings.

For deep understanding, what does "convergence" mean in the Z-transform?

When we compute the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

This infinite sum **only converges** (i.e., gives a finite result) for certain values of z. The set of all such values is called the **Region of Convergence (ROC)**.

Now here's the key idea:

The ROC tells us **where** the Z-transform is valid — and it depends on the **signal $x[n]$.**

Examples to Understand how to find ROC for different signals:

1. Right-sided signal (Causal)

$$x[n] = a^n u[n], \quad a \in \mathbb{R}$$

That means:

- $x[n] = a^n$ for $n \geq 0$, and 0 elsewhere.

Z-transform:

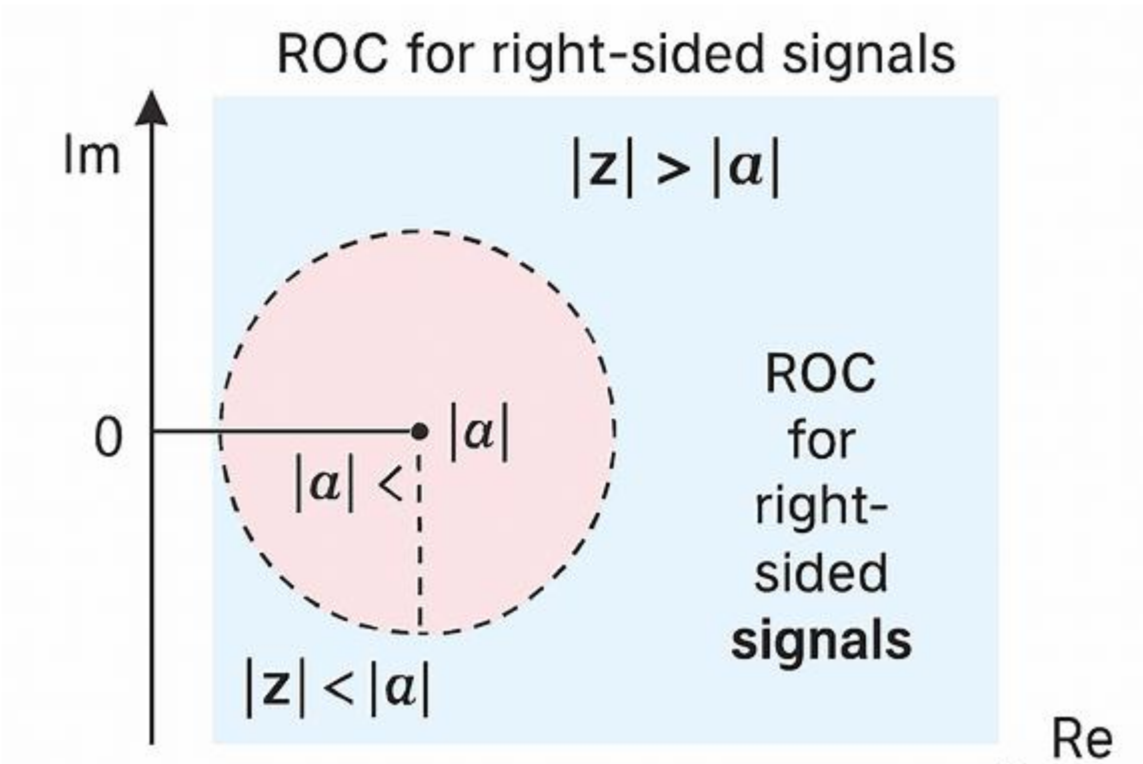
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

This is a **geometric series**, converges only when:

$$|az^{-1}| < 1 \Rightarrow |z| > |a|$$

✅ So the **ROC is**:

$$\boxed{|z| > |a|} \quad \text{(outside a circle of radius } |a|)$$



This diagram shows the Region of Convergence (ROC) for a **right-sided signal** in the Z-transform. The ROC is the **blue-shaded area outside** the dashed circle of radius | a |, where the Z-transform converges. Inside the circle (pink), the series diverges.

🔹 2. **Left-sided signal** (Anti-causal)

$$x[n] = -a^n u[-n-1], \quad a \in \mathbb{R}$$

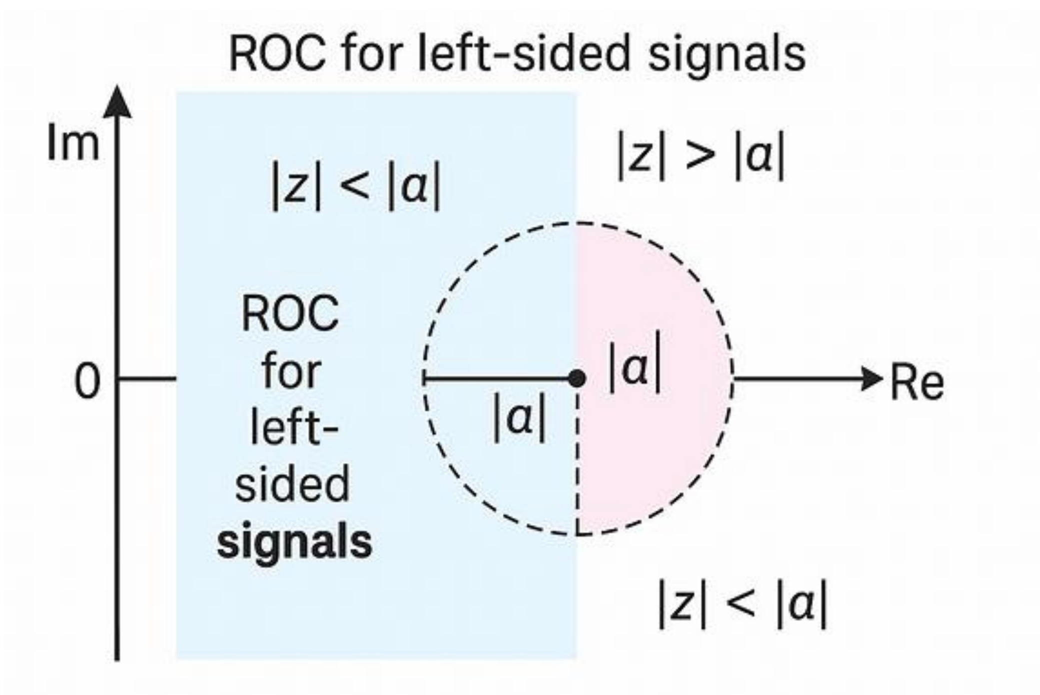
Z-transform:

$$X(z) = \sum_{n=-\infty}^{-1} a^n z^{-n} = \sum_{k=1}^{\infty} (az)^k$$

Converges if $|az| < 1 \Rightarrow |z| < \frac{1}{|a|}$

✅ ROC:

$$\boxed{|z| < \frac{1}{|a|}} \quad \text{(inside a circle)}$$



The **ROC for left-sided signals** is actually a **full circle** — meaning all points in the complex plane where $|z| < |a|$. But in the diagram, it might look like only half the circle is shaded. That's just a **visual simplification** to highlight the region of interest.

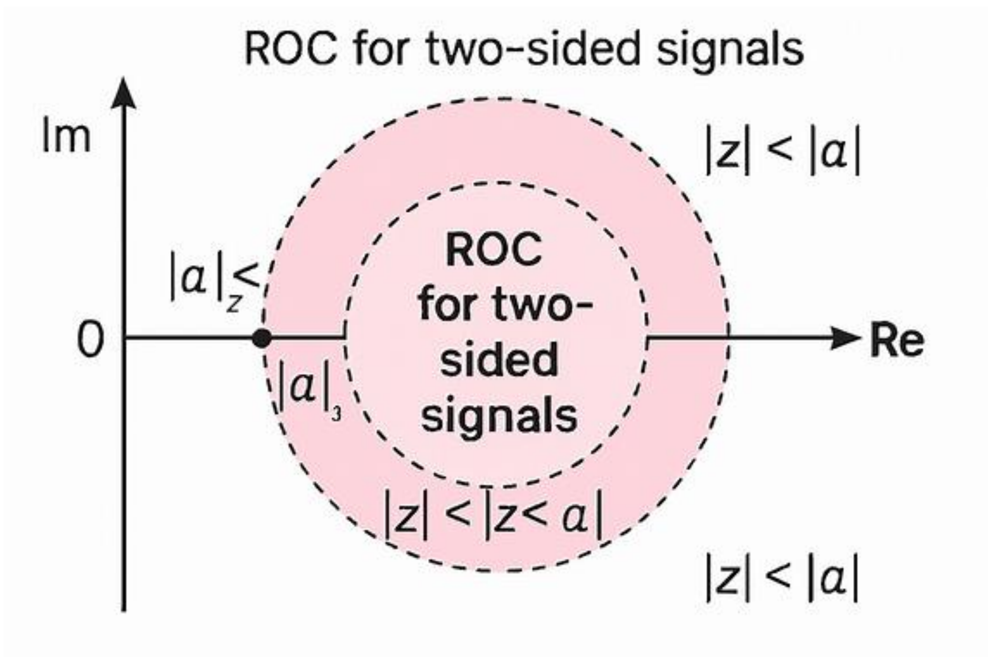
3. Two-sided signal

$$x[n] = a^n \quad \text{for all } n \in \mathbb{Z}$$

The Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n}$$

This diverges **everywhere** except possibly on an **annular ring** if both parts converge. But usually, **the ROC is empty** for exponentially growing two-sided signals.



This means the signal has nonzero values for both positive and negative n. For example:

$$\boldsymbol{x[n] = a^n u[n] + b^n u[-n-1]}$$

ROC for this case:

The Z-transform converges when:

$$|a| < |z| < |b|$$

So the ROC is the **ring-shaped region** between two circles — not inside or outside, but **in between.**

Hint:

A geometric series looks like this:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

Where:

- **r is the common ratio between terms**
- **The series starts at $n = 0$**

Convergence Condition

This series** converges **only when: $|r| < 1$

If $|r| \geq 1$, the series diverges **(goes to infinity).**

Formula for the Sum

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Variant Starting from $n = 1$

If the series starts from $n = 1$, like:

$$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \cdots$$

Then the formula becomes:

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1 - r}$$

Line No.	$x(n)$, $n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n) z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2 u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$d^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na} u(n)$	$\frac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$\sin(an) u(n)$	$\frac{z \sin(a)}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
10	$\cos(an) u(n)$	$\frac{z[z - \cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
11	$d^n \sin(bn) u(n)$	$\frac{[a \sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\|z\| > \|a\|$
12	$d^n \cos(bn) u(n)$	$\frac{z[z - a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\|z\| > \|a\|$
13	$e^{-an} \sin(bn) u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
14	$e^{-an} \cos(bn) u(n)$	$\frac{z[z - e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$

Problems:

Given signals, derive its Z-transform and determine the ROC?

1. Signal Expression: $x[n] = 2^n u[n]$

Step 1: Identify the Signal Type

- $u[n]$is the unit step function, which equals 1 for $n \geq 0$, and 0 otherwise.
- So $x[n] = 2^n$only for $n \geq 0$, and is zero elsewhere.

◇ Type: This is a right-sided (causal) signal.

Step 2: Compute the Z-Transform

Using the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n] = 0$for $n < 0$, the sum becomes:

$$X(z) = \sum_{n=0}^{\infty} 2^n z^{-n}$$

We rewrite this as a geometric series:

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

Step 3: Convergence Condition

For a geometric series to converge, the ratio must satisfy:

$$\left|\frac{2}{z}\right| < 1$$

Multiply both sides by $|z|$:

$$2 < |z| \Rightarrow |z| > 2$$

Step 4: Final Result

Z-Transform:

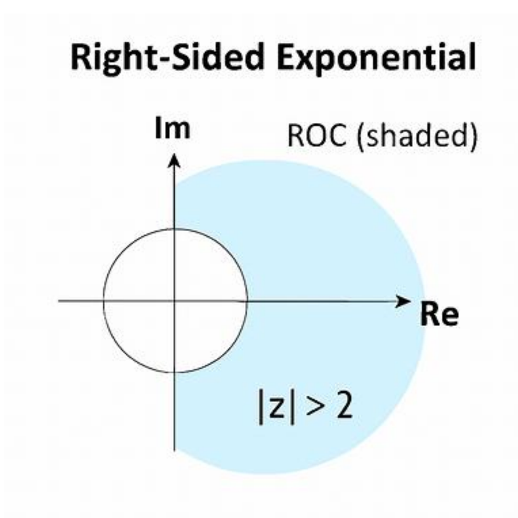
$$X(z) = \frac{1}{1 - \frac{2}{z}} = \frac{z}{z-2} \text{ for } |z| > 2$$

Region of Convergence (ROC):

$$|z| > 2$$

Step 5: ROC Diagram

- Draw a circle with radius 2 centered at the origin in the complex plane.
- Shade the region **outside** the circle — that's where the Z-transform converges.



The system is **not stable**, because the ROC **does not include** the unit circle.

- The unit circle is $|z| = 1$
- But the ROC is $|z| > 2$

So:

$$|z| = 1 \notin \text{ROC}$$

Notes:

- If **ROC includes unit circle** |z|=1, system is **stable**.
- If **ROC is |z|> r**, signal is **causal**.
- If **ROC is |z|< r**, signal is **anti-causal**.

2. Signal Expression: $x[n] = 3^n u[-n-1]$

Step 1: Understand the Signal

What is $\boldsymbol{u[-n-1]}$?

This is the left-sided unit step function, which equals 1 when:

$$\boldsymbol{n \leq -1}$$

and 0 otherwise.

So the signal becomes:

$$\boldsymbol{x[n] = 3^n \text{ for } n \leq -1}$$

Type: Left-sided (anti-causal)

Step 2: Apply the Z-Transform Definition

$$\boldsymbol{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

Since $x[n] = 0$ for $n > -1$, we simplify:

$$\boldsymbol{X(z) = \sum_{n=-\infty}^{-1} 3^n z^{-n}}$$

Let's rewrite this sum in a more manageable form.

Step 3: Change of Index

This means we're summing over all **negative integers** less than or equal to -1:

$$n = -1, -2, -3, -4, \ldots$$

Let's define a new variable:

$$k = -n \Rightarrow n = -k$$

When $n = -1$, $k = 1$; When $n = -\infty$, $k = \infty$

Let $\boldsymbol{k = -n}$

This flips the sign of each n, so:

- When $n = -1$, then $k = 1$
- When $n = -2$, then $k = 2$
- When $n = -3$, then $k = 3$
- ...
- When $n = -\infty$, then $k = \infty$

So the values of $k$become:

$$k = 1,2,3,4, \ldots, \infty$$

So the sum becomes:

$$X(z) = \sum_{k=1}^{\infty} 3^{-k} z^{k} = \sum_{k=1}^{\infty} (\frac{z}{3})^{k}$$

Step 4: Use Geometric Series Formula

We know:

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r} \quad \text{for } |r| < 1$$

Here, $r = \frac{z}{3}$

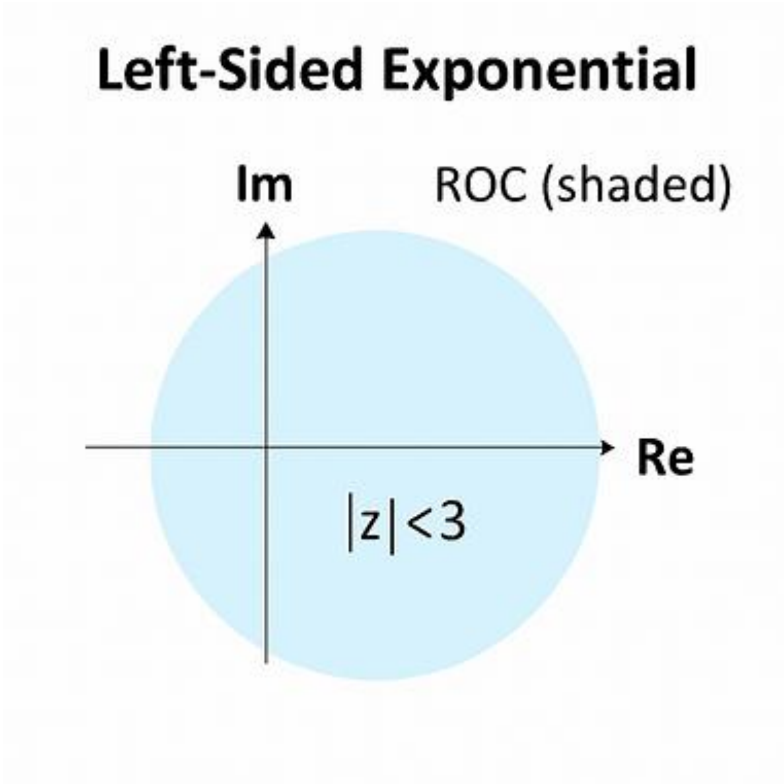
So:

$$X(z) = \frac{z/3}{1 - z/3} = \frac{z}{3 - z}$$

Step 5: Region of Convergence (ROC)

For convergence:

$$\left|\frac{\boldsymbol{z}}{\mathbf{3}}\right| < \mathbf{1} \Rightarrow |\boldsymbol{z}| < \mathbf{3}$$



Rules of ROC

Rule	Description
1	ROC is a ring or disk centered at the origin (in complex plane)
2	The Z-transform converges only inside the ROC
3	Right-sided signals ⇒ ROC is outside the outermost pole
4	Left-sided signals ⇒ ROC is inside the innermost pole
5	Two-sided signals ⇒ ROC is a ring between poles

Z-Transform Properties with Explanation and Example:

Property	Explanation	Example
Linearity	Z-transform of a linear combination is the same combination of transforms	$Z\{ax_1[n] + bx_2[n]\} = aX_1(z) + bX_2(z)$
Time Shifting	Shifting in time multiplies by z^{-m}	$Z\{x[n-m]\} = z^{-m}X(z)$
Time Reversal	Reversing time replaces z with z^{-1}	$Z\{x[-n]\} = X(z^{-1})$
Scaling in z-domain	Multiplying by a^n scales the argument of $X(z)$	$Z\{a^n x[n]\} = X(z/a)$
Multiplication by n	Multiplying by n corresponds to derivative in Z-domain	$Z\{nx[n]\} = -z\frac{dX(z)}{dz}$

🕮 H.W

Given signals, derive its Z-transform and determine the ROC?

Expression: $x[n] = (-0.5)^n u[n]$

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