Chapter one

Coordinate systems

1- General curvilinear coordinates

In the general case, the position of a point P having Cartesian coordinates x,y,z may be expressed in terms of the three curvilinear coordinates u_1,u_2,u_3, where:

$$x = x(u_1, u_2, u_3), \qquad y = y(u_1, u_2, u_3), \qquad z = z(u_1, u_2, u_3)$$

and similarly

$$u_1 = u_1\ (x,y,z),\ u_2 = u_2\ (x,y,z),\ u_3 = u_3\ (x,y,z).$$

The u_1, u_2 and u_3 coordinate curves of a general curvilinear system are analogous to the x, y and z axes of Cartesian coordinates. The surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$, where c_1, c_2, c_3 are constants, are called the coordinate surfaces and each pair of these surfaces has its intersection in a curve called a coordinate curve or line (see figure 1).

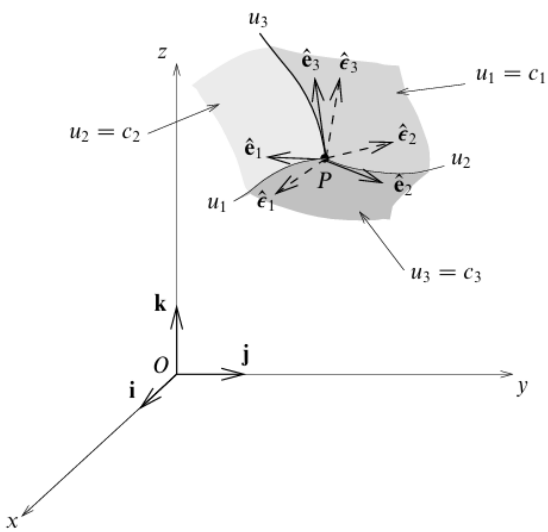

Figure (1): General curvilinear coordinates.

If at each point in space the three coordinate surfaces passing through the point meet at right angles then the curvilinear coordinate system is called orthogonal. For example, in spherical polars $u_1 = r$, $u_2 = \theta$, $u_3 = \varphi$ and the three coordinate surfaces passing through the point (R,Θ,Φ) are the sphere $r = R$, the circular cone $\theta = \Theta$ and the plane $\varphi = \Phi$, which intersect at right angles at that point. Therefore spherical polars form an orthogonal coordinate system (as do cylindrical polars).

If $\mathbf{r}(u_1, u_2, u_3)$ is the position vector of the point P then $e_1 = \partial\mathbf{r}/\partial u_1$ is a vector tangent to the u_1 curve at P (for which u_2 and u_3 are constants) in the direction of increasing u_1. Similarly, e_2 =

$\partial\mathbf{r}/\partial u_2$ and $e_3 = \partial\mathbf{r}/\partial u_3$ are vectors tangent to the u_2 and u_3 curves at P in the direction of increasing u_2 and u_3 respectively. Denoting the lengths of these vectors by h_1, h_2 and h_3, the unit vectors in each of these directions are given by:

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \boldsymbol{r}}{\partial u_1}, \qquad \hat{e}_2 = \frac{1}{h_2}\frac{\partial \boldsymbol{r}}{\partial u_2}, \qquad \hat{e}_3 = \frac{1}{h_3}\frac{\partial \boldsymbol{r}}{\partial u_3},$$

where

$$h_1 = \left|\frac{\partial \boldsymbol{r}}{\partial u_1}\right|, \qquad h_2 = \left|\frac{\partial \boldsymbol{r}}{\partial u_2}\right| \ \ and \ \ h_3 = \left|\frac{\partial \boldsymbol{r}}{\partial u_3}\right|$$

The quantities h_1, h_2, h_3 are the scale factors of the curvilinear coordinate system. The element of distance associated with an infinitesimal change du_i in one of the coordinates is $h_i\, du_i$. In the previous section we found that the scale factors for cylindrical and spherical polar coordinates were:

$$for \text{ cartesian coordinates} \quad h_x = 1, \qquad h_y = 1, \qquad h_z = 1,$$

$$for\ cylindrical\ polars \quad h_\rho = 1, \qquad h_\varphi = \rho, \qquad h_z = 1,$$

$$for\ spherical\ polars \quad h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\, sin\theta.$$

An infinitesimal vector displacement in general curvilinear coordinates is given by:

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u_1}\, du_1 + \frac{\partial \boldsymbol{r}}{\partial u_2}\, du_2 + \frac{\partial \boldsymbol{r}}{\partial u_3}\, du_3 \dots\dots\dots\dots\dots (1)$$

$$d\boldsymbol{r} = du_1\, \boldsymbol{e}_1 + du_2\, \boldsymbol{e}_2 + du_3\, \boldsymbol{e}_3 \dots\dots\dots\dots\dots (2)$$

$$d\boldsymbol{r} = h_1 du_1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3 \dots\dots\dots\dots\dots (3)$$

The element of arc length is given by:

$$(ds)^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = h_1^2\, (du_1)^2 + h_2^2\, (du_2)^2 + h_3^2 (du_3)^2$$

The volume element for the coordinate system is:

$$dV = |du_1\, \hat{e}_1 \cdot (du_2\, \hat{e}_2 \times du_3\, \hat{e}_3)|$$

$$= |h_1 \hat{e}_1 \cdot (h_2\, \hat{e}_2 \times h_3\, \hat{e}_3)| du_1\, du_2\, du_3$$

$$= h_1 h_2 h_3 du_1\, du_2\, du_3.$$

$$\begin{aligned}
\nabla\Phi &= \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\hat{\mathbf{e}}_3 \\
\nabla\cdot\mathbf{a} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3a_1) + \frac{\partial}{\partial u_2}(h_3h_1a_2) + \frac{\partial}{\partial u_3}(h_1h_2a_3)\right] \\
\nabla\times\mathbf{a} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]
\end{aligned}$$

Table (1): Vector operators in orthogonal curvilinear coordinates u_1,u_2,u_3. Φ is a scalar field and $\mathbf{a}$ is a vector field.

2- Gradient. Divergence, Curl and Laplacian in Cartesian Coordinate:

Certain differential operations may be performed on scalar and vector fields and have wideranging applications in the physical sciences. The most important operations are those of finding the gradient of a scalar field and the divergence and curl of a vector field.

Central to all these differential operations is the vector operator ∇, which is called del (or sometimes nabla) and in Cartesian coordinates is defined *by*:

$$\boldsymbol{\nabla} \equiv \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \quad \dots\dots\dots\dots\dots (4)$$

The gradient of a scalar field $\varphi(x,y,z)$ is defined by:

$$grad\ \varphi = \boldsymbol{\nabla}\varphi = \boldsymbol{i}\,\frac{\partial\varphi}{\partial x} + \boldsymbol{j}\,\frac{\partial\varphi}{\partial y} + \boldsymbol{k}\,\frac{\partial\varphi}{\partial z} \quad \dots\dots\dots\dots\dots (5)$$

Clearly, $\boldsymbol{\nabla}\varphi$ is a vector field whose x, y and z components are the first partial derivatives of $\varphi(x,y,z)$ with respect to x, y and z respectively. Also note that the vector field $\boldsymbol{\nabla}\varphi$ should not be confused with the vector operator $\varphi\boldsymbol{\nabla}$, which has components $(\varphi\partial/\partial x,\varphi\partial/\partial y,\varphi\partial/\partial z)$.

التدرج هو التغير المكاني بالنسبة للأبعاد الثلاثة، فعند دراسة تدرج المجال المغناطيسي فأننا نقيس معدل وجهة تغير المجال المغناطيسي في الفراغ وانتقاله من خلال الابعاد الثلاثة.

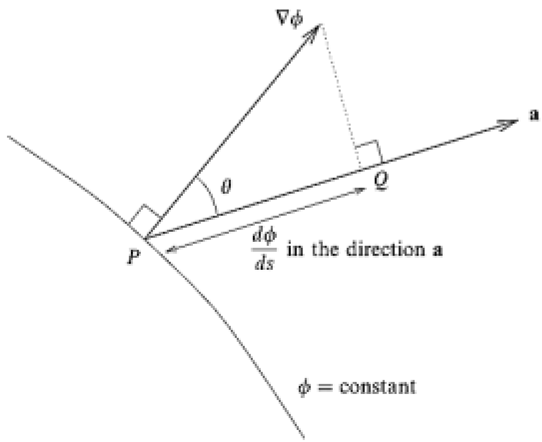

Figure (2): Geometrical properties of ∇φ. PQ gives the value of $d\varphi\ ds$ in the direction a

▶Find the gradient of the scalar field $\varphi = xy^2z^3$

$$\nabla\varphi = y^2z^3\boldsymbol{i} + 2xyz^3\boldsymbol{j} + 3xy^2z^2\boldsymbol{k}.\ ◀$$

The divergence of a vector field **a**(x,y,z) is defined by:

$$div\ \boldsymbol{a} = \nabla\cdot\boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \dots\dots\dots\dots\dots(6)$$

Where a_x, a_y and a_z are the x, y and z components of **a**. Clearly, $\nabla\cdot$**a** is a scalar field. Any vector field **a** for which $\nabla\cdot$ **a** = 0 is said to be solenoidal. (ملف لولبي)

التفريق هو معدل تباعد المجال عن نقطة معينة، ويدل ذلك على ان النقطة التي حسب فيها التفريق هي منبعا للمجال. عند دراسة تفريق المجال المغناطيسي فأننا نقيس معدل انتشار او تباعد المجال عند النقطة المفروضة للمجال.

▶Find the divergence of the vector field **a** = $x^2y^2\boldsymbol{i} + y^2z^2\boldsymbol{j} + x^2z^2\boldsymbol{k}$.

$$\nabla\cdot\mathbf{a} = 2xy^2 + 2yz^2 + 2x^2z = 2(xy^2 + yz^2 + x^2z).\ ◀$$

Now if some vector field **a** is itself derived from a scalar field via **a** = $\nabla\varphi$ then $\nabla\cdot$**a** has the form $\nabla\cdot\nabla\varphi$ or, as it is usually written, $\nabla^2\varphi$, where ∇^2 (del squared) is the scalar differential operator:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \dots\dots\dots\dots\dots(7)$$

$\nabla^2\varphi$ is called the Laplacian of φ and appears in several important partial differential equations of mathematical physics.

▶Find the Laplacian of the scalar field $\varphi = xy^2z^3$.

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 2xz^3 + 6xy^2z \blacktriangleleft$$

The curl of a vector field **a**(x,y,z) is defined by:

$$\text{curl } \mathbf{a} = \nabla \times \mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\boldsymbol{k} \ldots\ldots\ldots\ldots\ldots (8)$$

where a_x, a_y and a_z are the x, y and z components of **a**. The RHS can be written in **a** more memorable form as **a** determinant:

$$\nabla \times \boldsymbol{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (9)$$

Where it is understood that, on expanding the determinant, the partial derivatives in the second row act on the components of **a** in the third row. Clearly, ∇×**a** is itself a vector field. Any vector field **a** for which ∇×**a**=0 is said to be irrotational. (غير دوراني)

ان التدوير هو مدى دوران المجال عند أي نقطة، فعند دراسة دوران المجال المغناطيسي فأننا نقيس معدل دوران المجال المغناطيسي حول النقطة المفروضة. مثلا على ذلك خطوط المجل المغناطيسي للكرة الأرضية تخرج من القطب الجنوبي (المصدر) وتتجه الى القطب الشمالي (المصرف). وعند دراسة تفريق المجال المغناطيسي حول الأرض فالنتيجة صفر.

▶Find the curl of the vector field **a**$=x^2y^2z^2\,\boldsymbol{i} + y^2z^2\,\boldsymbol{j} + x^2z^2\,\boldsymbol{k}$.

$$\nabla\times a = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2z^2 & y^2z^2 & x^2z^2 \end{vmatrix} = -2\,y^2z\,\boldsymbol{i} - 2\,(xz^2 - x^2y^2z)\,\boldsymbol{j} - 2\,x^2yz^2\,\boldsymbol{k} \,.\blacktriangleleft$$

$$\nabla(\varphi+\psi)=\nabla\varphi+\nabla\psi$$

$$\nabla\cdot(\mathbf{a}+\mathbf{b})=\nabla\cdot\mathbf{a}+\nabla\cdot\mathbf{b}$$

$$\nabla\times(\mathbf{a}+\mathbf{b})=\nabla\times\mathbf{a}+\nabla\times\mathbf{b}$$

$$\nabla(\varphi\psi)=\varphi\nabla\psi+\psi\nabla\varphi$$

$$\nabla(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})+(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}$$

$$\nabla\cdot(\varphi\mathbf{a})=\varphi\nabla\cdot\mathbf{a}+\mathbf{a}\cdot\nabla\varphi$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\varphi\mathbf{a})=\nabla\varphi\times\mathbf{a}+\varphi\nabla\times\mathbf{a}$$

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$$

Table (2): Vector operators acting on sums and products. The operator ∇ is defined in eq.(4); φ and ψ are scalar fields, a and b are vector fields.

2-2 Combinations of grad, div and curl:

If φ is a scalar field and **a** is a vector field, these four combinations are grad(grad φ), div(div **a**), curl(div **a**) and grad(curl **a**). Of the five valid combinations of grad, div and curl, two are identically zero, namely

$$curl\ grad\ \varphi = \nabla\times\nabla\varphi = 0 \ldots\ldots\ldots\ldots\ldots (10)$$

$$div\ curl\ \boldsymbol{a} = \nabla\cdot(\nabla\times\boldsymbol{a}) = 0 \ldots\ldots\ldots\ldots\ldots (11)$$

We see that if **a** is derived from the gradient of some scalar function such that **a** = ∇φ then it is necessarily irrotational ($\nabla\times\mathbf{a} = 0$). We also note that if **a** is an irrotational vector field then another irrotational vector field is **a** +∇φ+c, where φ is any scalar field and c is a constant vector. This follows since:

$$\nabla\times(a+\nabla\varphi+c) = \nabla\times a+\nabla\times\nabla\varphi = 0 \ldots\ldots\ldots\ldots\ldots (12)$$

Similarly, from (8) we may infer that if **b** is the curl of some vector field **a** such that **b** = ∇×**a** then **b** is solenoidal ($\nabla\cdot\mathbf{b} = 0$). Obviously, if **b** is solenoidal and c is any constant vector then **b** + c is also solenoidal.

The three remaining combinations of grad, div and curl are:

$$div\ grad\ \varphi = \nabla \cdot \nabla\varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \dots\dots\dots\dots\dots (13)$$

$$\begin{aligned} Grad\ div\ a &= \nabla(\nabla \cdot a) \\ &= \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_y}{\partial x \partial y} + \frac{\partial^2 a_z}{\partial x \partial z}\right) i + \left(\frac{\partial^2 a_x}{\partial y \partial x} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_z}{\partial y \partial z}\right) j \\ &+ \left(\frac{\partial^2 a_x}{\partial z \partial x} + \frac{\partial^2 a_y}{\partial z \partial y} + \frac{\partial^2 a_z}{\partial z^2}\right) k \dots\dots\dots\dots\dots (14) \end{aligned}$$

$$curl\ curl\ \boldsymbol{a} = \nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a} \dots\dots\dots\dots\dots (15)$$

The term $\nabla^2\mathbf{a}$ has the linear differential operator ∇^2 acting on a vector (as opposed to a scalar as in (13)), which of course consists of a sum of unit vectors multiplied by components. Two cases arise.

(i) If the unit vectors are constants (i.e. they are independent of the values of the coordinates) then the differential operator gives a non-zero contribution only when acting upon the components, the unit vectors being merely multipliers.

إذا كانت متجهات الوحدة ثابتة (أي أنها مستقلة عن قيم الإحداثيات)، فإن عامل التفاضل يعطي مساهمة غير صفرية فقط عند العمل على المركبات، حيث تكون متجهات الوحدة مجرد مضاعفات.

(ii) If the unit vectors vary as the values of the coordinates change (i.e. are not constant in direction throughout the whole space) then the derivatives of these vectors appear as contributions to ∇^2a.

إذا تغيرت متجهات الوحدة مع تغير قيم الإحداثيات (أي أنها ليست ثابتة في الاتجاه في جميع أنحاء المساحة)، فإن مشتقات هذه المتجهات تظهر كمساهمات في ∇^2a

▶Show that $\nabla\cdot(\nabla\varphi\times\nabla\psi)=0$, where φ and ψ are scalar fields.

From the table(1) we have:

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b}).$$

If we let $\mathbf{a} = \nabla\varphi$ and $\mathbf{b} = \nabla\psi$ then we obtain

$$\nabla\cdot(\nabla\varphi\times\nabla\psi)=\nabla\psi\cdot(\nabla\times\nabla\varphi)-\nabla\varphi\cdot(\nabla\times\nabla\psi)=0,$$

since $\nabla\times\nabla\varphi=0=\nabla\times\nabla\psi$, from (7). ◄

3- Cylindrical polar coordinates:

As shown in figure (2), the position of a point in space *P* having Cartesian coordinates *x,y,z* may be expressed in terms of cylindrical polar coordinates $\rho,\ \varphi, z,$ where:

$$x = \rho cos\varphi, y = \rho sin\varphi, z = z \dots\dots\dots\dots\dots (16)$$

and $\rho \geq 0, 0 \leq \varphi < 2\pi,\ and -\infty < z < \infty$ .The position vector of *P* may therefore be written

$$\boldsymbol{r} = \rho\ cos\varphi\ \boldsymbol{i} + \rho\ sin\varphi\ \boldsymbol{j} + z\ \boldsymbol{k} \dots\dots\dots\dots\dots\ (17)$$

If we take the partial derivatives of **r** with respect to $\rho,\ \varphi$ and z respectively then we obtain the three vectors:

$$\boldsymbol{e_\rho} = \frac{\partial \boldsymbol{r}}{\partial \rho} = cos\varphi\ \boldsymbol{i} + sin\varphi\ \boldsymbol{j} \dots\dots\dots\dots\dots\dots (18)$$

$$\boldsymbol{e_\varphi} = \frac{\partial \boldsymbol{r}}{\partial \varphi} = -\rho\ sin\varphi\ \boldsymbol{i} + \rho\ cos\varphi\ \boldsymbol{j} \dots\dots\dots\dots\dots (19)$$

$$\boldsymbol{e_z} = \frac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{k} \dots\dots\dots\dots\dots\dots (20)$$

These vectors lie in the directions of increasing ρ, φ and z respectively but are not all of unit length. it is usual to work with the corresponding unit vectors, which are obtained by dividing each vector by its modulus to give:

$$\hat{e}_\rho = \boldsymbol{e_\rho} = cos\varphi\ \boldsymbol{i} + sin\varphi\ \boldsymbol{j} \dots\dots\dots\dots\dots\dots (21)$$

$$\hat{e}_\varphi = \frac{1}{\rho}\boldsymbol{e_\varphi} = -sin\varphi\ \boldsymbol{i} + cos\varphi\ \boldsymbol{j} \dots\dots\dots\dots\dots (22)$$

$$\hat{e}_z = \boldsymbol{e_z} = \boldsymbol{k} \dots\dots\dots\dots\dots\dots (23)$$

These three unit vectors, like the Cartesian unit vectors ***i***, ***j*** and ***k*,** form an orthonormal triad at each point in space.

The expression for a general infinitesimal vector displacement d**r** in the position of *P* is given by:

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \rho}\ d\rho + \frac{\partial \boldsymbol{r}}{\partial \varphi} d\varphi + \frac{\partial \boldsymbol{r}}{\partial z}\ dz.$$

$$d\boldsymbol{r} = d\rho\ \boldsymbol{e_\rho} + d\varphi\ \boldsymbol{e_\varphi} + dz\ \boldsymbol{e_z}$$

$$d\boldsymbol{r} = d\rho\, \hat{e}_\rho + \rho\, d\varphi\, \hat{e}_\varphi + dz\, \hat{e}_z \dots\dots\dots\dots\dots (24)$$

This expression illustrates an important difference between Cartesian and cylindrical polar coordinates (or non-Cartesian coordinates in general). In Cartesian coordinates, the distance moved in going from x to $x + dx$, with y and z held constant, is simply $ds = dx$. However, in cylindrical polars, if φ changes by $d\varphi$, with ρ and z held constant, then the distance moved is not $d\varphi$, but $ds = \rho d\varphi$.

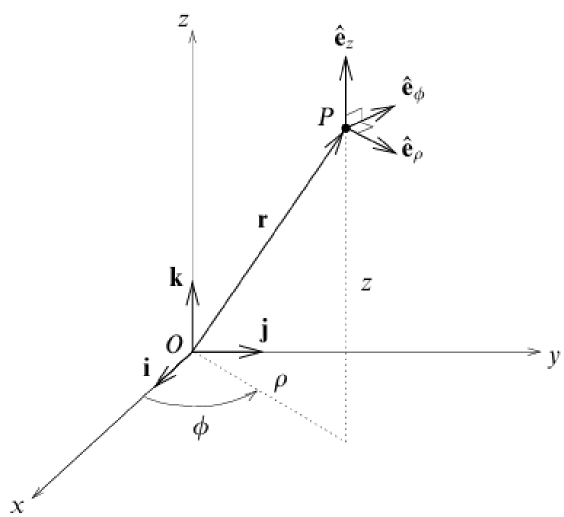

figure (3): Cylindrical polar coordinates ρ,φ,z.

Factors, such as the ρ in $\rho d\varphi$, that multiply the coordinate differentials to give distances are known as scale factors. From (24), the scale factors for the ρ, φ and z coordinates are therefore 1, ρ and 1 respectively.

The magnitude ds of the displacement d**r** is given in cylindrical polar coordinates by:

$$(ds)^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = (d\rho)^2 + \rho^2(d\varphi)^2 + (dz)^2 \dots\dots\dots\dots\dots (25)$$

Where in the second equality we have used the fact that the basis vectors are orthonormal. We can also find the volume element in a cylindrical polar system (see figure 4) by calculating the volume of the infinitesimal parallelepiped defined by the vectors $d\rho\hat{e}_\rho$, $\rho d\varphi\hat{e}_\varphi$ and $dz\hat{e}_z$:

$$dV = \left| d\rho\hat{e}_\rho \cdot \left(\rho d\varphi\hat{e}_\varphi \times dz\, \hat{e}_z\right)\right| = \rho d\rho d\varphi dz$$

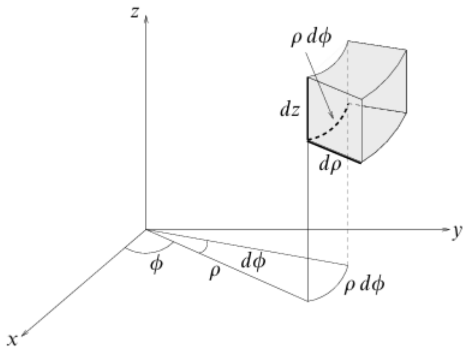

Figure (4): The element of volume in cylindrical polar coordinates is given by $\rho\ d\rho\ d\varphi\ dz$.

The expressions for grad, div, curl and ∇^2 can then be calculated and are given in table (3):

$$\nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z$$

$$\nabla\cdot\mathbf{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z}$$

$$\nabla\times\mathbf{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Table (3): Vector operators in cylindrical polar coordinates; Φ is a scalar field and **a** is a vector field.

Let us consider a vector field $\mathbf{a}(\rho,\varphi,z)$ and a scalar field $\Phi(\rho,\varphi,z)$, where we use Φ for the scalar field to avoid confusion with the azimuthal angle φ. We must first write the vector field in terms of the basis vectors of the cylindrical polar coordinate system, i.e.

$$\boldsymbol{a} = a_\rho\ \hat{e}_\rho + \rho\ a_\varphi\ \hat{e}_\varphi\ + a_z\ \ \hat{e}_z$$

where a_ρ, a_φ and a_z are the components of a in the ρ, φ and z directions respectively.

▶Express the vector field $\mathbf{a} = yz\,\boldsymbol{i} - y\,\boldsymbol{j} + xz^2\,\boldsymbol{k}$ in cylindrical polar coordinates, and hence calculate its divergence. Show that the same result is obtained by evaluating the divergence in Cartesian coordinates.

The basis vectors of the cylindrical polar coordinate system are given in (21)–(23). Solving these equations simultaneously for $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ we obtain:

$$\boldsymbol{i} = cos\varphi\ \hat{e}_\rho - sin\varphi\ \hat{e}_\varphi$$

$$\boldsymbol{j} = sin\varphi\ \hat{e}_\rho + cos\varphi\ \hat{e}_\varphi$$

$$\boldsymbol{k} = \hat{e}_z$$

Substituting these relations and (13) into the expression for **a** we find:

$$\begin{aligned}\boldsymbol{a} &= z\rho sin\varphi\left(cos\varphi\ \hat{e}_\rho - sin\varphi\ \hat{e}_\varphi\right) - \rho sin\varphi\left(sin\varphi\ \hat{e}_\rho + cos\varphi\ \hat{e}_\varphi\right) + z^2\rho\ cos\varphi\ \hat{e}_z \\ &= (z\ \rho\ sin\varphi\ cos\varphi - \rho\ sin^2\varphi)\hat{e}_\rho - (z\ \rho\ sin^2\ \varphi + \rho\ sin\varphi\ cos\varphi)\hat{e}_\varphi \\ &\quad + z^2\rho\ cos\varphi\ \hat{e}_z.\end{aligned}$$

Substituting into the expression for $\nabla \cdot$ **a** given in table 2:

$$\begin{aligned}\nabla\cdot\boldsymbol{a} &= 2z\ sin\varphi\ cos\varphi - 2\ sin^2\varphi - 2z\ sin\varphi\ cos\varphi - cos^2\varphi + sin^2\varphi + 2z\ \rho\ cos\varphi \\ &= 2z\ \rho\ cos\varphi - 1.\end{aligned}$$

Alternatively, and much more quickly in this case, we can calculate the divergence directly in Cartesian coordinates. We obtain

$$\nabla\cdot\boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2\ z\ x - 1$$

which on substituting $x = \rho\ cos\varphi$ yields the same result as the calculation in cylindrical polars. ◀

4- Spherical polar coordinates:

As shown in figure (4), the position of a point in space P, with Cartesian coordinates x,y,z, may be expressed in terms of spherical polar coordinates r, θ, φ,where:

$$x = rsin\theta cos\varphi, \qquad y = rsin\theta sin\varphi, \qquad z = rcos\theta \ldots\ldots\ldots\ldots\ldots (26)$$

and $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. The position vector of ` may therefore be written as:

$$\boldsymbol{r} = r\ sin\theta\ cos\varphi\ \boldsymbol{i} + r\ sin\theta\ sin\varphi\ \boldsymbol{j} + r\ cos\theta\ \boldsymbol{k} \ldots\ldots\ldots\ldots\ldots (27)$$

If, in a similar manner to that used in the previous section for cylindrical polars, we find the partial derivatives of r with respect to r, θ and φ respectively and divide each of the resulting vectors by its modulus then we obtain the unit basis vectors:

$$\hat{e}_r = \sin\theta\, \cos\varphi\, \boldsymbol{i} + \sin\theta\, \sin\varphi\, \boldsymbol{j} + \cos\theta\, \boldsymbol{k},$$

$$\hat{e}_\theta = \cos\theta\, \cos\varphi\, \boldsymbol{i} + \cos\theta\, \sin\varphi\, \boldsymbol{j} - \sin\theta\, \boldsymbol{k},$$

$$\hat{e}_\varphi = -\sin\varphi\, \boldsymbol{i} + \cos\varphi\, \boldsymbol{j}.$$

These unit vectors are in the directions of increasing *r, θ* and *φ* respectively and are the orthonormal basis set for spherical polar coordinates, as shown in figure (5).

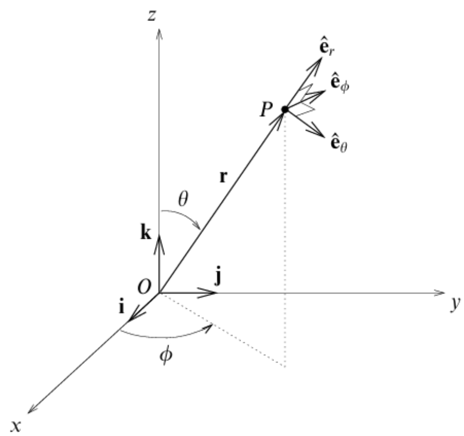

Figure (5): Spherical polar coordinates *r,θ,φ*.

A general infinitesimal vector displacement in spherical polars is:

$$d\boldsymbol{r} = dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta + r\, \sin\theta\, d\varphi\, \hat{e}_\varphi \dots\dots\dots\dots\dots (28)$$

Thus the scale factors for the *r*, *θ* and *φ* coordinates are 1, *r* and *r sinθ* respectively. The magnitude ds of the displacement d**r** is given by:

$$(ds)^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\varphi)^2$$

since the basis vectors form an orthonormal set. The element of volume in spherical polar coordinates (see figure 6) is the volume of the infinitesimal parallelepiped defined by the vectors $dr\, \hat{e}_r$, $rd\theta\, \hat{e}_\theta$ and $r\, \sin\theta\, d\varphi\ \hat{e}_\varphi$ and is given by:

$$dV = \left| dr\, \hat{e}_r \cdot \left(r\, d\theta\, \hat{e}_\theta \times r\, \sin\theta\, d\varphi\, \hat{e}_\varphi \right) \right| = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Where again we use the fact that the basis vectors are orthonormal. The expressions for $(ds)^2$ and *dV* in spherical polars can be obtained from the geometry of this coordinate system.

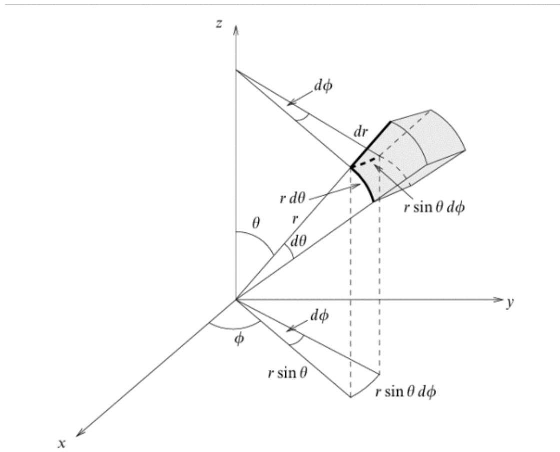

Figure (6): The element of volume in spherical polar coordinates is given by $r^2 \sin\theta \, dr \, d\theta \, d\varphi$.

We will now express the standard vector operators in spherical polar coordinates, using the same techniques as for cylindrical polar coordinates. We consider a scalar field $\Phi(r,\theta,\varphi)$ and a vector field $\mathbf{a}(r,\theta,\varphi)$. The latter may be written in terms of the basis vectors of the spherical polar coordinate system as:

$$\boldsymbol{a} = a_r \, \hat{e}_r + a_\theta \, \hat{e}_\theta + a_\varphi \, \hat{e}_\varphi,$$

where a_r, a_θ and a_φ are the components of a in the r, θ and φ directions respectively. The expressions for grad, div, curl and ∇^2 are given in table (4). The derivations of these results are given in the next section.

$$\begin{aligned}
\nabla\Phi &= \frac{\partial \Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \; a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & r a_\theta & r\sin\theta\; a_\phi \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

Table (4): Vector operators in spherical polar coordinates; Φ is a scalar field and a is a vector field.

We can rewrite the first term on the RHS as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)$$

Which can often be useful in shortening calculations.