Complex Integrals: Foundations and Applicatio

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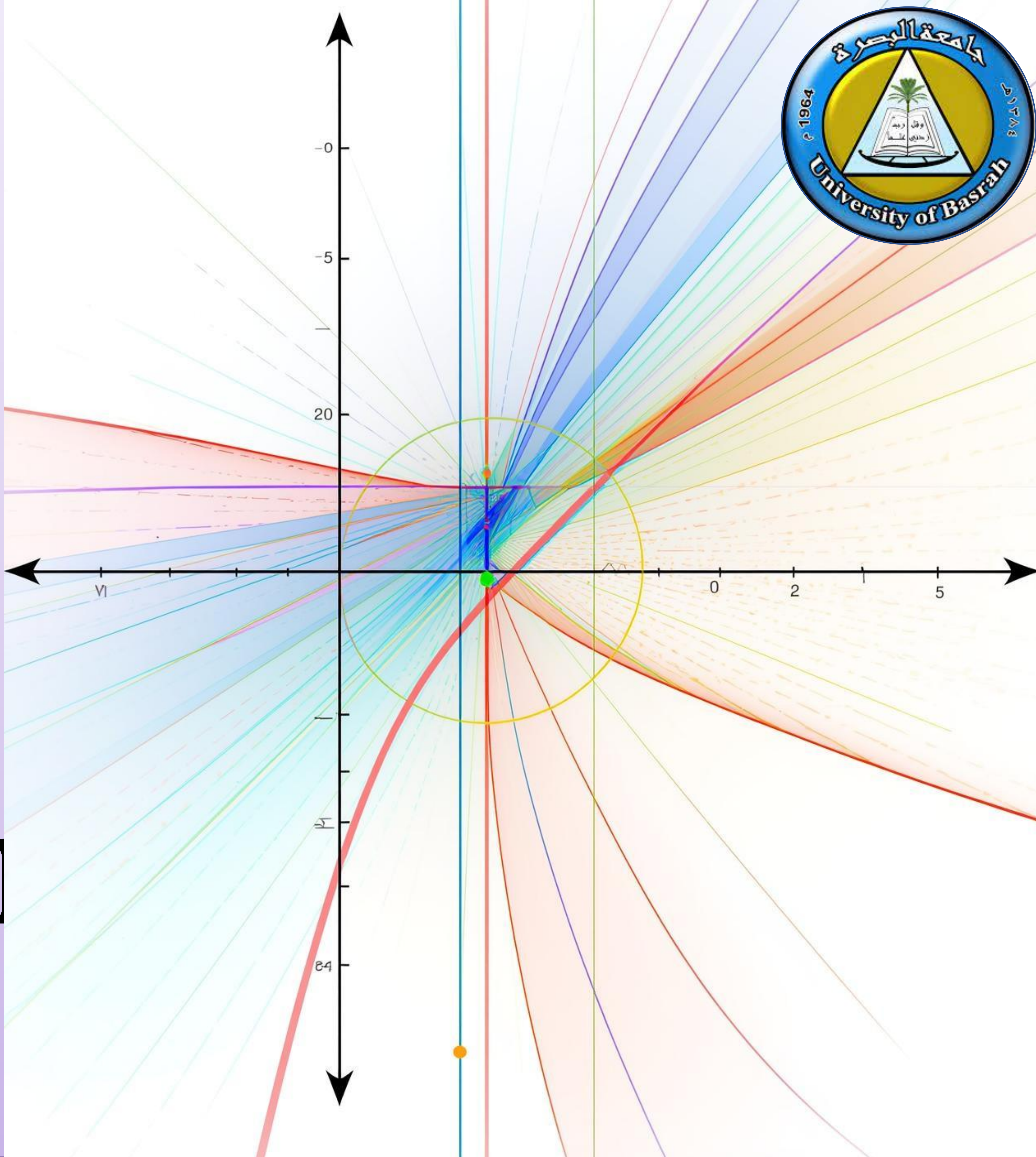


What Defines a Complex Integral?

Complex Integral

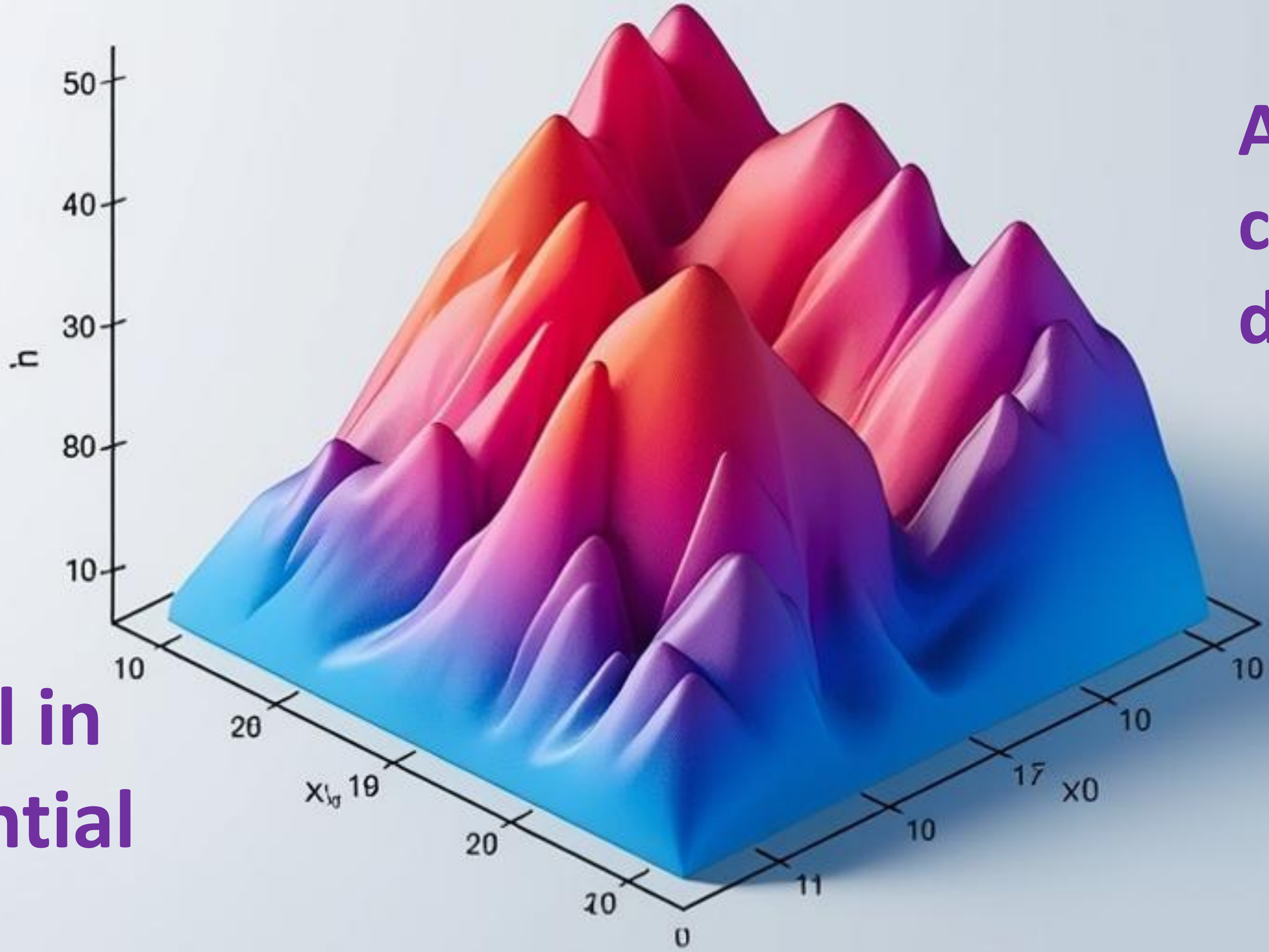
A complex integral is an integral where the integrand is a complex function, evaluated along a line or contour in the complex plane

Contour Integrals

Contour integrals are integrals taken over a specified path in the complex plane, differing from real integrals by considering complex functions' behavior and properties along these paths.

Key Distinction: All contour integrals are complex integrals, but the term "contour integral" emphasizes the path's properties and the application of powerful theorems like Cauchy's Residue Theorem

Applications of Complex Integrals



Analyzing electrical circuits, and in fluid dynamics.

In instrumental in solving differential equations

Complex Function

A complex function $f(z)$ can be written as

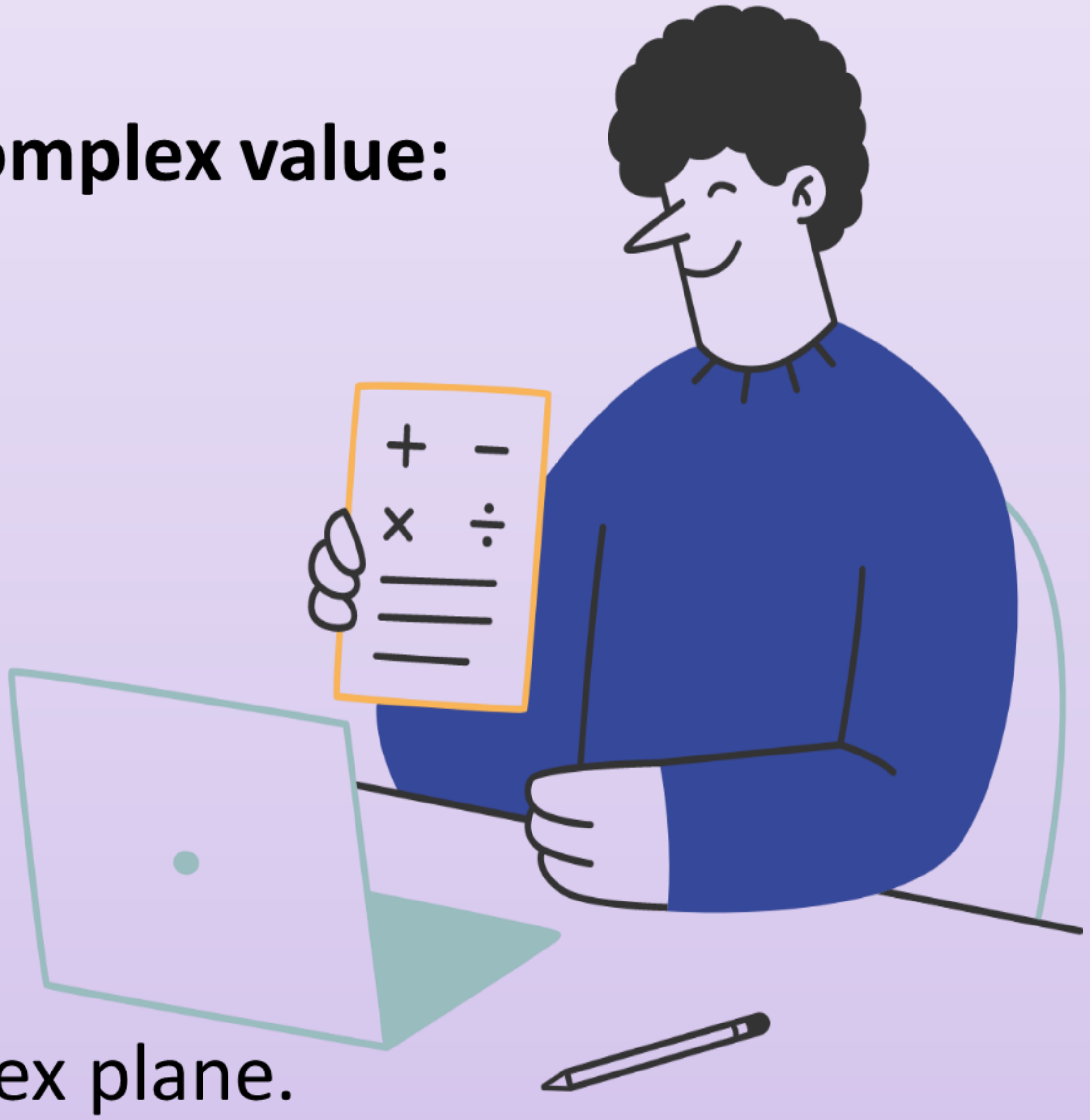
A complex function maps each complex number $z = x + iy$ to another complex value:

$$f(z) = u(x, y) + i\,v(x, y)$$

Where:

$u(x, y)$ **=real part**

$v(x, y)$ **=imaginary part**



The integral is often taken along a **contour** C, which is a curve in the complex plane.

Contour Integration: Integration along a path in the complex plane. The path can be a line, arc, or closed loop.

Types of Complex Functions

Types of Complex Functions

Polynomial Functions: f(z)=z^2+3z+5

Exponential Function: f(z)=e^z

Logarithmic Function: f(z)=log z= ln|z|+iarg(z)

Complex Trigonometric Functions: f(z)=sinz,f(z)=cosz

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Key Point

Analytic Functions

The concept of analytic functions emerged, highlighting their significance in complex analysis, leading to deeper understanding of singularities and the behavior of functions in the complex plane.

Cauchy's Integral Formula

The Cauchy's Integral Formula provided a powerful method for evaluating integrals of analytic functions, linking values of functions inside a contour to their integrals around that contour.

Cauchy's Integral Theorem

Cauchy's Integral Theorem established path independence for integrals of analytic functions over closed contours, a significant breakthrough in complex analysis, emphasizing the importance of contour choice.

Analytic Functions

Definition of an Analytic Function

- In mathematics, particularly in complex analysis:

An analytic function is a function that is complex differentiable at every point in some open set of the complex plane.

- This means it has a well-defined derivative that satisfies the Cauchy-Riemann equations in that region.

Equivalently, an analytic function can be represented by its Taylor power series expansion around any point in its domain.

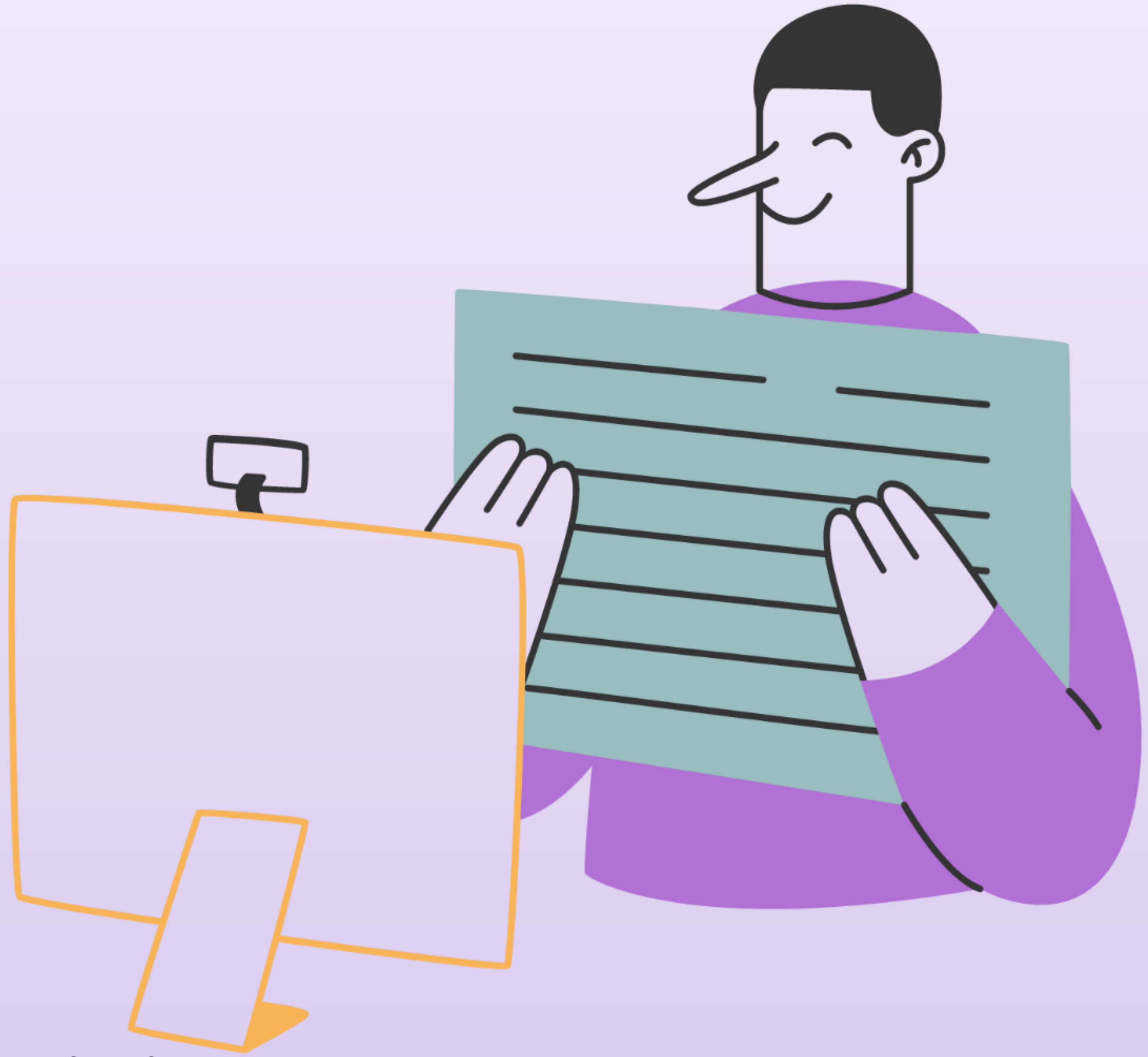
Cauchy–Riemann Conditions

A function is analytic if it satisfies:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad , \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

EX: use Cauchy–Riemann proof to verify that the complex exponential function $f(z) = e^z$

is analytic everywhere.



$f(z) = e^z$

$e^Z = e^{X+iy} = e^X \,.\, \boldsymbol{e^{iy}}$

note: $\boldsymbol{e^{iy}}$ =cos(y)+i sin(y)

$f(z) = e^X$ (cosy+isiny)

u(x,y) = $\mathbf{e^x}$ cosy
$\mathbf{v(x, y) = e^x \sin y}$

Now check:

1) First condition:

$u_x = e^x \cos y$
$v_y = e^x \cos y$

✓They match

2) Second condition:

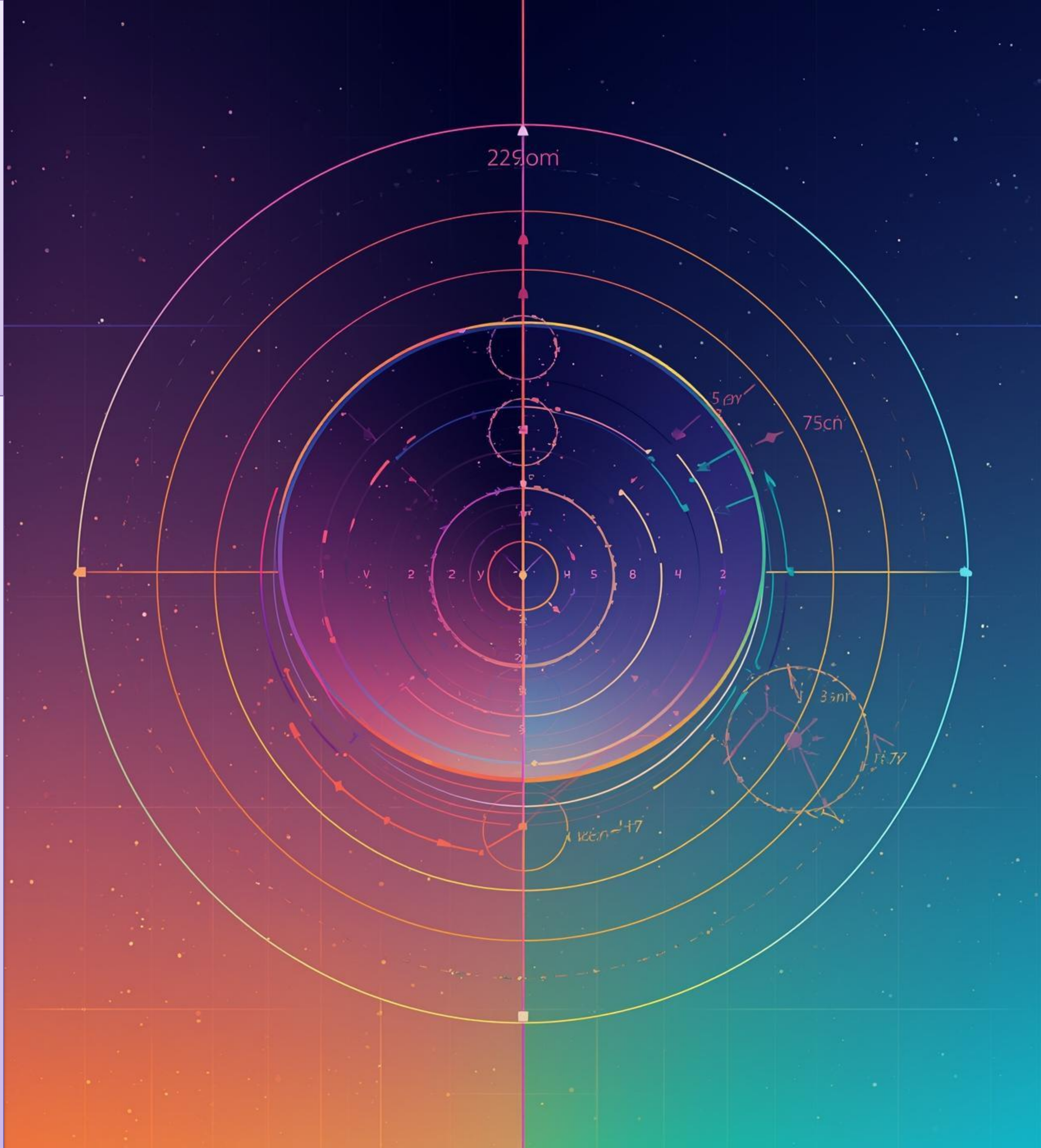
$u_y = -e^x \sin y$
$-v_x = -e^x \sin y$

✓They match

Contour Integrals Pathes

Examples of contours:

- Circle
- Rectangle
- Triangle
- Any closed curve that starts and ends at the same point.

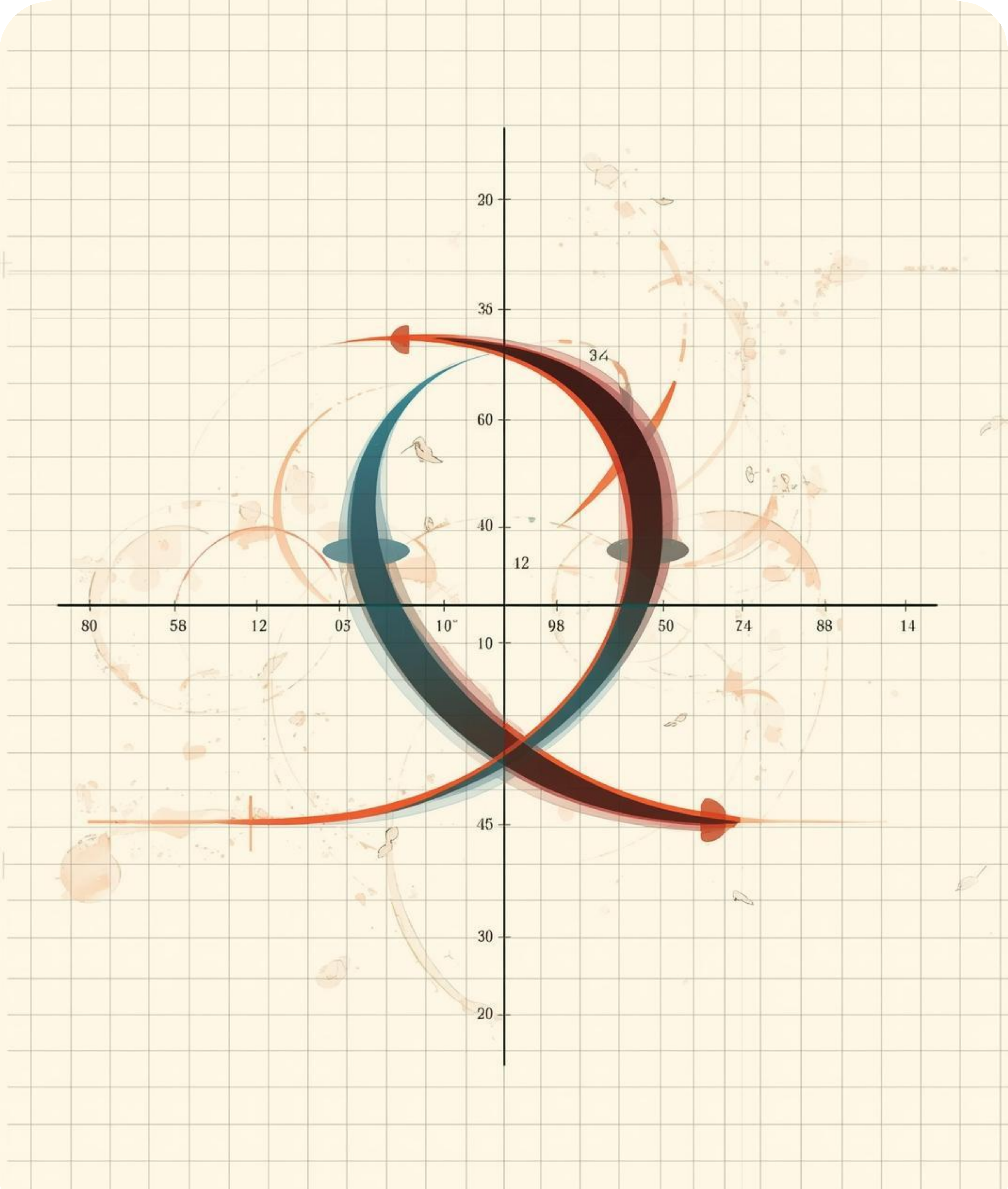


Class work Activity:

Each team will classify the following functions as **analytic** or **not analytic** why.

Functions:

1. $f(z) = z^2 + 3z + 5$
2. $f(z) = \overline{z}$



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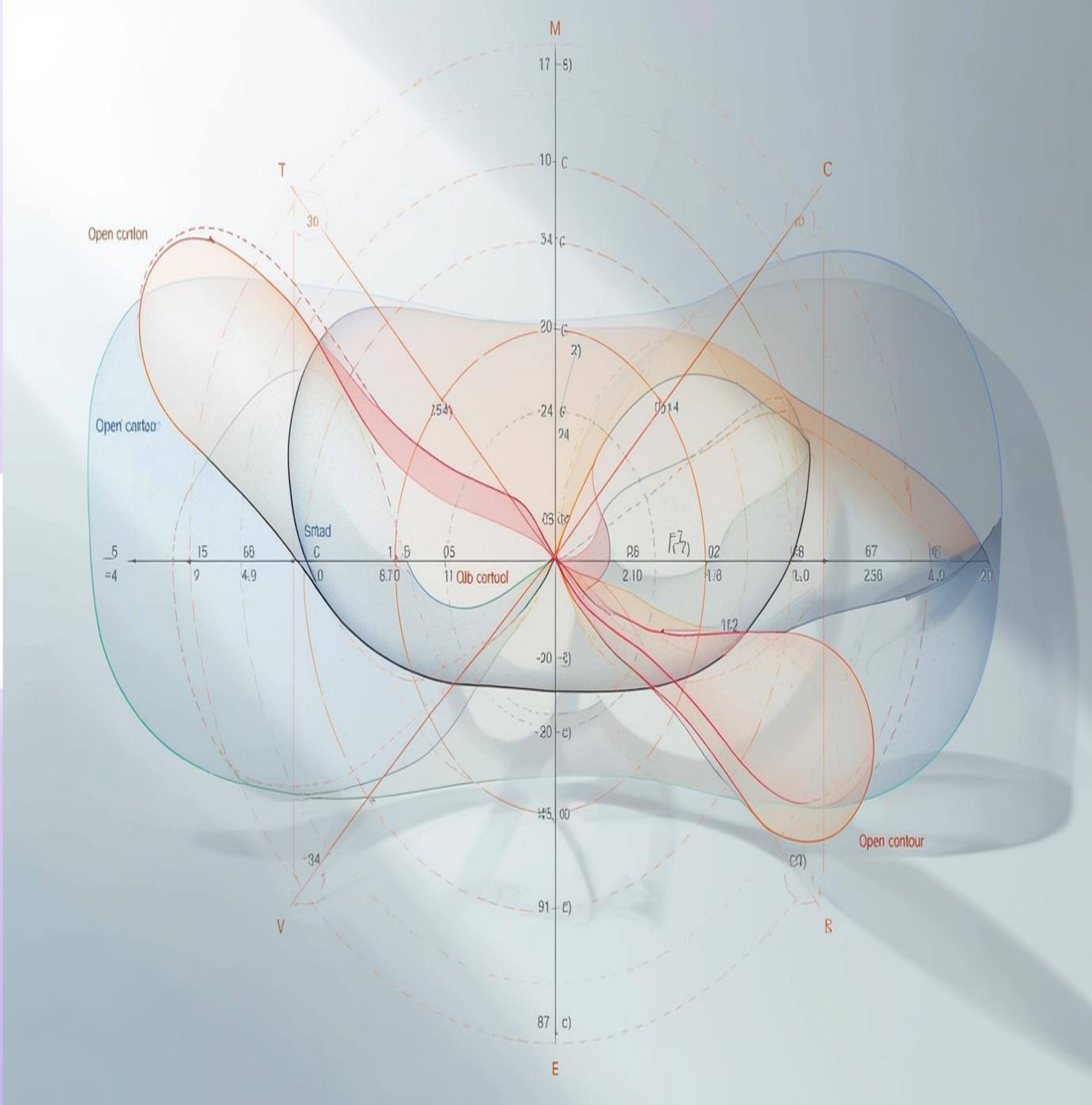
- **classify the following functions as analytic or not analytic in the complex plane?**

4. $f(z) = |z|$
5. $f(z) = \frac{1}{z}$ (analytic where?)
6. $f(z) = \sin z$

- **Taylor power series expansion?**



THANK YOU

For your Attention !