

Chapter Two

Complex Numbers

1- Complex Numbers:

Complex numbers of this chapter, underscoring their wide application in the mathematics of the physical sciences. The application of complex numbers to the description of physical systems is left the basic tools are use it.

Although complex numbers occur in many branches of mathematics, they arise most directly out of solving polynomial equations. We examine a specific quadratic equation as an example.

Consider the quadratic equation

$$z^2 - 4z + 5 = 0 \dots\dots\dots (29)$$

Equation (29) has two solutions, z_1 and z_2 , such that

$$(z - z_1)(z - z_2) = 0 \dots\dots\dots (30)$$

Using the familiar formula for the roots of a quadratic equation, the solutions z_1 and z_2 , written in brief as $z_{1,2}$, are

$$z_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2} = 2 \pm \frac{\sqrt{-4}}{2} \dots\dots\dots (31)$$

Both solutions contain the square root of a negative number. However, it is not true to say that there are no solutions to the quadratic equation. The fundamental theorem of algebra states that a quadratic equation will always have two solutions and these are in fact given by (31). The second term on the RHS of (31) is called an imaginary term since it contains the square root of a negative number; the first term is called a real term. The full solution is the sum of a real term and an imaginary term and is called a complex number. A plot of the function $f(z) = z^2 - 4z + 5$ is shown in figure (1). It will be seen that the plot does not intersect the z-axis, corresponding to the fact that the equation $f(z)=0$ has no purely real solutions.

In our particular example, $\sqrt{-4} = 2\sqrt{-1} = 2i$, and hence the two solutions of (31) are

$$z_{1,2} = 2 \pm i = x \pm iy \dots\dots\dots (32)$$

For compactness a complex number is sometimes written in the form

$$z_{1,2} = (x, y) = (2, \pm 1) \dots\dots\dots (33)$$

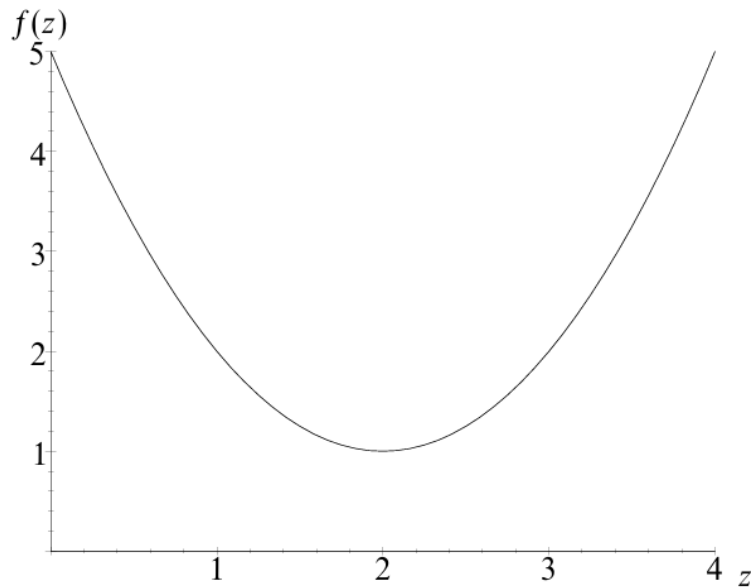


Figure (1): The function $f(z) = z^2 - 4z + 5$

Where the components of z may be thought of as coordinates in an xy -plot. Such a plot is called an Argand diagram and is a common representation of complex numbers; an example is shown in figure (2).

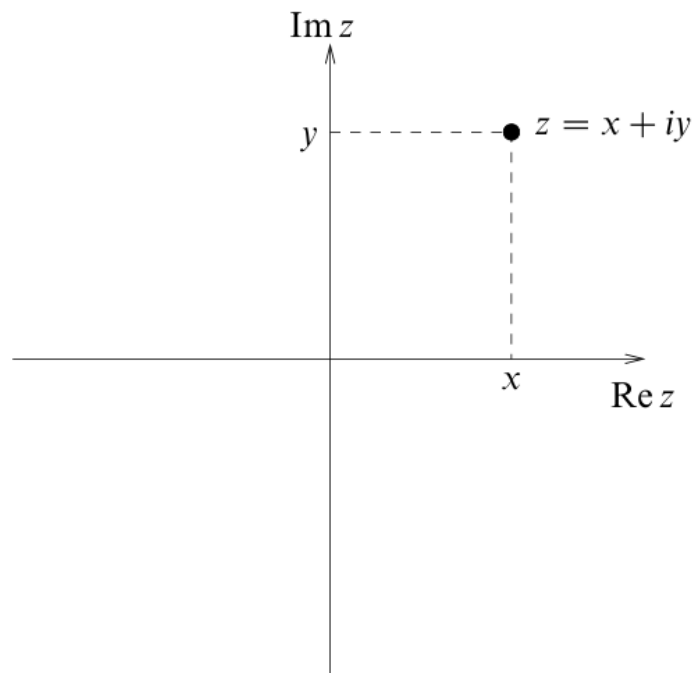


Figure (2): The Argand diagram.

1-1 Addition and subtraction:

The addition of two complex numbers, z_1 and z_2 , in general gives another complex number.

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \dots \dots \dots (34)$$

or in component notation

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \dots \dots \dots (35)$$

The Argand representation of the addition of two complex numbers is shown in figure (3).

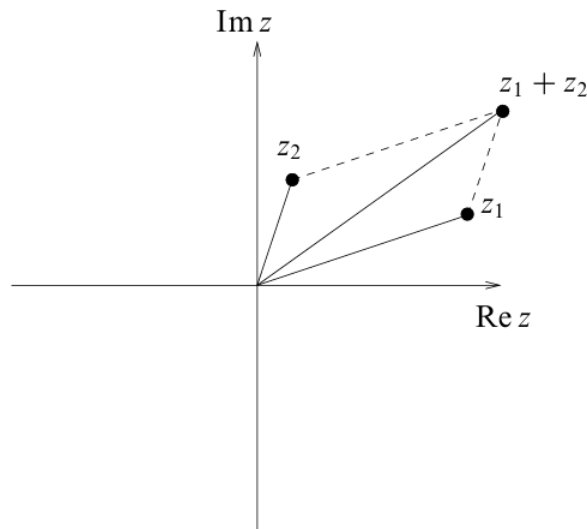


Figure (3): The addition of two complex numbers.

The subtraction of complex numbers is very similar to their addition. As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

► Sum the complex numbers $1 + 2i, 3 - 4i, -2 + i$.

Summing the real terms we obtain

$$1 + 3 - 2 = 2,$$

and summing the imaginary terms we obtain

$$2i - 4i + i = -i.$$

Hence

$$(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i. \blacktriangleleft$$

1-2 Multiplication:

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers z_1 and z_2 is found by multiplying them out in full and remembering that $i^2 = -1$,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \dots \dots \dots (36)$$

► Multiply the complex numbers $z_1 = 3+2i$ and $z_2 = -1-4i$.

By direct multiplication we find

$$z_1 z_2 = (3+2i)(-1-4i) = -3-2i-12i-8i^2 = 5-14i. \blacktriangleleft$$

The multiplication of complex numbers is both commutative and associative, i.e.

$$z_1 z_2 = z_2 z_1 \dots \dots \dots (37)$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3) \dots \dots \dots (38)$$

The product of two complex numbers also has the simple properties:

$$|z_1 z_2| = |z_1| |z_2| \dots \dots \dots (39)$$

► Verify that holds for the product of $z_1 = 3+2i$ and $z_2 = -1-4i$.

$$|z_1 z_2| = |5 - 14i| = \sqrt{5^2 + 14^2} = \sqrt{221}$$

We also find and hence

$$|z_1| = |3 - 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$|z_2| = |-1 - 4i| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$$

$$|z_1| |z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} \blacktriangleleft$$

1-3 Complex conjugate

If z has the convenient form $x + iy$ then the complex conjugate, denoted by z^* , may be found simply by changing the sign of the imaginary part, i.e. if $z = x+iy$ then $z^* = x-iy$. More generally, we may define the complex conjugate of z as the (complex) number having the same magnitude as z that when multiplied by z leaves a real result, i.e. there is no imaginary component in the product.

In the case where z can be written in the form $x + iy$ it is easily verified, by direct multiplication of the components, that the product zz^* gives a real result:

$$zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 = |z|^2 \dots \dots \dots (40)$$

Complex conjugation corresponds to a reflection of z in the real axis of the Argand diagram, as may be seen in figure (4).

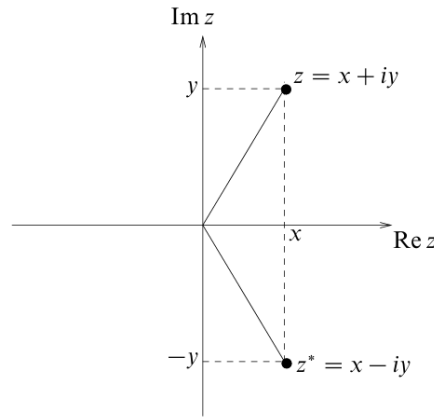


Figure (4) The complex conjugate as a mirror image in the real axis.

► Find the complex conjugate of $z = a + 2i + 3ib$.

The complex number is written in the standard form

$$z = a + i(2 + 3b);$$

then, replacing i by $-i$, we obtain

$$z^* = a - i(2 + 3b). \blacktriangleleft$$

Nevertheless, given two complex numbers, z_1 and z_2 , it is straightforward to show that the complex conjugate of their sum (or difference) is equal to the sum (or difference) of their complex conjugates, i.e. $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$. Similarly, it may be shown that the complex conjugate of the product (or quotient) of z_1 and z_2 is equal to the product (or quotient) of their complex conjugates, i.e. $(z_1 z_2)^* = z_1^* z_2^*$ and $(z_1 / z_2)^* = z_1^* / z_2^*$.

► Find the complex conjugate of the complex number $z = w^{(3y+2ix)}$, where $w = x + 5i$.

In this case w itself contains real and imaginary components and so must be written out in full, i.e.

$$z = w^{3y+2ix} = (x + 5i)^{3y+2ix}.$$

Now we can replace each i by $-i$ to obtain

$$z^* = (x - 5i)^{(3y-2ix)}.$$

It can be shown that the product zz^* is real, as required. $\blacktriangleleft$

The following properties of the complex conjugate are easily proved and others may be derived from them. If $z = x + iy$ then

$$(z^*)^* = z \dots \dots \dots (41)$$

$$z + z^* = 2\operatorname{Re}z = 2x \dots \dots \dots (42)$$

$$z - z^* = 2i \operatorname{Im}z = 2iy \dots \dots \dots (43)$$

$$\frac{z}{z^*} = \left(\frac{x^2 - y^2}{x^2 + y^2} \right) + i \left(\frac{2xy}{x^2 + y^2} \right) \dots \dots \dots (44)$$

1-4 Division

The division of two complex numbers z_1 and z_2 bears some similarity to their multiplication. Writing the quotient in component form we obtain

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \dots \dots \dots (45)$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator. By definition, this process will leave the denominator as a real quantity. Equation (45) gives

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i \frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Hence we have separated the quotient into real and imaginary components, as required.

In the special case where $z_2 = z_1^*$, so that $x_2 = x_1$ and $y_2 = -y_1$, the general result reduces to (44).

► Express z in the form $x + iy$, when $z = \frac{3-2i}{-1+4i}$.

Multiplying numerator and denominator by the complex conjugate of the denominator we obtain

$$\begin{aligned} z &= \frac{(3-2i)(-1-4i)}{(-1+4i)(-1-4i)} \\ &= \frac{-11-10i}{17} \\ &= \frac{-11}{17} - \frac{10}{17}i. \blacktriangleleft \end{aligned}$$

In analogy to (39), which describe the multiplication of two complex numbers, the following relations apply to division:

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \dots \dots \dots (46)$$

1-5 Modulus and argument:

The modulus of the complex number z is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2} \dots \dots \dots (47)$$

Hence the modulus of the complex number is the distance of the corresponding point from the origin in the Argand diagram, as may be seen in figure (5). The argument of the complex number z is denoted by $\arg z$ and is defined as:

$$\arg z = \tan^{-1} \frac{y}{x} \dots \dots \dots (48)$$

It can be seen that $\arg z$ is the angle that the line joining the origin to z on the Argand diagram makes with the positive x -axis.

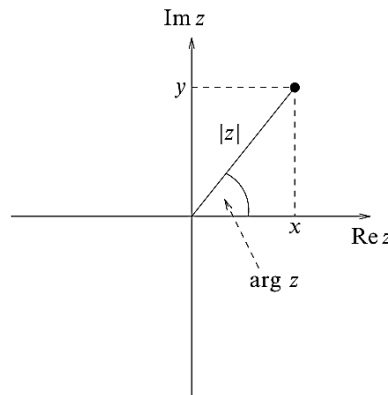


Figure (5): The modulus and argument of a complex number.

The anticlockwise direction is taken to be positive by convention. The angle $\arg z$ is shown in figure (5). Account must be taken of the signs of x and y individually in determining in which quadrant $\arg z$ lies. Thus, for example, if x and y are both negative then $\arg z$ lies in the range $-\pi < \arg z < -\frac{\pi}{2}$ rather than in the first quadrant ($0 < \arg z < \frac{\pi}{2}$), though both cases give the same value for the ratio of y to x .

► Find the modulus and the argument of the complex number $z = 2 - 3i$.

The modulus is given by

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

The argument is given by

$$\arg z = \tan^{-1} \frac{-3}{2}$$

The two angles whose tangents equal -1.5 are -0.9828 rad and 2.1588 rad. Since $x=2$ and $y=-3$, z clearly lies in the fourth quadrant; therefore $\arg z = -0.9828$ is the appropriate answer.



2- Polar representation of complex numbers

Although considering a complex number as the sum of a real and an imaginary part is often useful, sometimes the polar representation proves easier to manipulate. This makes use of the complex exponential function, which is defined by

$$e^z = \exp z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \quad \dots \dots \dots (49)$$

We also note that, using (49), by multiplying together the appropriate series we may show that

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} \quad \dots \dots \dots (50)$$

From (49), it immediately follows that for $z = i\theta$, θ real,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \quad \dots \dots \dots (51)$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \quad \dots \dots \dots (52)$$

and hence that

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \dots \dots \dots (53)$$

This last relationship is called Euler's equation. It also follows from (53) that

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

For all n . From Euler's equation (53) and figure (6) we deduce that

$$\begin{aligned} re^{i\theta} &= r(\cos \theta + i \sin \theta) \\ &= x + iy. \end{aligned}$$

Thus a complex number may be represented in the polar form

$$z = re^{i\theta} \quad \dots \dots \dots (54)$$

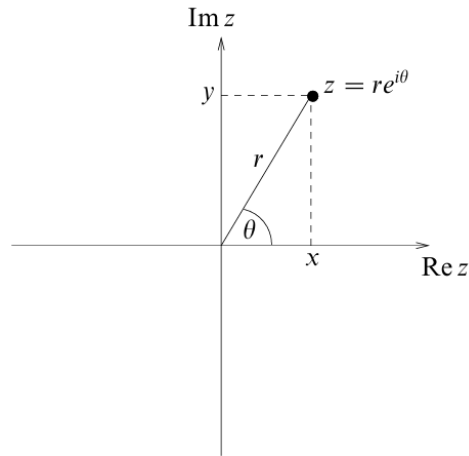


Figure (6): The polar representation of a complex number.

Referring again to figure (6), we can identify r with $|z|$ and θ with $\arg z$. The simplicity of the representation of the modulus and argument is one of the main reasons for using the polar representation.

► Write $z = -1 - i$ in the polar form.

Now $x = -1, y = -1$

The modulus is:

$$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

The argument is:

$$\theta = \arg z = \tan^{-1} \frac{-1}{-1} = \frac{5\pi}{4} \text{ or } 225^\circ$$

Its principle value is $\theta = -\frac{3\pi}{4}$

Hence $z = \left(\sqrt{2}, -\frac{3\pi}{4}\right)$ in the polar form. ◀

2-1 Multiplication and division in polar form:

Multiplication and division in polar form are particularly simple. The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ is given by

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \dots \dots \dots (55) \end{aligned}$$

The relations $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ follow immediately. An example of the multiplication of two complex numbers is shown in figure (7).

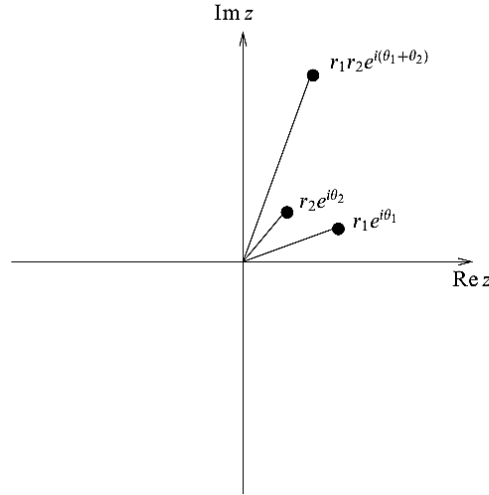


Figure (7): The multiplication of two complex numbers. In this case r_1 and r_2 are both greater than unity.

Division is equally simple in polar form; the quotient of z_1 and z_2 is given by:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \dots \dots \dots (56)$$

The relations $|z_1 / z_2| = |z_1| / |z_2|$ and $\arg(z_1 / z_2) = \arg z_1 - \arg z_2$ are again immediately apparent. The division of two complex numbers in polar form is shown in figure (8).

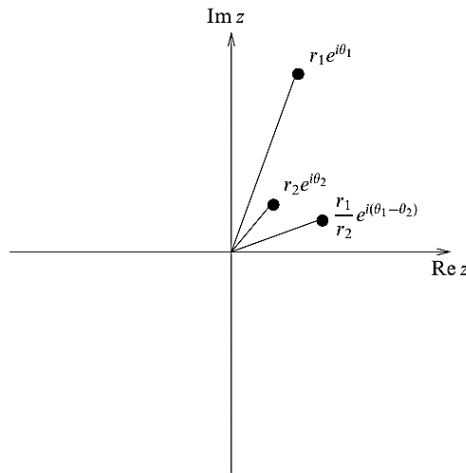


Figure (8): The division of two complex numbers. As in the previous figure, r_1 and r_2 are both greater than unity.

3- Roots of complex Numbers:

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r . As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π , we arrive at the original point; and the same is true when θ is decreased by 2π . It is, therefore, evident from Fig. (9) that two nonzero complex numbers:

$$z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2}$$

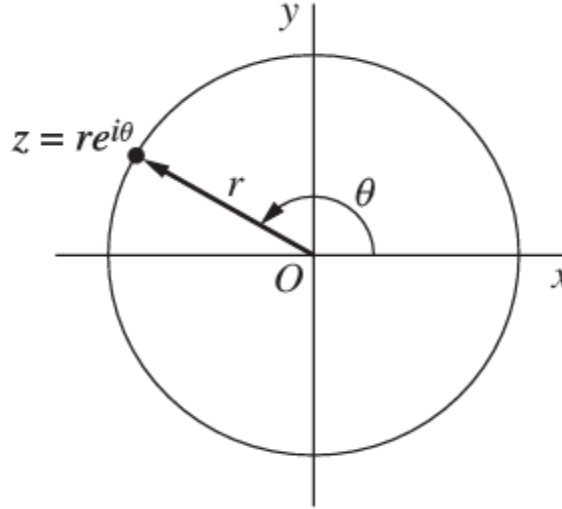


Figure (9) A point $z = re^{i\theta}$.

are equal if and only if

$$r_1 = r_2 \text{ and } \theta_1 = (\theta_2 + 2n\pi)$$

where k is any integer ($k=0, \pm 1, \pm 2, \dots$).

This observation, together with the expression $z^n = r^n e^{in\theta}$, is useful in finding the n th roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where n has one of the values $n=2, 3, \dots$. The method starts with the fact that an n th root of z_0 is a nonzero number $z = r e^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0} \dots \dots \dots (57)$$

According to the statement in italics just above, then,

$$r^n = r_0 \text{ and } n\theta = \theta_0 + 2k\pi \dots \dots \dots (58)$$

So $r = \sqrt[n]{r_0}$, where this radical denotes the unique positive n th root of the positive real number r_0 , and

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \quad (k = 0, \pm 1, \pm 2, \dots) \dots \dots \dots (59)$$

Consequently, the complex numbers

$$z = \sqrt[n]{r_0} \exp \left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] \quad (k = 0, \pm 1, \pm 2, \dots) \dots \dots \dots (60)$$

We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians, starting with argument θ_0/n .

► Let us find all four values of $(-16)^{1/4}$, or all of the fourth roots of the number -16 .

One need only write

$$z^4 = -16$$

$$r_0 = \sqrt{(-16)^2 + 0} = 16$$

$$\theta_0 = \tan^{-1} \frac{0}{-16} = \pi$$

$$\therefore z^4 = r^4 e^{i4\theta} = r_0 e^{i\theta_0}$$

$$\therefore r^4 = r_0 = 16 \quad \text{and} \quad 4\theta = \pi + 2k\pi$$

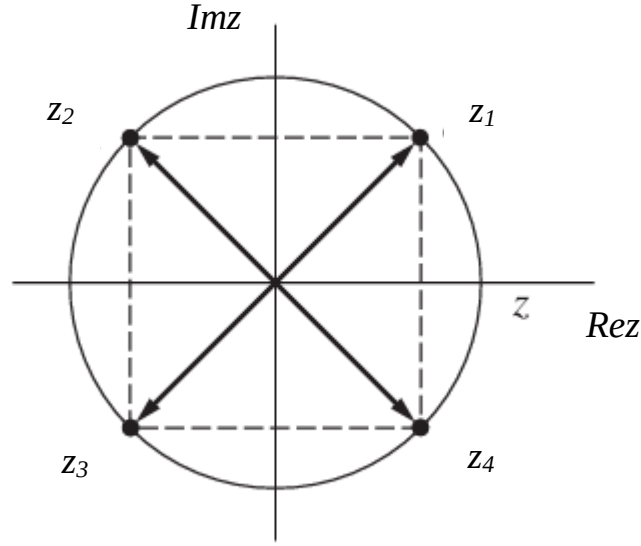
$$\therefore z = \sqrt[4]{16} \exp \left[i \left(\frac{\pi}{4} + \frac{k\pi}{2} \right) \right] \quad (k = 0, 1, 2, 3)$$

$$z_1 = 2 \exp \left[i \left(\frac{\pi}{4} \right) \right] = 2 \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \sqrt{2}(1 + i)$$

$$z_2 = 2 \exp \left[i \left(\frac{\pi}{4} + \frac{\pi}{2} \right) \right] = 2 \exp \left[i \left(\frac{3\pi}{4} \right) \right] = 2 \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \sqrt{2}(-1 + i)$$

$$z_3 = 2 \exp \left[i \left(\frac{\pi}{4} + \pi \right) \right] = 2 \exp \left[i \left(\frac{5\pi}{4} \right) \right] = 2 \left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = -\sqrt{2}(1 + i)$$

$$z_4 = 2 \exp \left[i \left(\frac{\pi}{4} + \frac{3\pi}{2} \right) \right] = 2 \exp \left[i \left(\frac{7\pi}{4} \right) \right] = 2 \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = \sqrt{2}(1 - i) \blacktriangleleft$$



► Let a denote any positive real number. In order to find the two square roots of $a + i$.

$$z^2 = a + i$$

$$r_0 = \sqrt{(a)^2 + 1^2} = (a^2 + 1)^{1/2}$$

$$\theta_0 = \tan^{-1} \frac{1}{a} = \beta$$

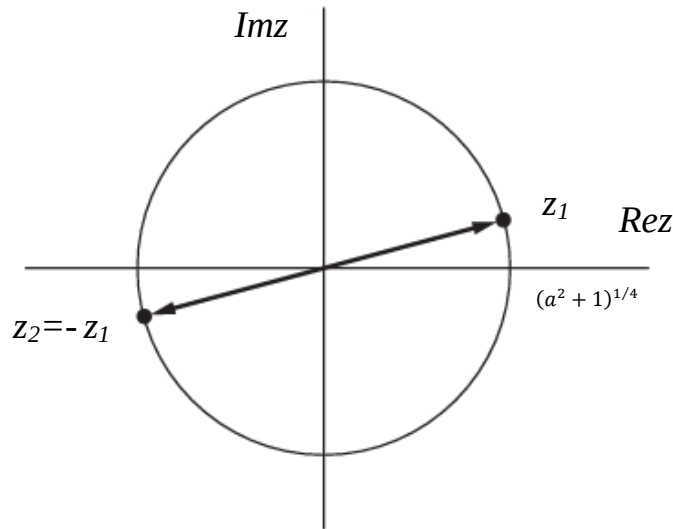
$$\therefore z^2 = r^2 e^{i2\theta} = r_0 e^{i\theta_0}$$

$$\therefore r^2 = r_0 = (a^2 + 1)^{1/2} \quad \text{and} \quad 2\theta = \beta + 2k\pi$$

$$\therefore z = \sqrt{(a^2 + 1)^{1/2}} \exp \left[i \left(\frac{\beta}{2} + k\pi \right) \right] \quad (k = 0, 1)$$

$$z_1 = (a^2 + 1)^{1/4} \exp \left[i \left(\frac{\beta}{2} \right) \right]$$

$$z_2 = (a^2 + 1)^{1/4} \exp \left[i \left(\frac{\beta}{2} + \pi \right) \right] = (a^2 + 1)^{1/4} \exp \left[i \left(\frac{\beta}{2} + \pi \right) \right] = -z_1 \blacktriangleleft$$



H.W. Find the square roots of (a) $2i$;(b) $1 - \sqrt{3}i$ and express them in Argand diagram.

4- de Moivre's theorem:

We now derive an extremely important theorem. Since $(e^{i\theta})^n = e^{in\theta}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta \dots \dots \dots (61)$$

where the identity $e^{in\theta} = \cos n\theta + i \sin n\theta$ follows from the series definition of $e^{in\theta}$. This result is called de Moivre's theorem and is often used in the manipulation of complex numbers. The theorem is valid for all n whether real, imaginary or complex.

There are numerous applications of de Moivre's theorem but this section examines just three: proofs of trigonometric identities; finding the nth roots of unity; and solving polynomial equations with complex roots.

3.1 Trigonometric identities:

The use of de Moivre's theorem in finding trigonometric identities is best illustrated by example. We consider the expression of a multiple-angle function in terms of a polynomial in the single-angle function, and its converse.

► Express $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Using de Moivre's theorem,

$$\begin{aligned}\cos 3\theta + i \sin 3\theta &= (\cos \theta + i \sin \theta)^3 \\ &= (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \sin \theta \cos^2 \theta - \sin^3 \theta).\end{aligned}$$

We can equate the real and imaginary coefficients separately, i.e.

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = 4 \cos^3 \theta - 3 \cos \theta$$

and

$$\sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta. \blacktriangleleft$$

This method can clearly be applied to finding power expansions of $\cos n\theta$ and $\sin n\theta$ for any positive integer n .

The converse process uses the following properties of $z = e^{i\theta}$,

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \dots \dots \dots (62)$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta \dots \dots \dots (63)$$

These equalities follow from simple applications of de Moivre's theorem, i.e.

$$\begin{aligned}z^n + \frac{1}{z^n} &= (\cos \theta + i \sin \theta)^n + (\cos \theta + i \sin \theta)^{-n} \\ &= \cos n\theta + i \sin n\theta + \cos - n\theta + i \sin - n\theta \\ &= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta \\ &= 2 \cos n\theta\end{aligned}$$

and

$$\begin{aligned}z^n - \frac{1}{z^n} &= (\cos \theta + i \sin \theta)^n - (\cos \theta + i \sin \theta)^{-n} \\ &= \cos n\theta + i \sin n\theta - \cos - n\theta - i \sin - n\theta \\ &= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta \\ &= 2i \sin n\theta\end{aligned}$$

In the particular case where $n=1$,

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2 \cos \theta \dots \dots \dots (64)$$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i \sin\theta \dots\dots\dots (65)$$

► Find an expression for $\cos^3 \theta$ in terms of $\cos 3\theta$ and $\cos \theta$

Using (60)

$$\begin{aligned} \cos^3 \theta &= \frac{1}{2^3} \left(z + \frac{1}{z} \right)^3 \\ &= \frac{1}{8} \left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \right) \\ &= \frac{1}{8} \left(z^3 + \frac{1}{z^3} \right) + \frac{3}{8} \left(z + \frac{1}{z} \right) \\ &= \frac{1}{8} * 2 \cos 3\theta + \frac{3}{8} * 2 \cos \theta \\ \cos^3 \theta &= \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta. \blacktriangleleft \end{aligned}$$

3-2 Finding the n th roots of unity:

The equation $z^2 = 1$ has the familiar solutions $z = \pm 1$. However, now that we have introduced the concept of complex numbers we can solve the general equation $z^n = 1$. Recalling the fundamental theorem of algebra, we know that the equation has n solutions. In order to proceed we rewrite the equation as

$$z^n = e^{2ik\pi} \dots\dots\dots (66)$$

where k is any integer. Now taking the n th root of each side of the equation we find

$$z = e^{2ik\pi/n} \dots\dots\dots (67)$$

Hence, the solutions of $z^n = 1$ are

$$z_{1,2,\dots,n} = 1, e^{2i\pi/n}, \dots, e^{2i(n-1)\pi/n} \dots\dots\dots (68)$$

corresponding to the values $0, 1, 2, \dots, n-1$ for k . Larger integer values of k do not give new solutions, since the roots already listed are simply cyclically repeated for $k = n, n+1, n+2$, etc.

► Find the solutions to the equation $z^3 = 1$.

By applying the above method we find

$$\begin{aligned} r &= \sqrt{1^2 + 0^2} = 1 \\ \arg z &= \tan^{-1} \frac{0}{1} = 0 \end{aligned}$$

$$z = [1,0]$$

$$z^3 = [1,0]^3 = [1,0]$$

$$z = e^{2ik\pi/3}$$

Hence the three solutions are $z_1 = e^{0i} = 1$, $z_2 = e^{2i\pi/3}$, $z_3 = e^{4i\pi/3}$. We note that, as expected, the next solution, for which $k=3$, gives $z_4 = e^{6i\pi/3} = 1 = z_1$, so that there are only three separate solutions. ◀

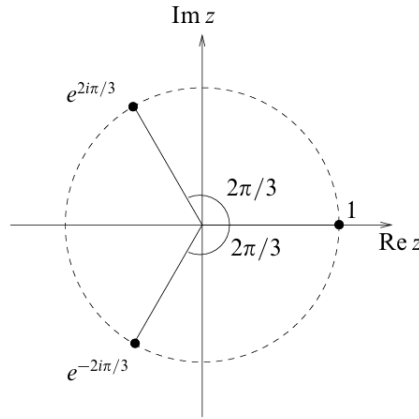


Figure (10) The solutions of $z^3 = 1$.

Not surprisingly, given that $|z^3| = |z|^3$ from (39), all the roots of unity have unit modulus, i.e. they all lie on a circle in the Argand diagram of unit radius. The three roots are shown in figure (10).

3-3 Solving polynomial equations:

A third application of de Moivre's theorem is to the solution of polynomial equations. Complex equations in the form of a polynomial relationship must first be solved for z in a similar fashion to the method for finding the roots of real polynomial equations. Then the complex roots of z may be found.

► Solve the equation $z^6 - z^5 + 4z^4 - 6z^3 + 2z^2 - 8z + 8 = 0$.

We first factorise to give

$$(z^3 - 2)(z^2 + 4)(z - 1) = 0.$$

Hence $z^3 = 2$ or $z^2 = -4$ or $z = 1$. The solutions to the quadratic equation are $z = \pm 2i$; to find the complex cube roots, we first write the equation in the form

$$z^3 = 2 = 2e^{2ik\pi},$$

where k is any integer. If we now take the cube root, we get

$$z = 2^{1/3} e^{2ik\pi/3}.$$

To avoid the duplication of solutions, we use the fact that $-\pi < \arg z < \pi$ and find

$$z_1 = 2^{1/3}$$

$$z_2 = 2^{1/3} e^{2i\pi/3} = 2^{1/3} \left(-\frac{1}{2} + \frac{3}{2}i \right)$$

$$z_3 = 2^{1/3} e^{-2i\pi/3} = 2^{1/3} \left(-\frac{1}{2} - \frac{3}{2}i \right)$$

The complex numbers z_1, z_2 and z_3 , together with $z_4 = 2i, z_5 = -2i$ and $z_6 = 1$ are the solutions to the original polynomial equation. As expected from the fundamental theorem of algebra, we find that the total number of complex roots (six, in this case) is equal to the largest power of z in the polynomial. ◀

5- Function of Complex Number:

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w . The number w is called the value of f at z and is denoted by $f(z)$, so that $w = f(z)$. The set S is called the domain of definition of f .

Suppose that $u + iv$ is the value of a function f at $z = x + iy$; that is,

$$u + iv = f(x + iy) \dots \dots \dots (69)$$

Each of the real numbers u and v depends on the real variables x and y , and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y :

$$f(z) = u(x, y) + iv(x, y) \dots \dots \dots (70)$$

If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

If the function v in equation (69) always has value zero, then the value of f is always real. Thus f is a real-valued function of a complex variable.

If the polar coordinates r and θ are used instead of x and y , then

$$u + iv = f(re^{i\theta}) \dots \dots \dots (71)$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

Hence

$$f(z) = u(r, \theta) + iv(r, \theta) \dots \dots \dots (72)$$

► Consider the function $w = z^2$ when $z = re^{i\theta}$.

$$w = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + i r^2 \sin 2\theta.$$

Hence

$$u(r, \theta) = r^2 \cos 2\theta \text{ and } v(r, \theta) = r^2 \sin 2\theta \blacktriangleleft$$

► Write the following function $f(z)$ in the forms $f(z) = u(x, y) + iv(x, y)$

$$(a) f(z) = z^3 + z + 1$$

$$(b) f(z) = \frac{1}{i - z}$$

If

$$z = x + iy$$

then

$$\begin{aligned} (a) f(z) &= (x + iy)^3 + (x + iy) + 1 = (x + iy)(x^2 - y^2 + 2ixy) + x + iy + 1 \\ &= x^3 - xy^2 + 2ix^2y + ix^2y - iy^3 - 2xy^2 + x + iy + 1 \\ &= x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y) \\ u(x, y) &= x^3 - 3xy^2 + x + 1 \text{ and } v(x, y) = 3x^2y - y^3 + y \end{aligned}$$

$$(b) f(z) = \frac{1}{i - (x + iy)} = \frac{1}{-x + i(1 - y)} \times \frac{-x - i(1 - y)}{-x - i(1 - y)} = \frac{-x - i(1 - y)}{x^2 - (1 - y)^2} = \frac{-x}{x^2 - (1 - y)^2} - i \frac{(1 - y)}{x^2 - (1 - y)^2}$$

H.W. Write the following function $f(z)$ in the forms $f(z) = u(x, y) + iv(x, y)$

$$f(z) = (\exp(z^2))^*$$

$$u(x, y) = \frac{-x}{x^2 - (1-y)^2} \text{ and } v(x, y) = \frac{(1-y)}{x^2 - (1-y)^2} \blacktriangleleft$$

6- Analytic Function:

Suppose that $f(z) = u(x, y) + iv(x, y)$ is analytic in the domain D and suppose that the partial derivatives of the component functions u and v are continuous in D also. Differentiation of the Cauchy–Riemann equations then gives:

$$u_x = v_y \Rightarrow u_{xx} = v_{xy} = v_{yx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0 \dots \dots \dots (73)$$

$$v_x = -u_y \Rightarrow v_{xx} = -u_{xy} = -u_{yx} = -v_{yy} \Rightarrow v_{xx} + v_{yy} = 0 \dots \dots \dots (74)$$

Therefore, both the real and imaginary components of f are harmonic functions that satisfy Laplace's equation. Furthermore, comparing the two- dimensional gradients:

$$\nabla u = \mathbf{i} \frac{\partial u}{\partial x} + \mathbf{j} \frac{\partial u}{\partial y} = \mathbf{i} \frac{\partial v}{\partial y} - \mathbf{j} \frac{\partial v}{\partial x} = \hat{n} \times \nabla v \dots \dots \dots (75)$$

$$\nabla v = \mathbf{i} \frac{\partial v}{\partial x} + \mathbf{j} \frac{\partial v}{\partial y} = -\mathbf{i} \frac{\partial u}{\partial y} + \mathbf{j} \frac{\partial u}{\partial x} = \hat{n} \times \nabla u \dots \dots \dots (76)$$

We find that lines of constant u (level curves) are orthogonal to lines of constant v anywhere that $f'(z) \neq 0$.

► Verify that the following functions $f(z)$ are analytic function.

(a) $f(z) = x^2 - y^2 + i2xy$

(b) $f(z) = 2y + ix$

(a) $f(z) = x^2 - y^2 + i2xy$

$$u_x = 2x, \quad v_y = 2x, \quad u_{xx} = 2, \quad v_{xy} = 2, \quad v_{yx} = 2, \quad u_{yy} = -2$$

$$u_y = -2y, \quad v_x = 2y, \quad v_{xx} = 0, \quad u_{xy} = 0, \quad u_{yx} = 0, \quad v_{yy} = 0$$

then

$$u_x = v_y \text{ and } v_x = -u_y$$

$$u_{xx} + u_{yy} = 2 - 2 = 0$$

$$v_{xx} + v_{yy} = 0 + 0 = 0$$

Hence, $f(z)$ is analytic.

$$(b) \quad f(z) = 2y + ix$$

$$u_x = 0, \quad v_y = 0, \quad u_{xx} = 0, \quad v_{xy} = 0, \quad v_{yx} = 0, \quad u_{yy} = 0$$

$$u_y = 2, \quad v_x = 1, \quad v_{xx} = 0, \quad u_{xy} = 0, \quad u_{yx} = 0, \quad v_{yy} = 0$$

then

$$u_x = v_y \quad \text{and} \quad v_x \neq -u_y$$

$$u_{xx} + u_{yy} = 0 + 0 = 0$$

$$v_{xx} + v_{yy} = 0 + 0 = 0$$

Hence, $f(z)$ is not analytic. ◀

7- HARMONIC FUNCTIONS:

A real-valued function H of two real variables x and y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation:

$$H_{xx}(x, y) + H_{yy}(x, y) = 0 \dots \dots \dots (77)$$

known as Laplace's equation. Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

► Verify that the following functions u are harmonic:

$$(a) \quad u(x, y) = 3x^2 y + 2x^2 - y^3 - 2y^2$$

$$(b) \quad u(x, y) = \ln(x^2 + y^2)$$

$$(a) \quad u(x, y) = 3x^2 y + 2x^2 - y^3 - 2y^2$$

$$u_x = 6xy + 4x, \quad u_y = 3x^2 - 3y^2 - 4y$$

$$u_{xx} = 6y + 4, \quad u_{yy} = -6y - 4$$

From eq.(72), then

$$u_{xx} + u_{yy} = 6y + 4 + (-6y - 4) = 0$$

Hence, u is harmonic.

(b) $u(x, y) = \ln(x^2 + y^2)$

$$u_x = \frac{2x}{x^2 + y^2}, \quad u_y = \frac{2y}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2) * 2 - 2x * 2x}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}, \quad u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

From eq.(72), then

$$u_{xx} + u_{yy} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0$$

Hence, u is harmonic. ◀

► Verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane.

$$T_x = e^{-y} \cos x, \quad T_y = -e^{-y} \sin x$$

$$T_{xx} = -e^{-y} \sin x, \quad T_{yy} = e^{-y} \sin x$$

From eq.(72), then

$$T_{xx} + T_{yy} = -e^{-y} \sin x + e^{-y} \sin x = 0$$

Hence, T is harmonic. ◀

Theorem. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D , then its component functions u and v are harmonic in D .

H.W. Verify that the function $T(x, y) = e^{-y} \sin x - ie^{-y} \cos x$ is harmonic in any domain of the xy plane.

We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function is:

$$u(x, y) = y^3 - 3x^2y$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is related to $u(x, y)$ by means of the Cauchy – Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

The first of these equations tells us that

$$v_y(x, y) = -6xy$$

Holding x fixed and integrating each side here with respect to y , we find that

$$v(x, y) = -3xy^2 + \phi(x)$$

where ϕ is, at present an arbitrary function of x . Using the second of equations

$$3y^2 - 3x^2 = 3y^2 - \phi'(x)$$

$$\phi'(x) = 3x^2$$

Thus

$$\phi(x) = x^3 + C$$

Where C is an arbitrary real number. According to above equation then, the function:

$$v(x, y) = -3xy^2 + x^3 + C$$

8- Complex integrals:

Corresponding to integration with respect to a real variable, it is possible to define integration with respect to a complex variable between two complex limits. If a complex function $f(z)$ is single-valued and continuous in some region R in the complex plane, then we can define the complex integral of $f(z)$ between two points A and B along some curve in R ; its value will depend, in general, upon the path taken between A and B .

Let a particular path C be described by a continuous (real) parameter t ($\alpha \leq t \leq \beta$) that gives successive positions on C by means of the equations

$$x = x(t), \quad y = y(t) \dots \dots \dots (78)$$

with $t = \alpha$ and $t = \beta$ corresponding to the points A and B , respectively. Then the integral along path C of a continuous function $f(z)$ is written:

$$\int f(z) dz \dots \dots \dots (79)$$

and can be given explicitly as a sum of real integrals as follows:

$$\begin{aligned} \int f(z) dz &= \int (u + iv)(dx + idy) = \int u dx - \int v dy + i \int u dy + i \int v dx \\ &= \int_{\alpha}^{\beta} u \frac{dx}{dt} dt - \int_{\alpha}^{\beta} v \frac{dy}{dt} dt + i \int_{\alpha}^{\beta} u \frac{dy}{dt} dt + i \int_{\alpha}^{\beta} v \frac{dx}{dt} dt \dots \dots \dots (80) \end{aligned}$$

► Evaluate the complex integral of $f(z) = z^{-1}$ along the circle $|z| = R$, starting and finishing at $z = R$.

The path C_1 is parameterized as follows:

$$z(t) = R \cos t + iR \sin t \quad 0 \leq t \leq 2\pi$$

whilst $f(z)$ is given by:

$$f(z) = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

Thus the real and imaginary parts of $f(z)$ are:

$$u = \frac{x}{x^2 + y^2} = \frac{R \cos t}{R^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2} = \frac{-R \sin t}{R^2}$$

Hence, using expression (79)

$$\begin{aligned} \int \frac{1}{z} dz &= \int_0^{2\pi} \frac{R \cos t}{R^2} (-R \sin t) dt - i \int_0^{2\pi} \frac{-R \sin t}{R^2} (R \cos t) dt + i \int_0^{2\pi} \frac{R \cos t}{R^2} (R \cos t) dt \\ &\quad + \int_0^{2\pi} \frac{-R \sin t}{R^2} (-R \sin t) dt \end{aligned}$$

$$\int \frac{1}{z} dz = 0 + 0 + i\pi + i\pi = i2\pi \blacktriangleleft$$

With a bit of experience, the reader may be able to evaluate integrals directly without having to write them as four separate real integrals. In the present case,

$$\int \frac{1}{z} dz = \int_0^{2\pi} \frac{-R \sin t + iR \cos t}{R \cos t + iR \sin t} dt = \int_0^{2\pi} i dt = 2\pi i$$

► Evaluate the complex integral of $f(z) = \operatorname{Re} z$ along the paths $0 \leq t \leq 2\pi$ and $0 \leq t \leq \pi$

$$\int \operatorname{Re} z dz = \int_0^{2\pi} R \cos t (-R \sin t + iR \cos t) dt = i\pi R^2$$

$$\int \operatorname{Re} z dz = \int_0^{\pi} R \cos t (-R \sin t + iR \cos t) dt = \frac{1}{2} i\pi R^2 \blacktriangleleft$$

► Evaluate the following integral:

$$\int_1^2 (t^2 + i)^2 dt$$

$$\int_1^2 (t^2 + i)^2 dt = \int_1^2 (t^4 + 2it^2 + i^2) dt = \frac{t^5}{5} + 2i \frac{t^3}{3} - t \Big|_1^2 = \frac{31}{5} + \frac{14}{3}i - 1 = \frac{26}{5} + \frac{14}{3}i \blacktriangleleft$$

H.W. Prove the following integral:

$$\int_0^{\pi/4} e^{-2it} dt = \frac{1}{2} - \frac{i}{2}$$

9- Cauchy's theorem and integral formula

Cauchy's theorem states that if $f(z)$ is an analytic function, and $\hat{f}(z)$ is continuous at each point within and on a closed contour C , then

$$\oint f(z) dz = 0 \dots \dots \dots (81)$$

We will need the two-dimensional form of the divergence theorem, known as Green's theorem in a plane. This says that if p and q are two functions with continuous first derivatives within and on a closed contour C (bounding a domain R) in the xy -plane, then

$$\iint \left(\frac{dp}{dx} + \frac{dq}{dy} \right) dx dy = \oint (p dy - q dx) \dots \dots \dots (82)$$

With $f(z) = u + iv$ and $dz = dx + i dy$, this can be applied to:

$$I = \oint f(z) dz = \oint (u dx - v dy) + i \oint (v dx + u dy)$$

To gives:

$$\iint \left(\frac{d(-u)}{dy} + \frac{d(-v)}{dx} \right) dx dy + i \iint \left(\frac{d(-v)}{dy} + \frac{d(u)}{dx} \right) dx dy \dots \dots \dots (83)$$

Now, recalling that $f(z)$ is analytic and therefore that the Cauchy–Riemann relations (73,74) apply, we see that each integrand is identically zero and thus I is also zero; this proves Cauchy's theorem.

Another very important theorem in the theory of complex variables is Cauchy's integral formula, which states that if $f(z)$ is analytic within and on a closed contour C and z_0 is a point within C then:

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz \dots \dots \dots (84)$$

This formula is saying that the value of an analytic function anywhere inside a closed contour is uniquely determined by its values on the contour. We then use the fact that any point z on γ is given by $z = z_0 + \rho \exp i\theta$ (and so $dz = i\rho \exp i\theta d\theta$). Thus the value of the integral around γ is given by:

$$I = \oint \frac{f(z)}{z - z_0} dz = \oint \frac{f(z_0 + \rho \exp i\theta)}{\rho \exp i\theta} i \rho \exp i\theta d\theta$$

$$I = i \oint f(z_0 + \rho \exp i\theta) d\theta$$

If the radius of the circle γ is now shrunk to zero, i.e. $\rho \rightarrow 0$, then $I \rightarrow 2\pi i f(z_0)$. An extension to Cauchy's integral formula can be made, yielding an integral expression for $\hat{f}(z)$:

$$\hat{f}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz \dots \dots \dots (85)$$

► Prove Cauchy's integral formula for $\hat{f}(z)$ given in (84).

$$\hat{f}(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

$$\hat{f}(z_0) = \lim_{h \rightarrow 0} \left[\frac{1}{2\pi i} \oint \frac{f(z)}{h} \left(\frac{1}{z - z_0 - h} - \frac{1}{z - z_0} \right) dz \right]$$

$$\hat{f}(z_0) = \lim_{h \rightarrow 0} \left[\frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0 - h)(z - z_0)} dz \right]$$

$$\hat{f}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz$$

which establishes result (84) ◀

Further, it may be proved by induction that the n th derivative of $f(z)$ is also given by a Cauchy integral,

$$f^n(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz \dots \dots \dots (86)$$

► Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals.

(a) $\int \frac{z dz}{z+1}$

(b) $\int \frac{\cosh z}{z^2 + z} dz$

(c) $\int \frac{\tan(z/2)}{z - \frac{\pi}{2}} dz$

(a) By Cauchy Integral Theorem

$$\int \frac{z dz}{z+1} = 2\pi i f(-1) = -2\pi i \blacktriangleleft$$

(b) By Cauchy Integral Theorem

$$\int \frac{\cosh z}{z^2 + z} dz = \int \frac{\cosh z}{z(z+1)} dz$$

$$f(z_0) = \frac{\cosh z_0}{(z_0 + 1)} \Big|_{z_0=0} \quad \text{and} \quad \frac{\cosh z_0}{z_0} \Big|_{z_0=-1}$$

$$\int \frac{\cosh z}{z^2 + z} dz = 2\pi i \frac{\cosh z}{(z+1)} \Big|_{z_0=0} = 2\pi i$$

$$\int \frac{\cosh z}{z^2 + z} dz = 2\pi i \frac{\cosh z}{z} \Big|_{z_0=-1} = -2\pi i \cosh(-1)$$

Hence

$$\int \frac{\cosh z}{z^2 + z} dz = 2\pi i (1 - \cosh(-1)) \blacktriangleleft$$

(c) $\int \frac{\tan(z/2)}{z - \frac{\pi}{2}} dz = 2\pi i \tan\left(\frac{\pi}{4}\right) = 2\pi i \blacktriangleleft$

► Compute the integral of the following for functions along the curves $C_1=\{|z| = 1\}$ and $C_2=\{|z - 2| = 1\}$, both oriented counterclockwise:

(a) $\frac{1}{2z-z^2}$
 (b) $\frac{\sinh z}{(2z-z^2)^2}$

(a)

$$\int \frac{1}{2z-z^2} dz = \int \frac{(2-z)^{-1}}{z} dz = 2\pi i(2-0)^{-1} = \pi i \blacktriangleleft$$

(b)

$$\int \frac{\sinh z}{(2z-z^2)^2} dz = \int \frac{(\sinh z)(2-z)^{-2}}{z^2} dz = 2\pi i \left((\sinh z)(2-z)^{-2} \right)' \Big|_{z=0} = \frac{\pi i}{2} \blacktriangleleft$$

► Evaluate $\int \frac{\sin z}{(z+1)^7} dz$

Where C is the circle of radius 5, center 0, positively oriented.

Recall the extension of the Cauchy integral formula:

$$f^n(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Considering the function $f(z) = \sin z$, which is analytic on C , we have

$$f^6(-1) = \frac{6!}{2\pi i} \oint \frac{\sin z}{(z-(-1))^{6+1}} dz = \frac{6!}{2\pi i} \int \frac{\sin z}{(z+1)^7} dz$$

Since $f^6(z) = -\sin z$, then

$$\int \frac{\sin z}{(z+1)^7} dz = -\frac{2\pi i}{6!} \sin(-1) = \frac{2\pi \sin(1)}{6!} i$$

Let C be the circle $|z| = 1$ oriented counter-clockwise. $\blacktriangleleft$

10- Taylor and Laurent series:

We may establish Taylor's theorem for functions of a complex variable. If $f(z)$ is analytic inside and on a circle C of radius R centered on the point $z = z_0$, and z is a point inside C , then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \dots \dots \dots (87)$$

The Taylor expansion is valid inside the region of analyticity and, for any particular z_0 , can be shown to be unique. To prove Taylor's theorem (86), we note that, since $f(z)$ is analytic inside and on C , we may use Cauchy's formula to write $f(z)$ as

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta \dots \dots \dots (88)$$

where ζ lies on C . Now we may expand the factor $(\zeta - z)^{-1}$ as a geometric series in $(z - z_0)/(\zeta - z_0)$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

So (87) becomes

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \frac{2\pi i f^n(z_0)}{n!} \dots \dots \dots (89)$$

where we have used Cauchy's integral formula (85) for the derivatives of $f(z)$. Cancelling the factors of $2\pi i$, we thus establish the result (87) with $a_n = \frac{f^n(z_0)}{n!}$.

$$f^0(z_0) = f(z_0) \quad \text{and} \quad 0! = 1$$

The Taylor series is:

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \dots \dots (|z - z_0| < R_0)$$

Taylor theorem when $z_0 = 0$, in which case f is assumed to be analytic throughout a disk $|z| < R_0$ become a Macalurin series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n \quad |z| < R_0 \dots \dots \dots (90)$$

► Since the function $f(z) = e^z$ is entire, It has a Maclaurin series representation which is valid for all z .

Here

$$f^n(z) = e^z [0,1,2, \dots \dots \dots] \text{ and } f^n(0) = 1 [0,1,2, \dots \dots \dots]$$

It follows that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty)$$

Note that if $z = x + i0$, expansion becomes

$$e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty < |x| < \infty)$$

If replace z by $3z$ on each side of equation and multiply through the resulting equation by z^2

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} \quad (|z| < \infty)$$

Finally, if we replace n by $n-2$ here, we have

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \quad (|z| < \infty) \blacktriangleleft$$

► Find the Maclaurin series for the function $f(z) = \sin z$

One can use expansion and the definition:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

To give the details. We refer to expansion and write

$$\sin z = \frac{1}{2i} \left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right] = \frac{1}{2i} \sum_{n=0}^{\infty} (1 - (-1)^n) \frac{i^n z^n}{n!} \quad (|z| < \infty)$$

If n is even $1 - (-1)^n = 0$, and so we can replace n by $2n + 1$

$$\sin z = \frac{1}{2i} \sum_{n=0}^{\infty} (1 - (-1)^{2n+1}) \frac{i^{2n+1} z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

► Another Maclaurin series representation is:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

Since the derivative of the function $f(z) = 1/(1-z)$, which fails to be analytic at $z=1$, are

$$f^n(z) = \frac{n!}{(1-z)^{n+1}} \quad (n = 0, 1, 2, \dots)$$

In particular $f^n(0) = n!$ ($n = 0, 1, 2, \dots$). Note that expansion gives us the sum of an infinite geometric series, where z is the common ratio of adjacent terms:

$$\therefore f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n$$

$$1 + z + z^2 + z^3 + \dots = \frac{1}{1-z} \quad (|z| < 1)$$

If we substitute $-z$ for z in equation above and its condition of validity, and note that $|z| < 1$ when $|-z| < 1$, we see that:

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

If , on the other hand, we replace the variable z in equation above by $1-z$, we have the Taylor series representation:

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (|z-1| < 1)$$

This condition of validity follows from the one associated with expansion above since $|1-z| < 1$ is the same as $|z-1| < 1$.

H.W. Prove the following series:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad (|z| < \infty)$$

and

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

► Find the Taylor series of the following functions and their radius of convergence:

(a) $z \sinh(z^2)$ at $z = 0$

(b) e^z at $z = 2$

(a) $z \sinh(z^2)$

Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$z \sinh(z^2) = z \left(\frac{e^{z^2} - e^{-z^2}}{2} \right)$$

$$z \sinh(z^2) = \frac{z}{2} \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{n!} \right)$$

$$z \sinh(z^2) = \frac{1}{2} \left(\sum_{n=0}^{\infty} (1 - (-1)^n) \frac{z^{2n+1}}{n!} \right)$$

If $n=0,2,4,6,\dots$ $z \sinh(z^2) = 0$

$$\text{Or } n=1,3,5,7,\dots \quad z \sinh(z^2) = \sum_{n=1,3,\dots}^{\infty} \frac{z^{2n+1}}{n!}$$

$$\therefore n = 2m + 1$$

$$z \sinh(z^2) = \sum_{m=0}^{\infty} \frac{z^{4m+3}}{(2m+1)!}$$

Since $f(z)$ is entire, the radius of convergence is ∞ .

(b) e^z

Let $z=w+2$ and $w=z-2$

$$e^z = e^{w+2} = e^2 e^w = e^2 \sum_{n=0}^{\infty} \frac{w^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!}$$

Since $f(z)$ is entire, the radius of convergence is ∞ . ◀

If a function f fails to be analytic at a point z_0 , one cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z-z_0$. We now present the theory of such representation, and we begin with Laurent's theorem.

Suppose that a function $f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered on the point z_0 , and let C denote any positively oriented simple closed contour around and lying in the domain. Then each point in the domain, $f(z)$ has the series representation,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \dots \dots \dots (91) \quad (R_1 < |z - z_0| < R_2)$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz \dots \dots \dots (92) \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{-n+1}} dz \dots \dots \dots (93) \quad (n = 1, 2, \dots)$$

Note how replacing n by $-n$ in the second series in representation (91) enables us to write that series as:

$$\sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z - z_0)^{-n}}$$

where

$$b_{-n} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz \dots \dots \dots (94) \quad (n = -1, -2, \dots)$$

Thus

$$f(z) = \sum_{n=-\infty}^{-1} b_{-n}(z - z_0)^n + \sum_{n=0}^{\infty} a_n(z - z_0)^n + \dots \dots \dots (95) \quad (R_1 < |z - z_0| < R_2)$$

If

$$c_n = \begin{cases} b_{-n} & \text{when } n \leq -1 \\ a_n & \text{when } n \geq 0 \end{cases}$$

This becomes

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \dots \dots \dots (96) \quad (R_1 < |z - z_0| < R_2)$$

where

$$c_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz \dots \dots \dots (97) \quad (n = 0, \pm 1, \pm 2, \dots)$$

The representation of $f(z)$ is called a **Laurent series**. Observe that the integrand in expression (93) can be written $f(z)(z - z_0)^{n-1}$. Thus it is clear that when f is actually analytic throughout the disk $|z - z_0| < R_2$, this integrand is too. Hence all of the coefficient b_n are zero; and

$$\frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^n(z_0)}{n!} \quad (n = 0, 1, 2, \dots)$$

Expansion (90) reduces to a Taylor series about z_0 .

11- A singular point:

A singular point of a complex function $f(z)$ is any point in the Argand diagram at which $f(z)$ fails to be analytic. If $f(z)$ has a singular point at $z = z_0$ but is analytic at all points

in some neighbourhood containing z_0 but no other singularities, then $z = z_0$ is called an isolated singularity.

The most important type of isolated singularity is the pole. If $f(z)$ has the form

$$f(z) = \frac{g(z)}{(z - z_0)^n} \dots \dots \dots (98)$$

where n is a positive integer, $g(z)$ is analytic at all points in some neighbourhood containing $z = z_0$ and $g(z) \neq 0$, then $f(z)$ has a pole of order n at $z = z_0$. An alternative (though equivalent) definition is that:

$$\lim_{z \rightarrow z_0} [(z - z_0)^n f(z)] = a \dots \dots \dots (99)$$

where a is a finite, non-zero complex number. We note that if the above limit is equal to zero, then $z = z_0$ is a pole of order less than n , or $f(z)$ is analytic there; if the limit is infinite then the pole is of an order greater than n . It may also be shown that if $f(z)$ has a pole at $z = z_0$, then $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$ from any direction in the Argand diagram.

► Find the singularities of the functions:

$$(i) f(z) = \frac{1}{1-z} - \frac{1}{1+z}, \quad (ii) f(z) = \tanh z.$$

(i) If we write $f(z)$ as:

$$f(z) = \frac{1}{1-z} - \frac{1}{1+z} = \frac{2z}{(1-z)(1+z)}$$

we see immediately from either (89) or (90) that $f(z)$ has poles of order 1 (or simple poles) at $z = 1$ and $z = -1$.

(ii) In this case we write:

$$f(z) = \tanh z = \frac{\sinh z}{\cosh z} = \frac{\exp z - \exp(-z)}{\exp z + \exp(-z)}.$$

Thus $f(z)$ has a singularity when $\exp z = -\exp(-z)$ or, equivalently, when

$$\exp z = \exp[i(2n + 1)\pi] \exp(-z)$$

where n is any integer. Equating the arguments of the exponentials we find $z =$

$$(n + \frac{1}{2})\pi i, \text{ for integer } n$$

Furthermore, we have

$$\lim_{z \rightarrow (n + \frac{1}{2})\pi i} \left\{ \frac{[z - (n + \frac{1}{2})\pi i] \sinh z}{\cosh z} \right\} = 1$$

Therefore, from (90), each singularity is a simple pole. ◀

► Show that $f(z) = (\sin z)/z$ has a removable singularity at $z = 0$.

It is clear that $f(z)$ takes the indeterminate form $0/0$ at $z = 0$. However, by expanding $\sin z$ as a power series in z , we find

$$f(z) = \frac{1}{z} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots.$$

Thus $\lim_{z \rightarrow 0} f(z) = 1$ independently of the way in which $z \rightarrow 0$, and so $f(z)$ has a removable singularity at $z = 0$. ◀

Zeros are classified in a similar way to poles, in that if

$$f(z) = (z - z_0)^n g(z),$$

where n is a positive integer and $g(z) \neq 0$, then $z = z_0$ is called a zero of order n of $f(z)$. If $n = 1$ then $z = z_0$ is called a simple zero. It may further be shown that if $z = z_0$ is a zero of order n of $f(z)$ then it is also a pole of order n of the function $1/f(z)$.

12- The residue theorem:

Having seen from Cauchy's theorem that the value of an integral round a closed contour C is zero if the integrand is analytic inside the contour, it is natural to ask what value it takes when the integrand is not analytic inside C . The answer to this is contained in the residue theorem, which we now discuss.

Suppose the function $f(z)$ has a pole of order m at the point $z = z_0$, and so can be written as a Laurent series about z_0 of the form

$$f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^n$$

Now consider the integral I of $f(z)$ around a closed contour C that encloses $z = z_0$, but no other singular points. Using Cauchy's theorem, this integral has the same value as the integral around a circle γ of radius ρ centred on $z = z_0$, since $f(z)$ is analytic in the region between C and γ . On the circle we have $z = z_0 + \rho \exp i\theta$ (and $dz = i\rho \exp i\theta d\theta$), and so

$$I = \oint f(z) dz$$

$$I = \sum_{n=-m}^{\infty} c_n \oint (z - z_0)^n dz$$

$$I = \sum_{n=-m}^{\infty} c_n \int_0^{2\pi} i \rho^{n+1} e^{i(n+1)\theta} d\theta$$

For every term in the series with $n \neq -1$, we have

$$\int_0^{2\pi} i \rho^{n+1} e^{i(n+1)\theta} d\theta = 0$$

but for the $n = -1$ term we obtain

$$\int_0^{2\pi} i d\theta = 2\pi i$$

Therefore only the term in $(z - z_0)^{-1}$ contributes to the value of the integral around γ (and therefore C), and I takes the value

$$I = \oint f(z) dz = 2\pi i c_{-1} \dots \dots \dots (100)$$

Thus the integral around any closed contour containing a single pole of general order m (or, by extension, an essential singularity) is equal to $2\pi i$ times the residue of $f(z)$ at $z = z_0$.

If we extend the above argument to the case where $f(z)$ is continuous within and on a closed contour C and analytic, except for a finite number of poles, within C , then we arrive at the residue theorem

$$\oint f(z) dz = 2\pi i \sum_j R_j \dots \dots \dots (101)$$

where $\sum_j R_j$ is the sum of the residues of $f(z)$ at its poles within C .

► Let $f(z) = \frac{z}{z^2+1}$. Find the poles and residues of f .

Using partial fractions we write

$$f(z) = \frac{z}{(z+i)(z-i)}$$

The poles are at $z = \pm i$. We compute the residues at each pole:

At $z = i$

$$f(z) = \frac{1}{2} \cdot \frac{1}{(z-i)} + \text{something analytic at } i$$

Therefore the pole is simple and $\text{Res}(f, i) = 1/2$.

At $z = -i$

$$f(z) = \frac{1}{2} \cdot \frac{1}{(z+i)} + \text{something analytic at } -i$$

Therefore the pole is simple and $\text{Res}(f, -i) = 1/2$. ◀

► Let $f(z) = \frac{2+z+z^2}{(z-2)(z-3)(z-4)(z-5)}$. Show all the poles are simple and compute their residues.

The poles are at $z = 2, 3, 4, 5$. They are all isolated. We'll look at $z=2$ the others are similar. Multiplying by $z - 2$ we get:

$$g(z) = (z-2)f(z) = \frac{2+z+z^2}{(z-3)(z-4)(z-5)}$$

This is analytic at $z = 2$ and

$$g(z) = \frac{8}{-6} = -\frac{4}{3}$$

So the pole is simple and $\text{Res}(f, 2) = -\frac{4}{3}$ ◀