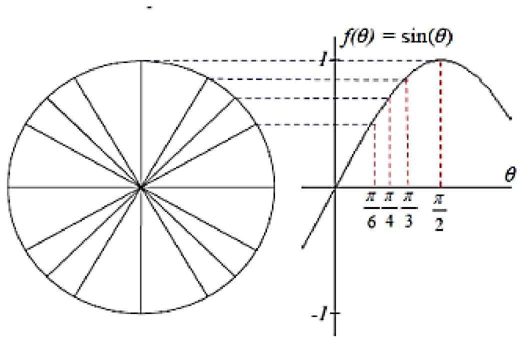
Chapter Three

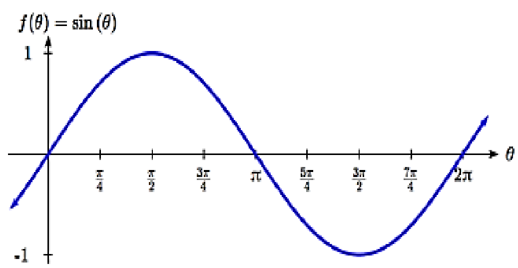
Fourier Series

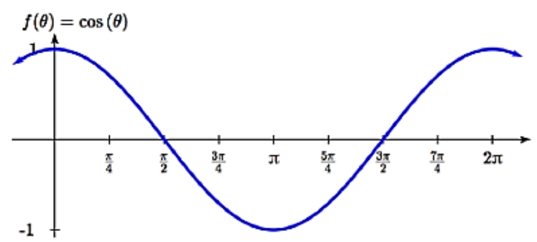
1- Periodic Function:

A periodic function is a function for which a specific horizontal shift, P, results in the original function $f(x+P)=f(x)$ for all values of x. When occurs we call the smallest such horizontal shift with $P>0$ the period of the function.

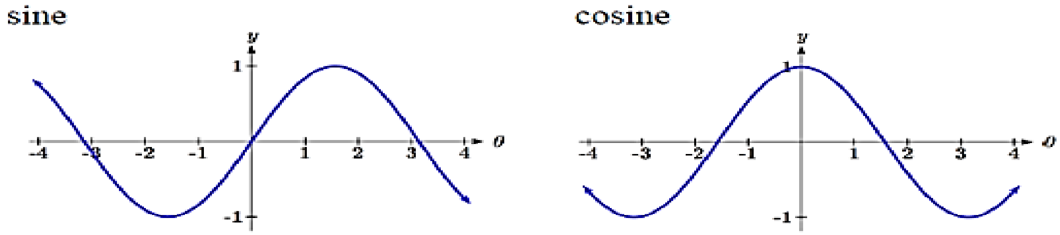


Notice how the sine values are positive between 0 and π, which correspond to the values of sine in quadrants 1 and 2 on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of sine in quadrants 3 and 4. Like the sine function, we can track the values of the cosine function through the 4 quadrants of the unit circle as we place it on a graph.





Both of these functions are defined for all real numbers, since we can evaluate the sine and cosine of any angle. By thinking of sine and cosine as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval [-1,1]. Both these graphs are called sinusoidal graphs. In both graphs, the shape of the graph begins repeating after 2π. Indeed, sine any conterminal angles will have the same sine and cosine values, we could conclude that $\sin(\theta+2\pi)=\sin(\theta)$ and $\cos(\theta+2\pi)=\cos(\theta)$. In other words, if you were to shift either graph horizontally by 2π, the resulting shape would be identical to the original function. Sinusoidal functions are a specific type of periodic function.



The sine function is symmetric about the origin, the same symmetry the cubic function has making it an odd function. The cosine function is clearly symmetric about the *y* axis, the same symmetry as the quadratic function, making it an even function.

$$The\ sine\ is\ an\ odd\ function, symmetric\ about\ the\ origin. So\ sine(-\theta) = -\sin(\theta)$$

$$The\ cosine\ is\ an\ even\ function, symmetric\ about\ the\ y - axis. So\ cos(-\theta) = \cos(\theta)$$

These identities can be used among other purposes. For helping with simplification and proving identities.

2- Even and odd functions

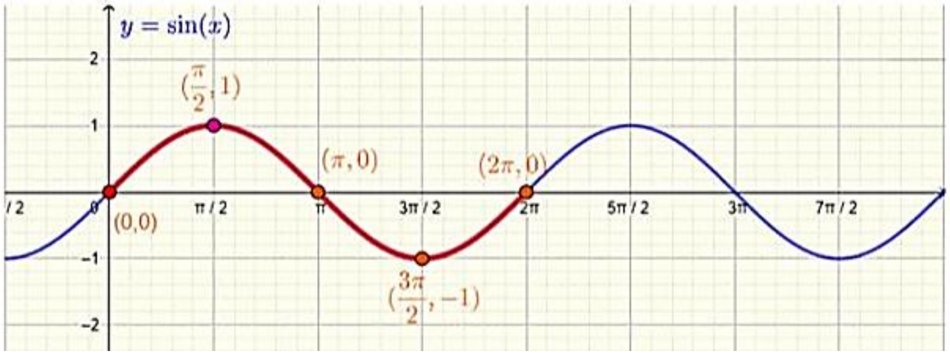
We have many an odd and an even functions, such as:

- $odd\ powers\ of\ x\ is\ odd:\ 5x^3 - 3x$
- $even\ powers\ of\ x\ is\ even:\ -x^6 + 4x^4 + x^2 - 3$
- $the\ product\ of\ two\ odd\ functions\ is\ even:\ xsinx$
- $the\ product\ of\ two\ even\ functions\ is\ even:\ x^2cosx$
- $the\ product\ of\ an\ odd\ function\ and\ an\ even\ function\ is\ odd:\ sinx\ cosx$

The all six trigonometric function are periodic:

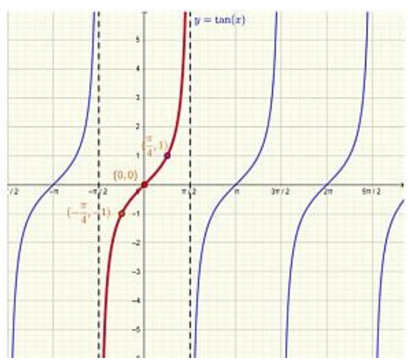
1. *sin(x+2π)=sin(x), the period sin(x) is equal to P=2π*

The graph of *sin(x)* is shown below with one cycle, in red, whose length over the *x* axis is equal to one period P given by: $P = 2\pi - 0 = 2\pi$



2. *cos(x+2π)=cos(x), the period cos(x) is equal to P=2π*
3. *sec(x+2π)=sec(x), the period sec(x) is equal to P=2π*
4. *csc(x+2π)= csc (x), the period csc (x) is equal to P=2π*
5. *tan(x+π)= tan (x), the period tan (x) is equal to P=π*
6. *cot(x+π)= cot (x), the period cot (x) is equal to P=π*

The graph of *tan(x)* is shown below with one cycle, in red, whose length over the *x axis* is equal to one period P given by: $P = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$



If P is the period of *f(x)*, then the period of $Af(bx + c) + D$ is given by $\frac{P}{|b|}$

If P is the period of *f(x),* then the period of $f(x + nP) = f(x)$, for n integer.

▶ Use the period of the trigonometric functions to find the period of each function given below:

1. $f(x) = \sin(0.5x)$
2. $g(x) = \tan(2x + \frac{\pi}{6})$
3. $h(x) = \cos(-\frac{2}{3}x - \pi)$

4. $j(x) = \sec(\pi x - 2)$
5. $k(x) = \cot(-\frac{2\pi}{3}x)$

1. The period of $\sin(x)$ is 2π. We used above formula to find the period:

$\because\ sin(x + 2\pi) = sin(x), the\ period\ sin(x)\ is\ equal\ to\ P = 2\pi$

$\therefore f(x) = \sin(0.5x) \rightarrow \dfrac{P}{|b|} = \dfrac{2\pi}{|0.5|} = 4\pi$

2. The period of $\tan(x)$ is π. We used above formula to find the period:

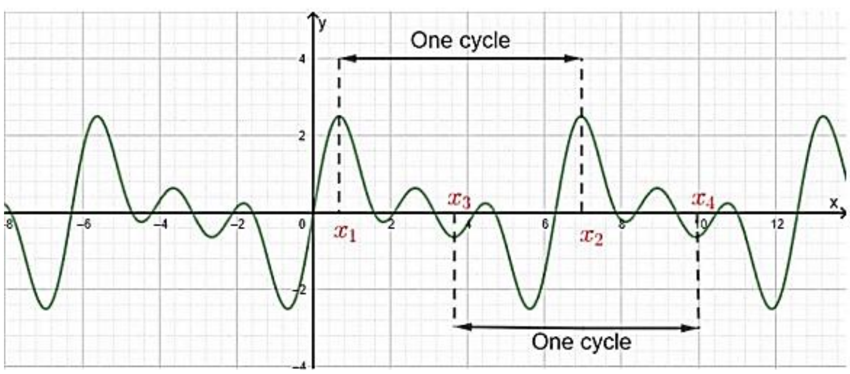
$\because tan(x + \pi) = tan\ (x), the\ period\ tan\ (x)\ is\ equal\ to\ P = \pi$

$\therefore g(x) = \tan\left(2x + \frac{\pi}{6}\right) \rightarrow \frac{P}{|b|} = \frac{\pi}{|2|} = \frac{\pi}{2}$ ◄

2.1- More on Periodic Function:

Any two points making a cycle as shown in the graph below:

$P = x_1 - x_2 = x_3 - x_4$

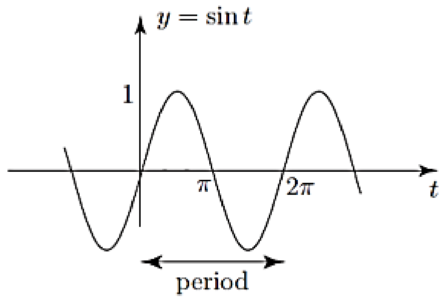


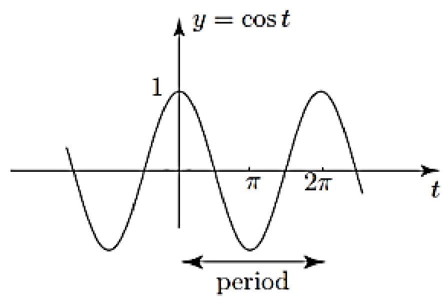
If P denotes the period for any value of t, we get

$$f(t + P) = f(t)$$

For example:

$$sin(t + 2\pi) = sin(t) \quad \text{and} \quad cos(t + 2\pi) = cos(t)$$

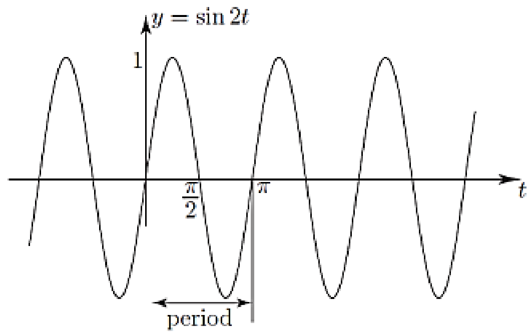




The fact that the period is π follows is a sinusoid of amplitude 1 and period $\frac{2\pi}{2} = \pi$ because

$$sin2(t + \pi) = sin\,(2t + 2\pi) = sin(2t)$$

For any value of *t*.



In general $y = A\ sin\ nt$ has amplitude A, period $\frac{2\pi}{n}$ and completes n oscillations when t changes by 2π. Formally, we define the frequency of a sinusoid as the reciprocal of the period:

$$\text{frequency} = \frac{1}{period}$$

and the angular frequency as:

$$\text{angular frequency} = 2\pi \times \text{frequency} = \frac{2\pi}{period}$$

Thus $y = A\ sin\ nt$ has frequency $\frac{n}{2\pi}$ and angular frequency n.

▶ State the amplitude, period, frequency, and angular frequency of:

a) $y = 5\cos 4t$
b) $y = 6\cos\frac{2t}{3}$

a) Amplitude 5, period $\frac{2\pi}{4} = \frac{\pi}{2}$, frequency $\frac{2}{\pi}$, and angular frequency 4

b) Amplitude 6, period 3π, frequency $\frac{1}{3\pi}$, and angular frequency $\frac{2}{3}$ ◀

2.2 Non-Sinusoidal period functions:

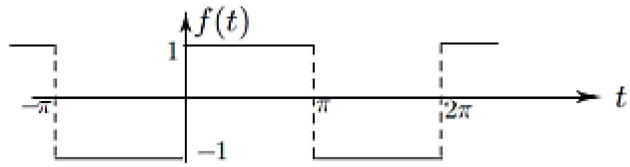
The following are examples of non- sinusoidal functions and they are often called waves.

1. Square Wave

Analytically we can describe this function as follows:

$$f(t) = \begin{Bmatrix} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{Bmatrix}$$

$f(t + 2\pi) = f(t)$ Which tell us that the function has period 2π.



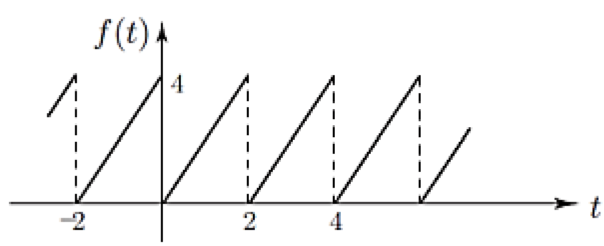
2. Saw-tooth Wave

In this case, we can describe the function as follows:

$$f(t) = 2t \qquad 0 < t < 2$$

$f(t + 2) = f(t)$ Which tell us that the function has period 2.

The frequency is $\frac{1}{2}$, and the angular frequency is π.



3. Triangular Wave

Here we can conveniently define the function using $-\pi < t < \pi$ as the basic period.

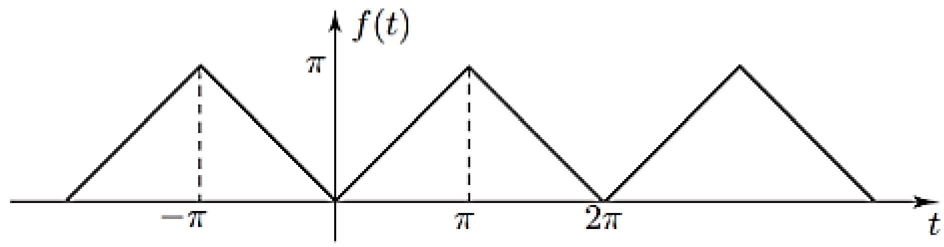
$$f(t) = \begin{Bmatrix} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{Bmatrix}$$

or, more concisely,

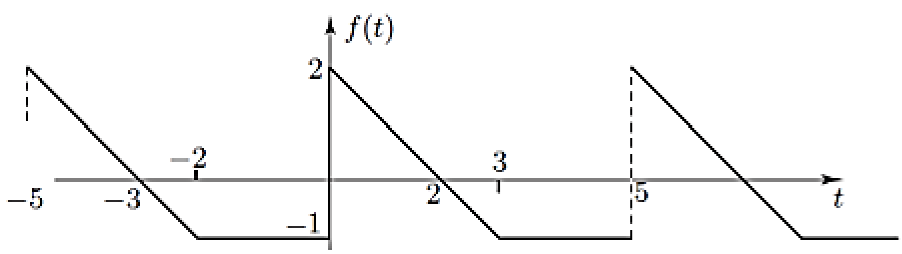
$$f(t) = |t| \qquad -\pi < t < \pi$$

Together with the usual statement on periodicity

$$f(t + 2\pi) = f(t)$$



▶ Write an analytic definition for the following periodic function:



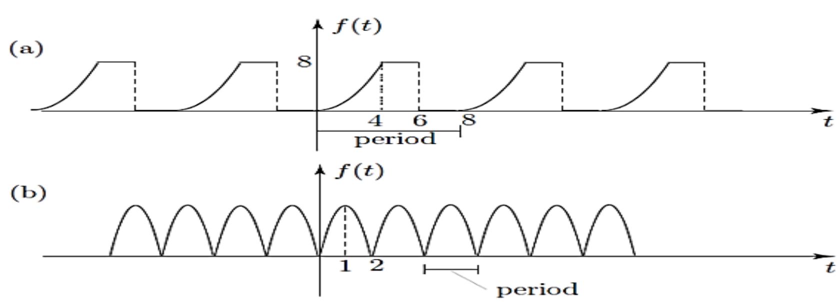
$$f(t) = \begin{Bmatrix} 2 - t & 0 < t < 3 \\ -1 & 3 < t < 5 \end{Bmatrix}$$

$f(t + 5) = f(t)$ ◀

▶ Sketch the graphs of the following periodic functions showing all relevant values:

(a) $f(t) = \begin{Bmatrix} t^2/2 & 0 < t < 4 \\ 8 & 4 < t < 6 \\ 0 & 6 < t < 8 \end{Bmatrix}$ *and* $f(t + 8) = f(t)$

(b) $f(t) = 2t - t^2 \qquad 0 < t < 2 \qquad f(t + 2) = f(t)$



3- Fourier Series:

We have already mentioned that Fourier series may be used to represent some functions for which a Taylor series expansion is not possible. The particular conditions that a function f(x) must fulfil in order that it may be expanded as a Fourier series are known as the Dirichlet conditions, and may be summarised by the following four points:

(i) the function must be periodic;

(ii) it must be single-valued and continuous, except possibly at a finite number of finite discontinuities;

(iii) it must have only a finite number of maxima and minima within one period;

(iv) the integral over one period of |f(x)| must converge.

If the above conditions are satisfied then the Fourier series converges to f(x) at all points where f(x) is continuous. The last three Dirichlet conditions are almost always met in real applications, but not all functions are periodic and hence do not fulfil the first condition. An example of a function that may, without modification, be represented as a Fourier series is shown in figure 3.1.

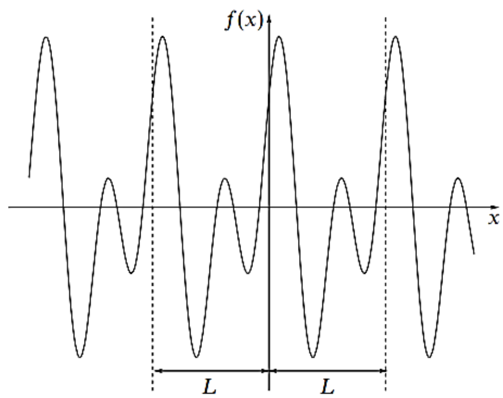


Figure 3.1: An example of a function that may be represented as a Fourier series without modification.

It is possible to represent all odd functions by a sine series and all even functions by a cosine series. Now, since all functions may be written as the sum of an odd and an even part,

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = feven(x) + fodd(x),$$

we can write any function as the sum of a sine series and a cosine series.

All the terms of a Fourier series are mutually orthogonal, i.e. the integrals, over one period, of the product of any two terms have the following properties:

$$\int_{x_0}^{x_0+L} sin\left(\frac{2\pi rx}{L}\right)\ cos\left(\frac{2\pi px}{L}\right)\ dx\ =\ 0 \qquad for\ all\ r\ and\ p\ \dots\dots\dots\dots\dots(102)$$

$$\int_{x_0}^{x_0+L} cos\left(\frac{2\pi rx}{L}\right)\ cos\left(\frac{2\pi px}{L}\right)\ dx\ =\begin{cases} L & for\ r=p=0 \\ \frac{1}{2}L & for\ r=p>0 \\ 0 & for\ r\neq p \end{cases} \dots\dots\dots\dots\dots(103)$$

$$\int_{x_0}^{x_0+L} sin\left(\frac{2\pi rx}{L}\right)\ sin\left(\frac{2\pi px}{L}\right)\ dx\ =\begin{cases} 0 & for\ r=p=0 \\ \frac{1}{2}L & for\ r=p>0 \\ 0 & for\ r\neq p \end{cases} \dots\dots\dots\dots\dots(104)$$

where r and p are integers greater than or equal to zero; these formulae are easily derived.

The Fourier series expansion of the function *f(x)* is conventionally written:

$$f(x)=\frac{a_0}{2}+\sum_{r=1}^{\infty}\left[a_r\ cos\left(\frac{2\pi rx}{L}\right)\ +\ b_r\ sin\left(\frac{2\pi rx}{L}\right)\right]\dots\dots\dots\dots\dots(105)$$

where a_0 , a_r, b_r are constants called the Fourier coefficients.

3.1- The Fourier coefficients:

We have indicated that a series that satisfies the Dirichlet conditions may be written in the form (105). We now consider how to find the Fourier coefficients for any particular function. For a periodic function *f(x)* of period *L* we will find that the Fourier coefficients are given by:

$$a_r=\frac{2}{L}\int_{x_0}^{x_0+L} f(x)\ cos\left(\frac{2\pi rx}{L}\right)\ dx\ \dots\dots\dots\dots\dots(106)$$

$$b_r=\frac{2}{L}\int_{x_0}^{x_0+L} f(x)\ sin\left(\frac{2\pi rx}{L}\right)\ dx\ \dots\dots\dots\dots\dots(107)$$

where x_0 is arbitrary but is often taken as 0 or −L/2. The apparently arbitrary factor $\frac{1}{2}$ which appears in the a_0 term in (104) is included so that (105) may apply for r = 0 as well as r > 0. The relations (105) and (106) may be derived as follows.

Suppose the Fourier series expansion of *f(x)* can be written as in (105)

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi r x}{L}\right) + b_r \sin\left(\frac{2\pi r x}{L}\right) \right]$$

Then, multiplying by *cos(2πpx/L)*, integrating over one full period in *x* and changing the order of the summation and integration, we get

$$\begin{aligned} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi p x}{L}\right) dx \\ &= \frac{a_0}{2} \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi p x}{L}\right) dx \\ &+ \sum_{r=1}^{\infty} \left[a_r \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx \right. \\ &\left. + \; b_r \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi r x}{L}\right) \sin\left(\frac{2\pi p x}{L}\right) dx \right] \dots\dots\dots\dots\dots (108) \end{aligned}$$

We can now find the Fourier coefficients by considering (107) as p takes different values. Using the orthogonality conditions (102) – (104) of the previous section, we find that when $p = 0$ (108) becomes

$$\int_{x_0}^{x_0+L} f(x) \; dx = \frac{a_0}{2} L$$

When $p \neq 0$ the only non-vanishing term on the RHS of (107) occurs when r = p, and so

$$\int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi r x}{L}\right) \ dx = \frac{a_r}{2} L$$

The other Fourier coefficients b_r may be found by repeating the above process but multiplying by *sin(2πpx/L)* instead of *cos(2πpx/L)*

$$\int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi r x}{L}\right) \ dx = \frac{b_r}{2} L$$

▶Express the square-wave function illustrated in figure 3.2 as a Fourier series.

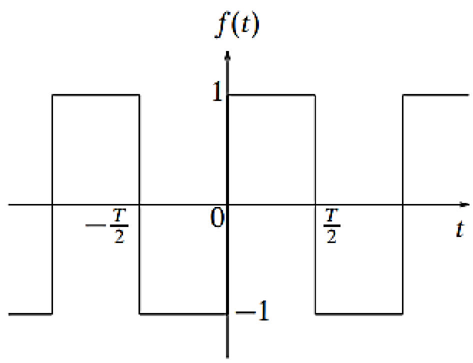


Figure 3.2: A square-wave function.

The square wave may be represented by

$$f(t) = \begin{cases} -1 & for \quad -\frac{1}{2}T < t < 0 \\ 1 & for \quad 0 < t < \frac{1}{2}T \end{cases}$$

In deriving the Fourier coefficients, we note firstly that the function is an odd function and so the series will contain only sine terms (this simplification is discussed further in the following section). To evaluate the coefficients in the sine series we use (106). Hence

$$b_r = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin\left(\frac{2\pi r t}{T}\right) \ dt$$

$$b_r = \frac{4}{T} \int_{0}^{T/2} \sin\left(\frac{2\pi r t}{T}\right) \ dt$$

$$b_r = \frac{2}{\pi r}[1-(-1)^r]$$

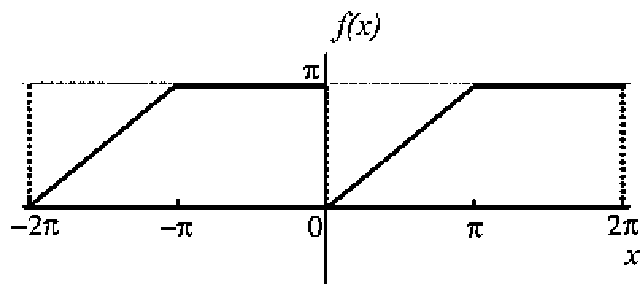
Thus the sine coefficients are zero if r is even and equal to 4/(πr) if r is odd. Hence the Fourier series for the square-wave function may be written as

$$f(t) = \frac{4}{\pi}\left(\sin wt + \frac{\sin 3wt}{3} + \frac{\sin 5wt}{5} + \cdots\right) \ldots\ldots\ldots\ldots\ldots (109)$$

where $\omega = 2\pi/T$ is called the angular frequency. ◀

▶Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$ and Fourier series representation of $f(x)$.

$$f(x) = \begin{Bmatrix} x & for & 0 < x < \pi \\ \pi & for & \pi < x < 2\pi \end{Bmatrix} \quad and\ has\ period\ 2\pi$$



$$a_0 = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\ \ dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x\ dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi.dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + [x]_{\pi}^{2\pi} = \frac{3\pi}{2}$$

$$a_n = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\ \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,.\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\, dx\right] + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi} = \frac{1}{n^2\pi}(\cos n\pi - 1)$$

$$a_n = \frac{1}{n^2\pi}((-1)^n - 1) = \begin{Bmatrix} -\dfrac{2}{n^2\pi} & n\ odd \\ 0 & n\ even \end{Bmatrix}$$

$$b_n = \frac{2}{L}\int_{x_0}^{x_0+L} f(x) \; \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi . \sin nx \, dx$$

$$b_n = \frac{1}{\pi}\left[\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right] - \frac{\pi}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi}^{2\pi} = -\frac{1}{n}(-1)^n - \frac{1}{n}(1-(-1)^n)$$

$$b_n = -\frac{1}{n}$$

We now have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \, cos \, nx + b_n \, sin \, nx\right]$$

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5,\ldots.}^{\infty}\frac{1}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

H.W. Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$ and show that the Fourier series in the interval $-\pi < x < \pi$.

$$f(t) = \begin{Bmatrix} 0 & for & -\pi < x < 0 \\ x & for & 0 < x < \pi \end{Bmatrix} \quad and \; has \; period \; 2\pi$$

4- Complex Fourier series:

As a Fourier series expansion in general contains both sine and cosine parts, it may be written more compactly using a complex exponential expansion. This simplification makes use of the property that $exp(irx) = \cos rx + i \sin rx$. The complex Fourier series expansion is written

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \, exp\left(\frac{2\pi irx}{L}\right) \quad \ldots \ldots \ldots \ldots \ldots (110)$$

where the Fourier coefficients are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\ \ exp\left(-\frac{2\pi irx}{L}\right)dx \ldots\ldots\ldots\ldots\ldots (111)$$

This relation can be derived, in a similar manner to that of section 3.1, by multiplying (108) by *exp(−2πipx/L)* before integrating and using the orthogonality relation:

$$\int_{x_0}^{x_0+L} exp\left(-\frac{2\pi ipx}{L}\right)\ exp\left(\frac{2\pi irx}{L}\right)dx = \begin{cases} L & for \quad r = p \\ 0 & for \quad r \neq p \end{cases} \ldots\ldots\ldots\ldots\ldots (112)$$

The complex Fourier coefficients in (108) have the following relations to the real Fourier coefficients:

$$c_r = \frac{1}{2}(a_r - i\ b_r)$$

$$\ldots\ldots\ldots\ldots\ldots(113)$$

$$c_{-r} = \frac{1}{2}(a_r + i\ b_r)$$

Note that if $f(x)$ is real then $c_{-r} = c_r^*$, where the asterisk represents complex conjugation.

▶Find a complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$.

Using (109), for $r \neq 0$,

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\ \ exp\left(-\frac{2\pi irx}{L}\right)dx = \frac{1}{4}\int_{-2}^{2} x\ \ exp\left(-\frac{\pi irx}{2}\right)dx$$

$$c_r = \left[-\frac{x}{2\pi ir}\ exp\left(-\frac{\pi irx}{2}\right)\right]_{-2}^{2} + \frac{1}{2\pi ir}\int_{-2}^{2} exp\left(-\frac{\pi irx}{2}\right)dx$$

$$c_r = -\frac{1}{\pi ir}\ [exp(-\pi ir) + exp(\pi ir)] + \left[-\frac{x}{\pi^2 r^2}\ exp\left(-\frac{\pi irx}{2}\right)\right]_{-2}^{2}$$

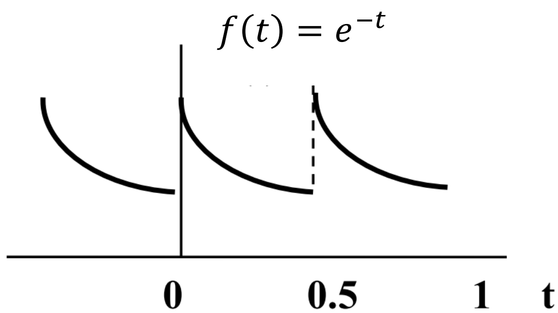
$$c_r = \frac{2i}{\pi r}\cos\pi r + \frac{2i}{\pi^2 r^2}\sin\pi r = \frac{2i}{\pi r}(-1)^r$$

For $r = 0$, we find $c_0 = 0$ and hence

$$f(x) = \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{2i}{\pi r}(-1)^r \, exp\left(\frac{\pi i r x}{2}\right)$$

We note that the Fourier series derived for *x* gives $a_r = 0$ for all *r* and $b_r = \frac{-4(-1)^r}{\pi r}$, and so, using (111), we confirm that c_r and c_{-r} have the forms derived above. It is also apparent that the relationship $c_{-r} = c_r^*$, holds, as we expect since $f(x)$ is real. ◀

▶Find the exponential Fourier series and corresponding frequency spectra for the function *f(t)* shown.



Using (109), for $r \neq 0$,

$$c_n = \frac{1}{L}\int_{x_0}^{x_0+L} f(t) \;\; exp\left(-\frac{2\pi i n t}{L}\right) dt = 2\int_0^{0.5} \mathrm{e}^{-t} \;\; exp(-4n\pi t i)\, dt$$

$$c_n = 2\int_0^{0.5} \mathrm{e}^{-(1+4n\pi i)t} \; dt = \; 2\left[\frac{\mathrm{e}^{-(1+4n\pi i)t}}{-(1+4n\pi i)}\right]_0^{0.5}$$

$$c_n = \frac{-2}{(1+4n\pi i)}\left\{\mathrm{e}^{-\frac{(1+4n\pi i)}{2}} - 1\right\} = \frac{2}{(1+4n\pi i)}\left\{1 - \mathrm{e}^{-\frac{1}{2}}\mathrm{e}^{-\,2n\pi i}\right\}$$

Since $\mathrm{e}^{-\,2n\pi i} = 1 \quad and \quad \mathrm{e}^{-\frac{1}{2}} = 0.607$

$$c_n = \frac{2(1-0.607)}{(1+4n\pi i)} \approx \frac{0.79}{(1+4n\pi i)}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{0.79}{(1+4n\pi i)} \;\; e^{4n\pi t i}$$