Chapter Four

Integral Transformation

1- Fourier Transforms:

The Fourier transform provides a representation of functions defined over an infinite interval and having no particular periodicity, in terms of a superposition of sinusoidal functions. It may thus be considered as a generalisation of the Fourier series representation of periodic functions. Since Fourier transforms are often used to represent time-varying functions, we shall present much of our discussion in terms of *f(t)*, rather than *f(x)*, although in some spatial examples *f(x)* will be the more natural notation.

In order to develop the transition from Fourier series to Fourier transforms, we first recall that a function of period T may be represented as a complex Fourier series,

$$f(t) = \sum_{r=-\infty}^{\infty} c_r \, exp\left(\frac{2\pi irt}{T}\right) = \sum_{r=-\infty}^{\infty} c_r \, exp(iw_r t) \; \ldots\ldots\ldots\ldots\ldots (114)$$

where $\omega_r = 2\pi r/T$. As the period T tends to infinity, the 'frequency quantum' $\Delta\omega = 2\pi/T$ becomes vanishingly small and the spectrum of allowed frequencies ω_r becomes a continuum. Thus, the infinite sum of terms in the Fourier series becomes an integral, and the coefficients c_r become functions of the continuous variable ω, as follows.

$$c_r = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \;\; exp\left(-\frac{2\pi irt}{T}\right) dt = \frac{\Delta w}{2\pi} \int_{-T/2}^{T/2} f(t) \;\; exp(-iw_r t) dt \ldots\ldots\ldots\ldots\ldots (115)$$

where we have written the integral in two alternative forms and, for convenience, made one period run from −T /2 to +T /2 rather than from 0 to T. Substituting from (115) into (114) gives

$$f(t) = \frac{\Delta w}{2\pi} \sum_{r=-\infty}^{\infty} \int_{-T/2}^{T/2} f(u) \;\; exp(-iw_r u) \, exp(iw_r t) \, du \ldots\ldots\ldots\ldots\ldots\ldots (116)$$

At this stage ω_r is still a discrete function of r equal to $2\pi r/T$. The solid points in figure 4.1 are a plot of (say, the real part of) $c_r\, exp(iw_r t)$ as a function of r (or equivalently of ω_r) and it is clear that $(2\pi/T)\, c_r\, exp(iw_r t)$ gives the area of the rth broken-line rectangle. If T tends to ∞ then $\Delta\omega$ ($= 2\pi/T$) becomes infinitesimal, the width of the rectangles tends to zero and, from the mathematical definition of an integral,

$$\sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\, g(\omega_r)\, e^{i\omega_r t} \rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega$$

In this particular case

$$\int_{-T/2}^{T/2} f(u)\ \ exp(-iw_r u)\, du$$

and (114) becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega \int_{-\infty}^{\infty} f(u)\ \ e^{-iwu}\ \ du \ldots\ldots\ldots\ldots\ldots (117)$$

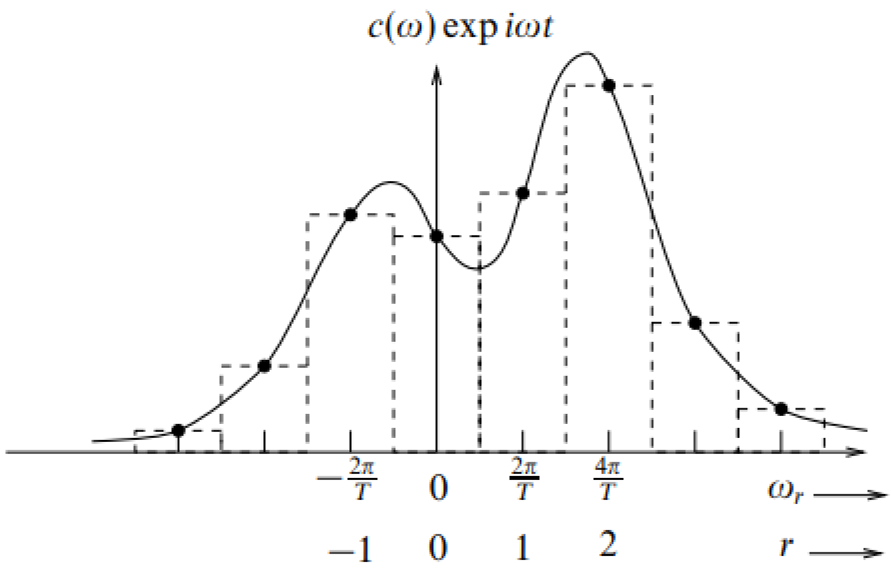

Figure 4.1 The relationship between the Fourier terms for a function of period T and the Fourier integral (the area below the solid line) of the function

This result is known as Fourier's inversion theorem. From it we may define the Fourier transform of *f(t)* by

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{-iwt}\ dt \dots\dots\dots\dots\dots (118)$$

and its inverse by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w)\ e^{iwt}\ dw \dots\dots\dots\dots\dots (119)$$

▶Find the Fourier transform of the exponential decay function $f(t)=0$ for $t < 0$ and $f(t) = A\, e^{-\lambda t}$ for $t \geq 0$ ($\lambda > 0$).

Using the definition (118) and separating the integral into two parts,

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} f(t)\ e^{-iwt}\ dt + \int_{0}^{\infty} f(t)\ e^{-iwt}\ dt \right]$$

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} (0)\ e^{-iwt}\ dt + A \int_{0}^{\infty} e^{-\lambda t}\ e^{-iwt}\ dt \right]$$

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \left[0 + A \left[-\frac{e^{-(\lambda + iw)t}}{(\lambda + iw)} \right]_0^{\infty} \right] = \frac{A}{\sqrt{2\pi}(\lambda + iw)}$$

which is the required transform. It is clear that the multiplicative constant A does not affect the form of the transform, merely its amplitude. ◀

▶Find the Fourier transform of the double sided exponential $f(t) = e^{-a|t|}$ with ($a > 0$).

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f(t)\ e^{-iwt}\ dt + \int_{0}^{\infty} f(t)\ e^{-iwt}\ dt\right]$$

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{at}\ \ e^{-iwt}\ dt + \int_{0}^{\infty} e^{-at}\ \ e^{-iwt}\ dt\right]$$

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a-iw} + \frac{1}{a+iw}\right] = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+w^2}$$

▶Find the Fourier transform of the normalised Gaussian distribution

$$f(t)\ = \frac{1}{\tau\ \sqrt{2\pi}}\ exp\,(-\frac{t^2}{2\tau^2})\ ,-\infty\ < t <\ \infty.$$

This Gaussian distribution is centered on $t = 0$ and has a root mean square deviation $\Delta t = \tau$.

Using the definition (116), the Fourier transform of $f(t)$ is given by

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{-iwt}\ dt$$

$$\tilde{f}(w) = \frac{1}{2\pi\tau} \int_{-\infty}^{\infty} e^{\,(-\frac{t^2}{2\tau^2})}\quad e^{-iwt}\ dt = \frac{1}{2\pi\tau}\int_{-\infty}^{\infty} e^{\,(-\frac{t^2}{2\tau^2}-iwt)}\ dt$$

$$\tilde{f}(w) = \frac{1}{2\pi\tau} \int_{-\infty}^{\infty} e^{\,-\frac{1}{2\tau^2}(t^2+i2\tau^2wt)}\quad dt = \frac{1}{2\pi\tau}\int_{-\infty}^{\infty} e^{\,-\frac{1}{2\tau^2}(t^2+i2\tau^2wt+(i\tau^2w)^2-(i\tau^2w)^2)}\ dt$$

$$\tilde{f}(w) = \frac{e^{-\frac{\tau^2w^2}{2}}}{2\pi\tau} \int_{-\infty}^{\infty} e^{\,-\frac{1}{2\tau^2}(t^2+i2\tau^2wt+(i\tau^2w)^2)}\ dt = \frac{e^{-\frac{\tau^2w^2}{2}}}{2\pi\tau}\int_{-\infty}^{\infty} e^{\,-\frac{(t+i\tau^2w)^2}{2\tau^2}}\ dt$$

$$\tilde{f}(w) = \frac{e^{-\frac{\tau^2w^2}{2}}}{\sqrt{2\pi}}$$

In the above example the root mean square deviation in t was τ, and so it is seen that the deviations or ‘spreads’ in t and in ω are inversely related:

$$\Delta\omega\ \Delta t = 1$$

H.W. Find the Fourier transform of the rectangular pulse:

$$f(t) = \begin{cases} 1 & for \quad -T \leq t \leq T \\ 0 & for \quad |t| > T \end{cases}$$

1.1 Odd and even functions:

If *f(t)* is odd or even then we may derive alternative forms of Fourier's inversion theorem, which lead to the definition of different transform pairs. Let us first consider an odd function *f(t) = −f(−t),* whose Fourier transform is given by:

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ \ e^{-iwt}\ \ dt$$

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ \ (\cos wt - i \sin wt)\ \ dt$$

$$\tilde{f}(w) = \frac{-2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \sin wt\ \ dt$$

where in the last line we use the fact that *f(t)* and *sin ωt* are odd, whereas *cos ωt* is even.

We note that $\tilde{f}(-\omega) = -\tilde{f}(\omega)$, i.e. $\tilde{f}(\omega)$, is an odd function of ω. Hence

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w)\ \ e^{iwt}\ \ dw = \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(w) \sin wt\ \ dw$$

$$f(t) = \frac{2}{\pi} \int_{0}^{\infty} \sin wt\ \ dw \int_{0}^{\infty} f(u) \sin wu\ \ du$$

Thus we may define the Fourier sine transform pair for odd functions:

$$\tilde{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin wt\ \ dt \ldots\ldots\ldots\ldots\ldots (120)$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_s(w) \sin wt \;\; dw \ldots\ldots\ldots\ldots\ldots (121)$$

For an even function, i.e. one for which *f(t) = f(−t),* we can define the Fourier cosine transform pair in a similar way, but with *sin ωt* replaced by *cos ωt*.

$$\tilde{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos wt \;\; dt \ldots\ldots\ldots\ldots\ldots (122)$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_c(w) \cos wt \;\; dw \ldots\ldots\ldots\ldots\ldots (123)$$

▶Find the Fourier sine transform of function $f(t) = e^{-at}$ with ($a > 0$).

$$\tilde{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin wt \;\; dt = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \sin wt \;\; dt$$

$$\tilde{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \left(\frac{e^{iwt} - e^{-iwt}}{2i}\right) \; dt = \frac{1}{2i}\sqrt{\frac{2}{\pi}} \int_0^\infty \left(e^{-(a-iw)t} - e^{-(a+iw)t}\right) \; dt$$

$$\tilde{f}_s(w) = \frac{1}{2i}\sqrt{\frac{2}{\pi}} \left[\frac{e^{-(a-iw)t}}{-(a-iw)} + \frac{e^{-(a+iw)t}}{(a+iw)}\right]_0^\infty = \frac{1}{2i}\sqrt{\frac{2}{\pi}} \left[\frac{1}{(a-iw)} - \frac{1}{(a+iw)}\right]$$

$$\therefore \; \tilde{f}_s(w) = \frac{1}{2i}\sqrt{\frac{2}{\pi}} \left[\frac{(a+iw) - (a-iw)}{(a^2+w^2)}\right] = \frac{1}{2i}\sqrt{\frac{2}{\pi}} \frac{2iw}{(a^2+w^2)} = \sqrt{\frac{2}{\pi}} \frac{w}{(a^2+w^2)}$$

2- Laplace transforms:

Often we are interested in functions *f(t)* for which the Fourier transform does not exist because $f \nrightarrow 0$ as $t \to \infty$, and so the integral defining $\tilde{f}$ does not converge. This would be the case for the function *f(t) = t*, which does not possess a Fourier transform. Furthermore, we might be interested in a given function only for *t > 0*, for example when we are given the value at *t = 0* in an initial-value problem. This leads us to consider the Laplace transform, $\bar{f}(s)$ or $\mathcal{L}[f(t)]$which is defined by

$$\bar{f}(s) = \int_0^\infty f(t)\ \ e^{-st}\ \ dt \dots\dots\dots\dots\dots (124)$$

provided that the integral exists. We assume here that s is real, but complex values would have to be considered in a more detailed study. In practice, for a given function *f(t)* there will be some real number s_0 such that the integral in (124) exists for $s > s_0$ but diverges for $s \leq s_0$.

Through (122) we define a linear transformation $\mathcal{L}$ that converts functions of the variable t to functions of a new variable s:

$$\mathcal{L}\,[af_1(t) + \ bf_2(t)] = \ a\mathcal{L}\,[f_1(t)] + \ b\mathcal{L}\,[f_2(t)] = \ a\,\overline{f_1}(s) + \ b\,\overline{f_2}(s) \dots\dots\dots\dots\dots (125)$$

▶Find the Laplace transforms of the functions $(i)f(t) = 1, (ii)f(t) = \ e^{at}, (iii)f(t) = t^n\ , for\ n\ = \ 0, 1, 2, \dots$

(i) By direct application of the definition of a Laplace transform (124), we find

$$\mathcal{L}[1] = \int_0^\infty (1)\ \ e^{-st}\ \ dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}, \quad if\ s\ > 0$$

(ii) Again using (122) directly, we find

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}\ \ e^{-st}\ \ dt = \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty = \frac{1}{s-a}, \quad if\ s\ > a$$

(iii) Once again using the definition (124) we have

$$\mathcal{L}[t^n] = \int_0^\infty t^n\ \ e^{-st}\ \ dt$$

Integrating by parts we find

$$\mathcal{L}[t^n] = \int_0^\infty t^n\ \ e^{-st}\ \ dt = \left[\frac{-t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}\ \ e^{-st}\ \ dt = 0 + \frac{n}{s}\mathcal{L}[t^{n-1}]\ if\ s\ > 0$$

If $n = 0 \ \rightarrow\ f(t) = t^0 = 1 \ \rightarrow\ \mathcal{L}[1] = \frac{1}{s}$

If $n = 1 \rightarrow f(t) = t \rightarrow \mathcal{L}[t] = \frac{1}{s^2}$

If $n = 2 \rightarrow f(t) = t^2 \rightarrow \mathcal{L}[t^2] = \frac{2!}{s^3}$

If $n = 3 \rightarrow f(t) = t^3 \rightarrow \mathcal{L}[t^3] = \frac{3!}{s^4}$

$\vdots$

If $n = n \rightarrow f(t) = t^n \rightarrow \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

▶Using table 4.1 find *f(t)* if

$$\bar{f}(s) = \frac{s+3}{s(s+1)}$$

Using partial fractions $\bar{f}(s)$ may be written

$$\bar{f}(s) = \frac{3}{s} - \frac{2}{(s+1)}$$

Comparing this with the standard Laplace transforms in table 4.1, we find that the inverse transform of 3/s is 3 for s > 0 and the inverse transform of 2/(s + 1) is $2e^{-t}$ for s > −1, and so

$$f(t) = 3 - 2e^{-t}, \qquad if\ s > 0$$

H.W. Find the Laplace transform to prove:

$$\mathcal{L}[\sin bt] = \frac{b}{(s^2 + b^2)}$$

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\lvert a \rvert$
$\cosh at$	$s/(s^2-a^2)$	$\lvert a \rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0)=\begin{cases}1 & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0\end{cases}$	e^{-st_0}/s	0

Table 4.1 Standard Laplace transforms. The transforms are valid for $s > s_0$