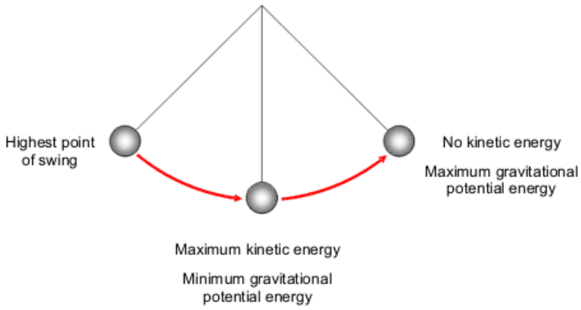
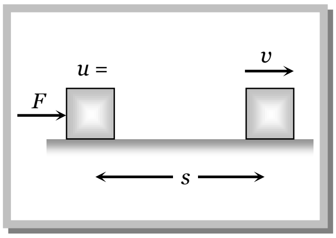
Gravitational Potential Energy, Kinetic Energy, and Conservation of



1. Kinetic Energy

The energy possessed by a body by virtue of its motion is called kinetic energy.
Examples : (i) Flowing water possesses kinetic energy which is used to run the water mills.
(ii) Moving vehicle possesses kinetic energy.
(iii) Moving air (*i.e.* wind) possesses kinetic energy which is used to run wind mills.
(iv) The hammer possesses kinetic energy which is used to drive the nails in wood.
(v) A bullet fired from the gun has kinetic energy and due to this energy the bullet penetrates into a target.



(1) **Expression for kinetic energy :** Let
m = mass of the body, u = Initial velocity of the body (= 0)
F = Force acting on the body, a = Acceleration of the body
s = Distance travelled by the body, v = Final velocity of the body

From $v^2 = u^2 + 2as$

$\Rightarrow \quad v^2 = 0 + 2as \quad \therefore s = \dfrac{v^2}{2a}$

Since the displacement of the body is in the direction of the applied force, then work done by the force is

$$W = F \times s = ma \times \frac{v^2}{2a}$$

$$\Rightarrow \quad W = \frac{1}{2}mv^2$$

This work done appears as the kinetic energy of the body $KE = W = \dfrac{1}{2}mv^2$

(2) **Calculus method :** Let a body is initially at rest and force $\vec{F}$ is applied on the body to displace it through $d\bar{s}$ along its own direction then small work done

$$dW = \vec{F}.\vec{ds} = F\,ds$$

$$\Rightarrow \quad dW = m\,a\,ds \qquad [\text{As } F = ma]$$

$$\Rightarrow \quad dW = m\frac{dv}{dt}ds \qquad \left[\text{As } a = \frac{dv}{dt}\right]$$

$$\Rightarrow \quad dW = mdv.\frac{ds}{dt}$$

$$\Rightarrow \quad dW = m\,v\,dv \qquad \text{.......(i)} \quad \left[\text{As } \frac{ds}{dt} = v\right]$$

Therefore work done on the body in order to increase its velocity from zero to v is given by

$$W = \int_0^v mv\,dv = m\int_0^v v\,dv = m\left[\frac{v^2}{2}\right]_0^v = \frac{1}{2}mv^2$$

This work done appears as the kinetic energy of the body $KE = \frac{1}{2}mv^2$.

In vector form $KE = \frac{1}{2}m(\vec{v}.\vec{v})$

As m and $\vec{v}.\vec{v}$ are always positive, kinetic energy is always positive scalar *i.e.* kinetic energy can never be negative.

(3) **Kinetic energy depends on frame of reference :** The kinetic energy of a person of mass *m,* sitting in a train moving with speed *v,* is zero in the frame of train but $\frac{1}{2}mv^2$ in the frame of the earth.

(4) **Kinetic energy according to relativity :** As we know $E = \frac{1}{2}mv^2$.

But this formula is valid only for ($v << c$) If v is comparable to c (speed of light in free space = $3 \times 10^8\ m/s$) then according to Einstein theory of relativity

$$E = \frac{mc^2}{\sqrt{1-(v^2/c^2)}} - mc^2$$

(5) **Work-energy theorem:** From equation (i) $dW = mv\ dv$.

Work done on the body in order to increase its velocity from u to v is given by

$$W = \int_u^v mv\ dv = m\int_u^v v\ dv = m\left[\frac{v^2}{2}\right]_u^v$$

$$\Rightarrow \qquad W = \frac{1}{2}m[v^2 - u^2]$$

Work done = change in kinetic energy

$$W = \Delta E$$

This is work energy theorem, it states that work done by a force acting on a body is equal to the change produced in the kinetic energy of the body.

This theorem is valid for a system in presence of all types of forces (external or internal, conservative or non-conservative).

If kinetic energy of the body increases, work is positive *i.e.* body moves in the direction of the force (or field) and if kinetic energy decreases work will be negative and object will move opposite to the force (or field).

Examples : (i) In case of vertical motion of body under gravity when the body is projected up, force of gravity is opposite to motion and so kinetic energy of the body decreases and when it falls down, force of gravity is in the direction of motion so kinetic energy increases.

(ii) When a body moves on a rough horizontal surface, as force of friction acts opposite to motion, kinetic energy will decrease and the decrease in kinetic energy is equal to the work done against friction.

(6) **Relation of kinetic energy with linear momentum:** As we know

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\left[\frac{P}{v}\right]v^2 \qquad [\text{As } P = mv]$$

$$\therefore \qquad E = \frac{1}{2}Pv$$

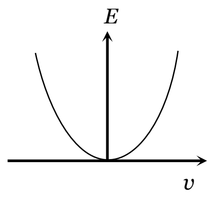
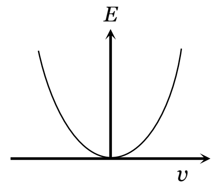
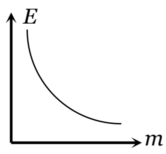
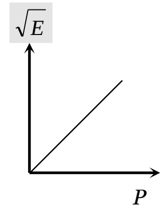
$$\text{or} \qquad E = \frac{P^2}{2m} \qquad \left[\text{As } v = \frac{P}{m}\right]$$

So we can say that kinetic energy $E = \frac{1}{2}mv^2 = \frac{1}{2}Pv = \frac{p^2}{2m}$

and $\qquad$ Momentum $P = \frac{2E}{v} = \sqrt{2mE}$.

From above relation it is clear that a body can not have kinetic energy without having momentum and vice-versa.

(7) **Various graphs of kinetic energy**

 	$E \propto v^2$ m = constant	 	$E \propto P^2$ m = constant
 	$E \propto \dfrac{1}{m}$ P = constant	 	$P \propto \sqrt{E}$ m = constant

Sample problem based on kinetic energy

***Problem* 1.** Consider the following two statements

1. Linear momentum of a system of particles is zero

2. Kinetic energy of a system of particles is zero

Then **[AIEEE 2003]**

(a) 1 implies 2 and 2 implies 1 (b) 1 does not imply 2 and 2 does not imply 1

(c) 1 implies 2 but 2 does not imply 1 (d) 1 does not imply 2 but 2 implies 1

Solution : (d) Momentum is a vector quantity whereas kinetic energy is a scalar quantity. If the kinetic energy of a system is zero then linear momentum definitely will be zero but if the momentum of a system is zero then kinetic energy may or may not be zero.

***Problem* 2.** A running man has half the kinetic energy of that of a boy of half of his mass. The man speeds up by 1 *m/s* so as to have same *K.E.* as that of boy. The original speed of the man will be

(a) $\sqrt{2}\, m/s$ (b) $(\sqrt{2}-1) m/s$ (c) $\dfrac{1}{(\sqrt{2}-1)} m/s$ (d) $\dfrac{1}{\sqrt{2}} m/s$

Solution : (c) Let m = mass of the boy, M = mass of the man, v = velocity of the boy and V = velocity of the man

Initial kinetic energy of man $= \dfrac{1}{2}MV^2 = \dfrac{1}{2}\left[\dfrac{1}{2}mv^2\right] = \dfrac{1}{2}\left[\dfrac{1}{2}\left(\dfrac{M}{2}\right)v^2\right]$ $\left[\text{As } m = \dfrac{M}{2} \text{ given}\right]$

$\Rightarrow \quad V^2 = \dfrac{v^2}{4} \Rightarrow V = \dfrac{v}{2}$ .....(i)

When the man speeds up by 1 *m/s* , $\dfrac{1}{2}M(V+1)^2 = \dfrac{1}{2}mv^2 = \dfrac{1}{2}\left(\dfrac{M}{2}\right)v^2 \Rightarrow (V+1)^2 = \dfrac{v^2}{2}$

$\Rightarrow V+1 = \dfrac{v}{\sqrt{2}}$ .....(ii)

From (i) and (ii) we get speed of the man $V = \dfrac{1}{\sqrt{2}-1} m/s$.

Problem 3. A body of mass 10 *kg* at rest is acted upon simultaneously by two forces 4*N* and 3*N* at right angles to each other. The kinetic energy of the body at the end of 10 sec is **[Kerala (Engg.) 2001]**

(a) 100 *J* (b) 300 *J* (c) 50 *J* (d) 125 *J*

Solution : (d) As the forces are working at right angle to each other therefore net force on the body $F = \sqrt{4^2 + 3^2} = 5N$

Kinetic energy of the body = work done = $F \times s$ $= F \times \frac{1}{2} a t^2 = F \times \frac{1}{2}\left(\frac{F}{m}\right) t^2 = 5 \times \frac{1}{2}\left(\frac{5}{10}\right)(10)^2 = 125\ J.$

Problem 4. If the momentum of a body increases by 0.01%, its kinetic energy will increase by

(a) 0.01% (b) 0.02 % (c) 0.04 % (d) 0.08 %

Solution : (b) Kinetic energy $E = \frac{P^2}{2m}$ $\therefore\ E \propto P^2$

Percentage increase in kinetic energy = 2(% increase in momentum) [If change is very small]

= 2(0.01%) = 0.02%.

Problem 5. If the momentum of a body is increased by 100 %, then the percentage increase in the kinetic energy is **[NCERT 1990; BHU 1999; Pb. PMT 1999; CPMT 1999, 2000; CBSE PMT 2001]**

(a) 150 % (b) 200 % (c) 225 % (d) 300 %

Solution : (d) $E = \frac{P^2}{2m} \Rightarrow \frac{E_2}{E_1} = \left(\frac{P_2}{P_1}\right)^2 = \left(\frac{2P}{P}\right)^2 = 4$

$E_2 = 4\,E_1 = E_1 + 3E_1 = E_1 + 300\ \%$ of E_1.

Problem 6. A body of mass 5 *kg* is moving with a momentum of 10 *kg-m/s*. A force of 0.2 *N* acts on it in the direction of motion of the body for 10 seconds. The increase in its kinetic energy is

(a) 2.8 *J* (b) 3.2 *J* (c) 3.8 *J* (d) 4.4 *J*

Solution : (d) Change in momentum $= P_2 - P_1 = F \times t \Rightarrow P_2 = P_1 + F \times t = 10 + 0.2 \times 10 = 12\,kg\text{-}m/s$

Increase in kinetic energy $E = \frac{1}{2m}[P_2^2 - P_1^2]$

$= \frac{1}{2m}[(12)^2 - (10)^2] = \frac{1}{2 \times 5}[144 - 100] = \frac{44}{10} = 4.4\,J.$

Problem 7. Two masses of 1*g* and 9*g* are moving with equal kinetic energies. The ratio of the magnitudes of their respective linear momenta is

(a) 1 : 9 (b) 9 : 1 (c) 1 : 3 (d) 3 : 1

Solution : (c) $P = \sqrt{2mE}$ $\therefore\ P \propto \sqrt{m}$ if E = constant . So $\frac{P_1}{P_2} = \sqrt{\frac{m_1}{m_2}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

Problem 8. A body of mass 2 *kg* is thrown upward with an energy 490 *J*. The height at which its kinetic energy would become half of its initial kinetic energy will be **[** $g = 9.8\ m/s^2$ **]**

(a) 35 *m* (b) 25 *m* (c) 12.5 *m* (d) 10 *m*

Solution : (c) If the kinetic energy would become half, then Potential energy = $\frac{1}{2}$(Initial kinetic energy)

$\Rightarrow mgh = \frac{1}{2}[490] \Rightarrow 2 \times 9.8 \times h = \frac{1}{2}[490] \Rightarrow h = 12.5\,m$

***Problem* 9.** A 300 *g* mass has a velocity of $(3\hat{i} + 4\hat{j})\,m/sec$ at a certain instant. What is its kinetic energy

(a) 1.35 *J* (b) 2.4 *J* (c) 3.75 *J* (d) 7.35 *J*

Solution : (c) $\vec{v} = (3\hat{i} + 4\hat{j})$ $\therefore\ v = \sqrt{3^2 + 4^2} = 5\,m/s$. So kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.3 \times (5)^2 = 3.75\ J$

2. Stopping of Vehicle by Retarding Force

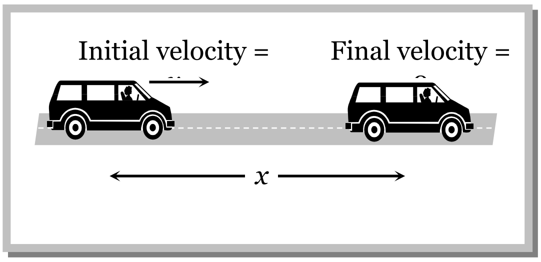
If a vehicle moves with some initial velocity and due to some retarding force it stops after covering some distance after some time.

(1) **Stopping distance :** Let m = Mass of vehicle, v = Velocity, P = Momentum, E = Kinetic energy

F = Stopping force, x = Stopping distance, t = Stopping time

Then, in this process stopping force does work on the vehicle and destroy the motion.

By the work- energy theorem



$$W = \Delta K = \frac{1}{2}mv^2$$

$\Rightarrow$ Stopping force (F) × Distance (x) = Kinetic energy (E)

$$\Rightarrow \text{Stopping distance } (x) = \frac{\text{Kinetic energy } (E)}{\text{Stopping force } (F)}$$

$$\Rightarrow \qquad x = \frac{mv^2}{2F} \qquad .....(i)$$

(2) **Stopping time :** By the impulse-momentum theorem

$$F \times t = \Delta P \Rightarrow F \times t = P$$

$$\therefore \qquad t = \frac{P}{F}$$

$$\text{or} \qquad t = \frac{mv}{F} \qquad .....(ii)$$

(3) **Comparison of stopping distance and time for two vehicles :** Two vehicles of masses m_1 and m_2 are moving with velocities v_1 and v_2 respectively. When they are stopped by the same retarding force (F).

The ratio of their stopping distances $\frac{x_1}{x_2} = \frac{E_1}{E_2} = \frac{m_1 v_1^2}{m_2 v_2^2}$

and the ratio of their stopping time $\frac{t_1}{t_2} = \frac{P_1}{P_2} = \frac{m_1 v_1}{m_2 v_2}$

If vehicles possess same velocities		
$v_1 = v_2$	$\frac{x_1}{x_2} = \frac{m_1}{m_2}$	$\frac{t_1}{t_2} = \frac{m_1}{m_2}$
If vehicle possess same kinetic momentum		

$P_1 = P_2$	$\frac{x_1}{x_2} = \frac{E_1}{E_2} = \left(\frac{P_1^2}{2m_1}\right)\left(\frac{2m_2}{P_2^2}\right) = \frac{m_2}{m_1}$	$\frac{t_1}{t_2} = \frac{P_1}{P_2} = 1$
If vehicle possess same kinetic energy		
$E_1 = E_2$	$\frac{x_1}{x_2} = \frac{E_1}{E_2} = 1$	$\frac{t_1}{t_2} = \frac{P_1}{P_2} = \frac{\sqrt{2m_1E_1}}{\sqrt{2m_2E_2}} = \sqrt{\frac{m_1}{m_2}}$

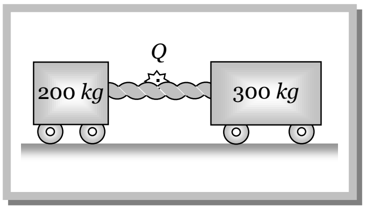
Note : ❑ If vehicle is stopped by friction then

Stopping distance $x = \frac{\frac{1}{2}mv^2}{F} = \frac{\frac{1}{2}mv^2}{ma} = \frac{v^2}{2\mu g}$ [As $a = \mu g$]

Stopping time $t = \frac{mv}{F} = \frac{mv}{m\mu g} = \frac{v}{\mu g}$

Sample problems based on stopping of vehicle

***Problem* 1.** Two carts on horizontal straight rails are pushed apart by an explosion of a powder charge Q placed between the carts. Suppose the coefficients of friction between the carts and rails are identical. If the 200 *kg* cart travels a distance of 36 *metres* and stops, the distance covered by the cart weighing 300 *kg* is **[CPMT 1989]**



(a) 32 *metres* (b) 24 *metres*
(c) 16 *metres* (d) 12 *metres*

Solution : (c) Kinetic energy of cart will goes against friction. $\therefore\ E = \frac{P^2}{2m} = \mu mg \times s \Rightarrow s = \frac{P^2}{2\mu gm^2}$

As the two carts pushed apart by an explosion therefore they possess same linear momentum and coefficient of friction is same for both carts (given). Therefore the distance covered by the cart before coming to rest is given by

$$s \propto \frac{1}{m^2} \quad \therefore \frac{s_2}{s_1} = \left(\frac{m_1}{m_2}\right)^2 = \left(\frac{200}{300}\right)^2 = \frac{4}{9} \Rightarrow S_2 = \frac{4}{9} \times 36 = 16 \textit{ metres}.$$

***Problem* 2.** An unloaded bus and a loaded bus are both moving with the same kinetic energy. The mass of the latter is twice that of the former. Brakes are applied to both, so as to exert equal retarding force. If x_1 and x_2 be the distance covered by the two buses respectively before coming to a stop, then

(a) $x_1 = x_2$ (b) $2x_1 = x_2$ (c) $4x_1 = x_2$ (d) $8x_1 = x_2$

Solution : (a) If the vehicle stops by retarding force then the ratio of stopping distance $\frac{x_1}{x_2} = \frac{E_1}{E_2}$.

But in the given problem kinetic energy of bus and car are given same *i.e.* $E_1 = E_2$. $\therefore x_1 = x_2$.

***Problem* 3.** A bus can be stopped by applying a retarding force F when it is moving with a speed v on a level road. The distance covered by it before coming to rest is s. If the load of the bus increases by 50 % because of passengers, for the same speed and same retarding force, the distance covered by the bus to come to rest shall be

(a) 1.5 s (b) 2 s (c) 1 s (d) 2.5 s

Solution : (a) Retarding force (F) × distance covered (x) = Kinetic energy $\left(\frac{1}{2}mv^2\right)$

If v and F are constants then $x \propto m$ $\therefore \frac{x_2}{x_1} = \frac{m_2}{m_1} = \frac{1.5\,m}{m} = 1.5 \Rightarrow x_2 = 1.5\,s$.

***Problem* 4.** A vehicle is moving on a rough horizontal road with velocity v. The stopping distance will be directly proportional to

(a) $\sqrt{v}$ (b) v (c) v^2 (d) v^3

Solution : (c) As $s = \frac{v^2}{2a}$ $\therefore$ $s \propto v^2$.

3. Potential Energy

Potential energy is defined only for conservative forces. In the space occupied by conservative forces every point is associated with certain energy which is called the energy of position or potential energy. Potential energy generally are of three types : Elastic potential energy, Electric potential energy and Gravitational potential energy *etc.*

(1) **Change in potential energy :** Change in potential energy between any two points is defined in the terms of the work done by the associated conservative force in displacing the particle between these two points without any change in kinetic energy.

$$U_2 - U_1 = -\int_{r_1}^{r_2} \vec{F}.d\vec{r} = -W \qquad ......(i)$$

We can define a unique value of potential energy only by assigning some arbitrary value to a fixed point called the reference point. Whenever and wherever possible, we take the reference point at infinite and assume potential energy to be zero there, *i.e.* if take $r_1 = \infty$ and $r_2 = r$ then from equation (i)

$$U = -\int_{\infty}^{r} \vec{F}.d\vec{r} = -W$$

In case of conservative force (field) potential energy is equal to negative of work done in shifting the body from reference position to given position.

This is why in shifting a particle in a conservative field (say gravitational or electric), if the particle moves opposite to the field, work done by the field will be negative and so change in potential energy will be positive *i.e.* potential energy will increase. When the particle moves in the direction of field, work will be positive and change in potential energy will be negative *i.e.* potential energy will decrease.

(2) **Three dimensional formula for potential energy:** For only conservative fields $\vec{F}$ equals the negative gradient $(-\vec{\nabla})$ of the potential energy.

So $\vec{F} = -\vec{\nabla}U$ ($\vec{\nabla}$ read as Del operator or Nabla operator and $\vec{\nabla} = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}$)

$$\Rightarrow \qquad \vec{F} = -\left[\frac{dU}{dx}\hat{i} + \frac{dU}{dy}\hat{j} + \frac{dU}{dz}\hat{k}\right]$$

where $\frac{dU}{dx}$ = Partial derivative of U *w.r.t.* x (keeping y and z constant)

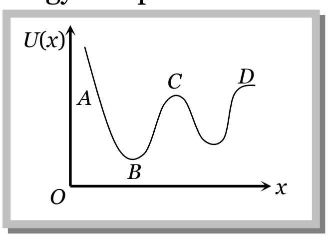
$\frac{dU}{dy}$ = Partial derivative of U *w.r.t.* y (keeping x and z constant)

$\frac{dU}{dz}$ = Partial derivative of U *w.r.t.* z (keeping x and y constant)

(3) **Potential energy curve :** A graph plotted between the potential energy of a particle and its displacement from the centre of force is called potential energy curve.

Figure shows a graph of potential energy function $U(x)$ for one dimensional motion.

As we know that negative gradient of the potential energy gives force.



$\therefore\ -\dfrac{dU}{dx} = F$

(4) **Nature of force :**

(i) Attractive force : On increasing x, if U increases $\dfrac{dU}{dx}$ = positive

then F is negative in direction *i.e.* force is attractive in nature. In graph this is represented in region BC.

(ii) Repulsive force : On increasing x, if U decreases $\dfrac{dU}{dx}$ = negative

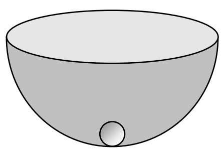
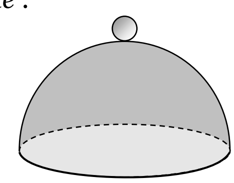
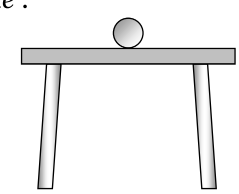
then F is positive in direction *i.e.* force is repulsive in nature. In graph this is represented in region AB.

(iii) Zero force : On increasing x, if U does not changes $\dfrac{dU}{dx} = 0$

then F is zero *i.e.* no force works on the particle. Point B, C and D represents the point of zero force or these points can be termed as position of equilibrium.

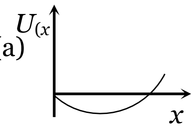
(5) **Types of equilibrium :** If net force acting on a particle is zero, it is said to be in equilibrium.

For equilibrium $\dfrac{dU}{dx} = 0$, but the equilibrium of particle can be of three types :

Stable	**Unstable**	**Neutral**
When a particle is displaced slightly from a position, then a force acting on it brings it back to the initial position, it is said to be in stable equilibrium position.	When a particle is displaced slightly from a position, then a force acting on it tries to displace the particle further away from the equilibrium position, it is said to be in unstable equilibrium.	When a particle is slightly displaced from a position then it does not experience any force acting on it and continues to be in equilibrium in the displaced position, it is said to be in neutral equilibrium.
Potential energy is minimum.	Potential energy is maximum.	Potential energy is constant.
$F = -\dfrac{dU}{dx} = 0$	$F = -\dfrac{dU}{dx} = 0$	$F = -\dfrac{dU}{dx} = 0$
$\dfrac{d^2U}{dx^2}$ = positive *i.e.* rate of change of $\dfrac{dU}{dx}$ is positive.	$\dfrac{d^2U}{dx^2}$ = negative *i.e.* rate of change of $\dfrac{dU}{dx}$ is negative.	$\dfrac{d^2U}{dx^2} = 0$ *i.e.* rate of change of $\dfrac{dU}{dx}$ is zero.
Example :  A marble placed at the bottom of a hemispherical bowl.	*Example* :  A marble balanced on top of a hemispherical bowl.	*Example* :  A marble placed on horizontal table.

***S**ample problems based on potential energy*

***Problem* 1.** A particle which is constrained to move along the x-axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin as $F(x) = -kx + ax^3$. Here k and a are positive constants. For $x \geq 0$, the functional from of the potential energy $U_{(x)}$ of the particle is **[IIT-JEE (Screening) 2002]**

(a)


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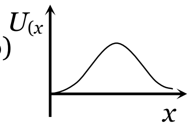

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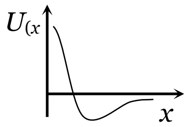
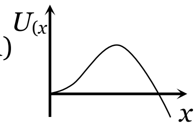


(d)


Solution : (d) $F = -\frac{dU}{dx} \Rightarrow dU = -F.dx \Rightarrow U = -\int_0^x (-kx + ax^3)dx \Rightarrow U = \frac{kx^2}{2} - \frac{ax^4}{4}$

$\therefore$ We get $U = 0$ at $x = 0$ and $x = \sqrt{\frac{2k}{a}}$ Also we get $U =$ negative for $x > \sqrt{\frac{2k}{a}}$

From the given function we can see that $F = 0$ at $x = 0$ *i.e.* slope of U-x graph is zero at $x = 0$.

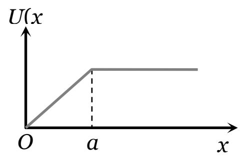
***Problem* 2.** The potential energy of a body is given by $A - Bx^2$ (where x is the displacement). The magnitude of force acting on the particle is

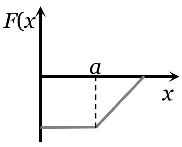
(a) Constant (b) Proportional to x

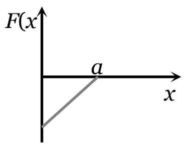
(c) Proportional to x^2 (d) Inversely proportional to x

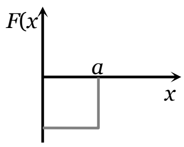
Solution : (b) $F = \frac{-dU}{dx} = -\frac{d}{dx}(A - Bx^2) = 2Bx \quad \therefore F \propto x$.

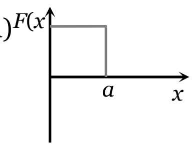
***Problem* 3.** The potential energy of a system is represented in the first figure. The force acting on the system will be represented by



(a)


(b)


(c)


(d)


Solution : (c) As slope of problem graph is positive and constant upto distance a then it becomes zero. Therefore from $F = -\frac{dU}{dx}$ we can say that upto distance a force will be constant (negative) and suddenly it becomes zero.

***Problem* 4.** A particle moves in a potential region given by $U = 8x^2 - 4x + 400$ J. Its state of equilibrium will be

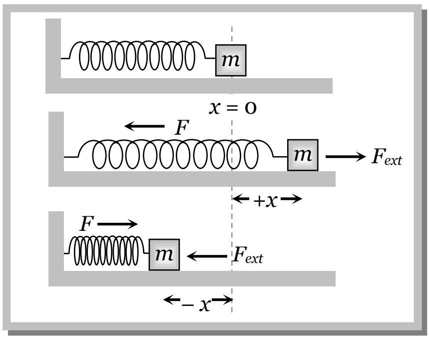
(a) $x = 25\,m$ (b) $x = 0.25\,m$ (c) $x = 0.025\,m$ (d) $x = 2.5\,m$

Solution : (b) $F = -\frac{dU}{dx} = -\frac{d}{dx}(8x^2 - 4x + 400)$

For the equilibrium condition $F = -\frac{dU}{dx} = 0 \Rightarrow 16x - 4 = 0 \Rightarrow x = 4/16 \therefore x = 0.25\,m$.

4. Elastic Potential Energy

(1) **Restoring force and spring constant :** When a spring is stretched or compressed from its normal position ($x = 0$) by a small distance x, then a restoring force is produced in the spring to bring it to the normal position.According to Hooke's law this restoring force is proportional to the displacement x and its direction is always opposite to the displacement.



i.e. $\vec{F} \propto -\vec{x}$

or $\vec{F} = -k\,\vec{x}$ .....(i)

where k is called spring constant.

If $x = 1$, $F = k$ (Numerically)

or $k = F$

Hence spring constant is numerically equal to force required to produce unit displacement (compression or extension) in the spring. If required force is more, then spring is said to be more stiff and vice-versa.

Actually k is a measure of the stiffness/softness of the spring.

Dimension : As $k = \dfrac{F}{x}$ $\therefore\ [k] = \dfrac{[F]}{[x]} = \dfrac{[MLT^{-2}]}{L} = [MT^{-2}]$

Units : S.I. unit *Newton/metre*, C.G.S unit *Dyne/cm*.

Note : ❑ Dimension of force constant is similar to surface tension.

(2) **Expression for elastic potential energy :** When a spring is stretched or compressed from its normal position ($x = 0$), work has to be done by external force against restoring force. $\vec{F}_{\text{ext}} = \vec{F}_{restoring} = k\vec{x}$

Let the spring is further stretched through the distance dx, then work done

$$dW = \vec{F}_{\text{ext}}.d\vec{x} = F_{\text{ext}}.dx\cos 0^o = kx\,dx \qquad [\text{As } \cos 0^o = 1]$$

Therefore total work done to stretch the spring through a distance x from its mean position is given by

$$W = \int_0^x dW = \int_0^x kx\,dx = k\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2}kx^2$$

This work done is stored as the potential energy of the stretched spring.

$\therefore$ Elastic potential energy $U = \dfrac{1}{2}kx^2$

$$U = \frac{1}{2}Fx \qquad \left[\text{As } k = \frac{F}{x}\right]$$

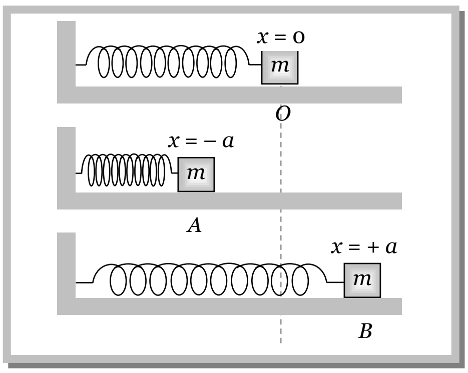
$$U = \frac{F^2}{2k} \qquad \left[\text{As } x = \frac{F}{k}\right]$$

$\therefore$ Elastic potential energy $U = \dfrac{1}{2}kx^2 = \dfrac{1}{2}Fx = \dfrac{F^2}{2k}$

Note : ❑If spring is stretched from initial position x_1 to final position x_2 then work done

= Increment in elastic potential energy $= \dfrac{1}{2}k(x_2^2 - x_1^2)$

(3) **Energy graph for a spring :** If the mass attached with spring performs simple harmonic motion about its mean position then its potential energy at any position (x) can be given by



$$U = \frac{1}{2}kx^2 \qquad ....(i)$$

So for the extreme position

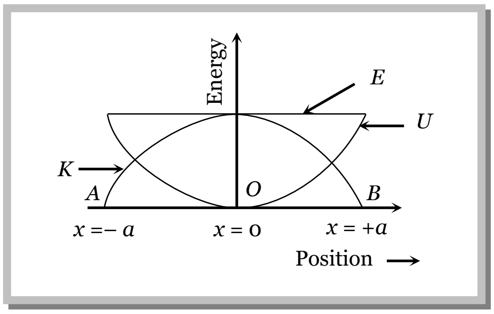
$$U = \frac{1}{2}ka^2 \qquad [\text{As } x = \pm a \text{ for extreme}]$$

This is maximum potential energy or the total energy of mass.

$\therefore$ Total energy $E = \frac{1}{2}ka^2$ ....(ii)

[Because velocity of mass = 0 at extreme $\therefore$ $K = \frac{1}{2}mv^2 = 0$]

Now kinetic energy at any position $K = E - U = \frac{1}{2}ka^2 - \frac{1}{2}kx^2$



$$K = \frac{1}{2}k(a^2 - x^2) \quad ....(iii)$$

From the above formula we can check that

$U_{max} = \frac{1}{2}ka^2$ [At extreme $x = \pm a$] and $U_{min} = 0$ [At mean $x = 0$]

$K_{max} = \frac{1}{2}ka^2$ [At mean $x = 0$]and $K_{min} = 0$ [At extreme $x = \pm a$]

$E = \frac{1}{2}ka^2$ = constant (at all positions)

It mean kinetic energy changes parabolically *w.r.t.* position but total energy remain always constant irrespective to position of the mass

Sample problems based on elastic potential energy

***Problem* 1.** A long spring is stretched by 2 *cm*, its potential energy is *U*. If the spring is stretched by 10 *cm*, the potential energy stored in it will be

(a) $U/25$ (b) $U/5$ (c) $5U$ (d) $25U$

Solution : (d) Elastic potential energy of a spring $U = \frac{1}{2}kx^2$ $\therefore U \propto x^2$

So $\frac{U_2}{U_1} = \left(\frac{x_2}{x_1}\right)^2 \Rightarrow \frac{U_2}{U} = \left(\frac{10\,cm}{2\,cm}\right)^2 \Rightarrow U_2 = 25\,U$

***Problem* 2.** A spring of spring constant $5 \times 10^3 N/m$ is stretched initially by 5 *cm* from the unstretched position. Then the work required to stretch it further by another 5 *cm* is

(a) 6.25 *N-m* (b) 12.50 *N-m* (c) 18.75 *N-m* (d) 25.00 *N-m*

Solution : (c) Work done to stretch the spring from x_1 to x_2

$W = \frac{1}{2}k(x_2^2 - x_1^2) = \frac{1}{2}5 \times 10^3[(10 \times 10^{-2})^2 - (5 \times 10^{-2})^2] = \frac{1}{2} \times 5 \times 10^3 \times 75 \times 10^{-4} = 18.75\ N.m$.

***Problem* 3.** Two springs of spring constants 1500 N/m and 3000 N/m respectively are stretched with the same force. They will have potential energy in the ratio

(a) 4 : 1 (b) 1 : 4 (c) 2 : 1 (d) 1 : 2

Solution : (c) Potential energy of spring $U = \frac{F^2}{2k} \Rightarrow \frac{U_1}{U_2} = \frac{k_2}{k_1} = \frac{3000}{1500} = 2:1$ [If *F* = constant]

***Problem* 4.** A body is attached to the lower end of a vertical spiral spring and it is gradually lowered to its equilibrium position. This stretches the spring by a length *x*. If the same body attached to the same spring is allowed to fall suddenly, what would be the maximum stretching in this case

(a) *x* (b) 2*x* (c) 3*x* (d) *x*/2

Solution : (b) When spring is gradually lowered to it's equilibrium position

$kx = mg \therefore x = \frac{mg}{k}$.

When spring is allowed to fall suddenly it oscillates about it's mean position

Let y is the amplitude of vibration then at lower extreme, by the conservation of energy

$$\Rightarrow \frac{1}{2}ky^2 = mgy \Rightarrow y = \frac{2mg}{k} = 2x.$$

***Problem* 5.** Two equal masses are attached to the two ends of a spring of spring constant k. The masses are pulled out symmetrically to stretch the spring by a length x over its natural length. The work done by the spring on each mass is

(a) $\frac{1}{2}kx^2$ (b) $-\frac{1}{2}kx^2$ (c) $\frac{1}{4}kx^2$ (d) $-\frac{1}{4}kx^2$

Solution : (d) If the spring is stretched by length x, then work done by two equal masses = $\frac{1}{2}kx^2$

So work done by each mass on the spring = $\frac{1}{4}kx^2$ $\therefore$ Work done by spring on each mass = $-\frac{1}{4}kx^2$.

5. Electrical Potential Energy

It is the energy associated with state of separation between charged particles that interact via electric force. For two point charge q_1 and q_2, separated by distance r.

$$U = \frac{1}{4\pi\varepsilon_0}.\frac{q_1q_2}{r}$$

While for a point charge q at a point in an electric field where the potential is V

$$U = qV$$

As charge can be positive or negative, electric potential energy can be positive or negative.

Sample problems based on electrical potential energy

***Problem* 1.** A proton has a positive charge. If two protons are brought near to one another, the potential energy of the system will

(a) Increase (b) Decrease

(c) Remain the same (d) Equal to the kinetic energy

Solution : (a) As the force is repulsive in nature between two protons. Therefore potential energy of the system increases.

***Problem* 2.** Two protons are situated at a distance of 100 *fermi* from each other. The potential energy of this system will be in eV

(a) 1.44 (b) 1.44×10^3 (c) 1.44×10^2 (d) 1.44×10^4

Solution : (d) $U = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r} = \frac{9\times10^9 \times (1.6\times10^{-19})^2}{100\times10^{-15}} = 2.304\times10^{-15}\ J = \frac{2.304\times10^{-15}}{1.6\times10^{-19}} eV = 1.44\times10^4\ eV$

***Problem* 3.** $_{80}Hg^{208}$ nucleus is bombarded by α-particles with velocity 10^7 m/s. If the α-particle is approaching the Hg nucleus head-on then the distance of closest approach will be

(a) $1.115\times10^{-13}\ m$ (b) $11.15\times10^{-13}\ m$ (c) $111.5\times10^{-13}\ m$ (d) Zero

Solution : (a) When α particle moves towards the mercury nucleus its kinetic energy gets converted in potential energy of the system. At the distance of closest approach $\frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}$

$$\Rightarrow \frac{1}{2}\times(1.6\times10^{-27})(10^7)^2 = 9\times10^9\frac{(2.e)(80\ e)}{r} \Rightarrow r = 1.115\times10^{-13}\ m.$$

Problem 4. A charged particle A moves directly towards another charged particle B. For the $(A+B)$ system, the total momentum is P and the total energy is E

(a) P and E are conserved if both A and B are free to move

(b) (a) is true only if A and B have similar charges

(c) If B is fixed, E is conserved but not P

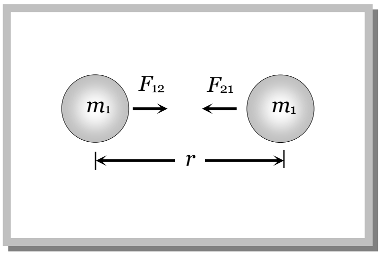
(d) If B is fixed, neither E nor P is conserved

Solution : (a, c) If A and B are free to move, no external forces are acting and hence P and E both are conserved but when B is fixed (with the help of an external force) then E is conserved but P is not conserved.

6. Gravitational Potential Energy

It is the usual form of potential energy and is the energy associated with the state of separation between two bodies that interact via gravitational force.

For two particles of masses m_1 and m_2 separated by a distance r



Gravitational potential energy $U = -\dfrac{Gm_1m_2}{r}$

(1) If a body of mass m at height h relative to surface of earth then

Gravitational potential energy $U = \dfrac{mgh}{1+\dfrac{h}{R}}$

Where R = radius of earth, g = acceleration due to gravity at the surface of the earth.

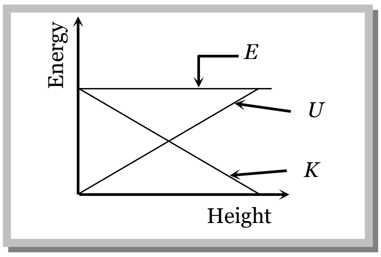
(2) If $h << R$ then above formula reduces to $U = mgh$.

(3) If V is the gravitational potential at a point, the potential energy of a particle of mass m at that point will be

$$U = mV$$

(4) Energy height graph : When a body projected vertically upward from the ground level with some initial velocity then it possess kinetic energy but its potential energy is zero.

As the body moves upward its potential energy increases due to increase in height but kinetic energy decreases (due to decrease in velocity). At maximum height its kinetic energy becomes zero and potential energy maximum but through out the complete motion total energy remains constant as shown in the figure.



Sample problems based on gravitational potential energy

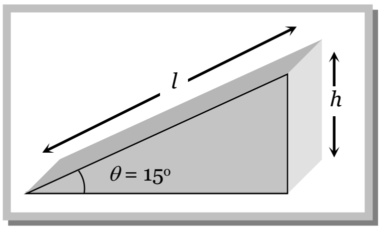
Problem 1. The work done in pulling up a block of wood weighing $2kN$ for a length of 10 m on a smooth plane inclined at an angle of 15^o with the horizontal is ($\sin 15^o = 0.259$)

(a) 4.36 kJ (b) 5.17 kJ (c) 8.91 kJ (d) 9.82 kJ

Solution : (b) Work done = $mg \times h$

$= 2\times 10^3 \times l \sin\theta$

$= 2\times 10^3 \times 10 \times \sin 15^o = 5176\ J = 5.17\ kJ$

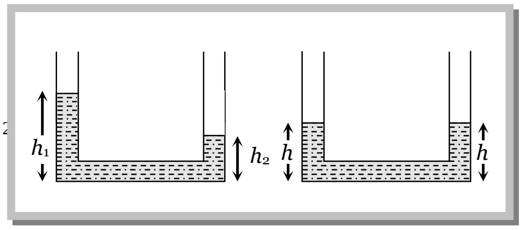


Problem 2. Two identical cylindrical vessels with their bases at same level each contains a liquid of density d. The height of the liquid in one vessel is h_1 and that in the other vessel is h_2. The area of

either vases is A. The work done by gravity in equalizing the levels when the two vessels are connected, is **[SCRA 1996]**

(a) $(h_1 - h_2)gd$ (b) $(h_1 - h_2)gAd$ (c) $\frac{1}{2}(h_1 - h_2)^2\, gAd$ (d) $\frac{1}{4}(h_1 - h_2)^2\, gAd$

Solution : (d) Potential energy of liquid column is given by $mg\frac{h}{2} = Vdg\frac{h}{2} = Ahdg\frac{h}{2} = \frac{1}{2}Adgh^2$



Initial potential energy $= \frac{1}{2}Adgh_1^2 + \frac{1}{2}Adgh_2^2$

Final potential energy = $\frac{1}{2}Adgh^2 + \frac{1}{2}Adh^2g = Adgh^2$

Work done by gravity = change in potential energy

$$W = \left[\frac{1}{2}Adgh_1^2 + \frac{1}{2}Adgh_2^2\right] - Adgh^2$$

$$= Adg\left[\frac{h_1^2}{2} + \frac{h_2^2}{2}\right] - Adg\left(\frac{h_1 + h_2}{2}\right)^2 \qquad \left[\text{As } h = \frac{h_1 + h_2}{2}\right]$$

$$= Adg\left[\frac{h_1^2}{2} + \frac{h_2^2}{2} - \left(\frac{h_1^2 + h_2^2 + 2h_1h_2}{4}\right)\right] = \frac{Adg}{4}(h_1 - h_2)^2$$

Problem **3.** If g is the acceleration due to gravity on the earth's surface, the gain in the potential energy of an abject of mass m raised from the surface of earth to a height equal to the radius of the earth R, is **[IIT-JEE1983]**

(a) $\frac{1}{2}mgR$ (b) $2mgR$ (c) mgR (d) $\frac{1}{4}mgR$

Solution : (a) Work done = gain in potential energy $= \frac{mgh}{1 + h/R} = \frac{mgR}{1 + R/R} = \frac{1}{2}mgR$ [As $h = R$ (given)]

Problem **4.** The work done in raising a mass of 15 gm from the ground to a table of 1m height is

(a) 15 J (b) 152 J (c) 1500 J (d) 0.15 J

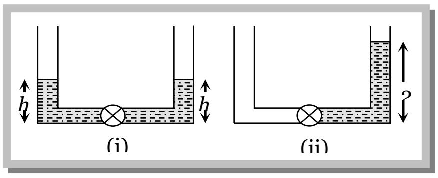
Solution : (d) $W = mgh = 15 \times 10^{-3} \times 10 \times 1 = 0.15\ J$.

Problem **5.** A body is falling under gravity. When it loses a gravitational potential energy by U, its speed is v. The mass of the body shall be

(a) $\frac{2U}{v}$ (b) $\frac{U}{2v}$ (c) $\frac{2U}{v^2}$ (d) $\frac{U}{2v^2}$

Solution : (c) Loss in potential energy = gain in kinetic energy $\Rightarrow U = \frac{1}{2}mv^2 \therefore m = \frac{2U}{v^2}$.

Problem **6.** A liquid of density d is pumped by a pump P from situation (i) to situation (ii) as shown in the diagram. If the cross-section of each of the vessels is a, then the work done in pumping (neglecting friction effects) is



(a) $2dgh$ (b) $dgha$

(c) $2dgh^2a$ (d) dgh^2a

Solution : (d) Potential energy of liquid column in first situation $= Vdg\frac{h}{2} + Vdg\frac{h}{2} = Vdgh = ahdgh = dgh^2a$

[As centre of mass of liquid column lies at height $\frac{h}{2}$]

Potential energy of the liquid column in second situation $= Vdg\left(\frac{2h}{2}\right) = (A \times 2h)dgh = 2dgh^2 a$

Work done pumping = Change in potential energy = $2dgh^2 a - dgh^2 a = dgh^2 a$.

Problem **7.** The mass of a bucket containing water is 10 *kg*. What is the work done in pulling up the bucket from a well of depth 10 *m* if water is pouring out at a uniform rate from a hole in it and there is loss of 2*kg* of water from it while it reaches the top $(g = 10\ m/\sec^2)$

(a) 1000 *J* (b) 800 *J* (c) 900 *J* (d) 500 *J*

Solution : (c) Gravitational force on bucket at starting position = *mg* = 10 × 10 = 100 *N*

Gravitational force on bucket at final position = 8 × 10 = 80 *N*

So the average force through out the vertical motion $= \frac{100 + 80}{2} = 90\ N$

$\therefore$ Work done = Force × displacement = 90 × 10 = 900 *J*.

Problem **8.** A rod of mass *m* and length *l* is lying on a horizontal table. The work done in making it stand on one end will be

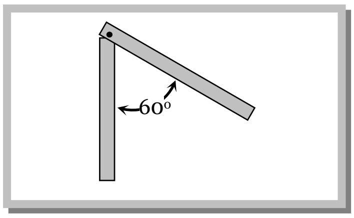
(a) mgl (b) $\frac{mgl}{2}$ (c) $\frac{mgl}{4}$ (d) $2mgl$

Solution : (b) When the rod is lying on a horizontal table, its potential energy = 0

But when we make its stand vertical its centre of mass rises upto high $\frac{l}{2}$. So it's potential energy $= \frac{mgl}{2}$

$\therefore$ Work done = charge in potential energy $= mg\frac{l}{2} - 0 = \frac{mgl}{2}$.

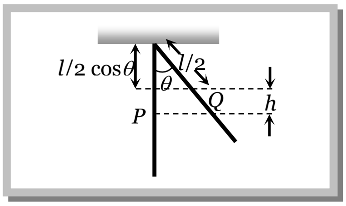
Problem **9.** A *metre* stick, of mass 400 *g*, is pivoted at one end displaced through an angle 60^o. The increase in its potential energy is



(a) 1 *J* (b) 10 *J*

(c) 100 *J* (d) 1000 *J*

Solution : (a) Centre of mass of a stick lies at the mid point and when the stick is displaced through an angle 60° it rises upto height '*h*' from the initial position.



From the figure $h = \frac{l}{2} - \frac{l}{2}\cos\theta = \frac{l}{2}(1 - \cos\theta)$

Hence the increment in potential energy of the stick = *mgh*

$$= mg\frac{l}{2}(1 - \cos\theta) = 0.4 \times 10 \times \frac{1}{2}(1 - \cos 60^o) = 1\ J$$

Problem **10.** Once a choice is made regarding zero potential energy reference state, the changes in potential energy

(a) Are same (b) Are different

(c) Depend strictly on the choice of the zero of potential energy

(d) Become indeterminate

Solution : (a) Potential energy is a relative term but the difference in potential energy is absolute term. If reference level is fixed once then change in potential energy are same always.

7. Law of Conservation of Energy

(1) **Law of conservation of energy**

For a body or an isolated system by work-energy theorem we have $K_2 - K_1 = \int \vec{F}.d\vec{r}$ .....(i)

But according to definition of potential energy in a conservative field $U_2 - U_1 = -\int \vec{F}.d\vec{r}$ .....(ii)

So from equation (i) and (ii) we have

$$K_2 - K_1 = -(U_2 - U_1)$$

or $$K_2 + U_2 = K_1 + U_1$$

i.e. $K + U$ = constant.

For an isolated system or body in presence of conservative forces the sum of kinetic and potential energies at any point remains constant throughout the motion. It does not depends upon time. This is known as the law of conservation of mechanical energy.

$$\Delta(K + U) = \Delta E = 0 \qquad \text{[As } E \text{ is constant in a conservative field]}$$

$\therefore$ $$\Delta K + \Delta U = 0$$

i.e. if the kinetic energy of the body increases its potential energy will decrease by an equal amount and vice-versa.

(2) **Law of conservation of total energy :** If some non-conservative force like friction is also acting on the particle, the mechanical energy is no more constant. It changes by the amount of work done by the frictional force.

$$\Delta(K + U) = \Delta E = W_f \qquad \text{[where } W_f \text{ is the work done against friction]}$$

The lost energy is transformed into heat and the heat energy developed is exactly equal to loss in mechanical energy.

We can, therefore, write $\Delta E + Q = 0$ [where Q is the heat produced]

This shows that if the forces are conservative and non-conservative both, it is not the mechanical energy alone which is conserved, but it is the total energy, may be heat, light, sound or mechanical *etc*., which is conserved.

In other words : "*Energy may be transformed from one kind to another but it cannot be created or destroyed. The total energy in an isolated system is constant*". This is the law of conservation of energy.

Sample problems based on conservation of energy

Problem 1. Two stones each of mass 5*kg* fall on a wheel from a height of 10*m*. The wheel stirs 2*kg* water. The rise in temperature of water would be

(a) $2.6°C$ (b) $1.2°C$ (c) $0.32°C$ (d) $0.12°C$

Solution : (d) For the given condition potential energy of the two masses will convert into heat and temperature of water will increase $W = JQ \Rightarrow 2m \times g \times h = J(m_w\ S\ \Delta t) \Rightarrow$

$$2 \times 5 \times 10 \times 10 = 4.2(2 \times 10^3 \times \Delta t)$$

$$\therefore\ \Delta t = \frac{1000}{8.4 \times 10^3} = 0.119\,^oC = 0.12\,^oC.$$

Problem 2. A boy is sitting on a swing at a maximum height of 5*m* above the ground. When the swing passes through the mean position which is 2*m* above the ground its velocity is approximately

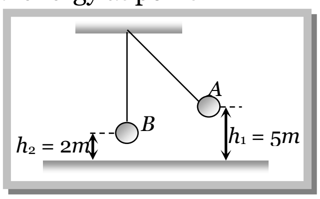
(a) 7.6 *m/s* (b) 9.8 *m/s* (c) 6.26 *m/s* (d) None of these

Solution : (a) By the conservation of energy Total energy at point *A* = Total energy at point *B*

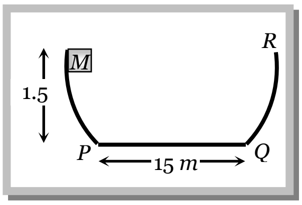
$$\Rightarrow\ mgh_1 = mgh_2 + \frac{1}{2}mv^2$$

$$\Rightarrow\ 9.8 \times 5 = 9.8 \times 2 + \frac{1}{2}v^2$$

$$\Rightarrow\ v^2 = 58.8\ \therefore\ v = 7.6\,m/s$$



Problem 3. A block of mass *M* slides along the sides of a bowl as shown in the figure. The walls of the bowl are frictionless and the base has coefficient of friction 0.2. If the block is released from the top of the side, which is 1.5 *m* high, where will the block come to rest ?
Given that the length of the base is 15 *m*



(a) 1 *m* from *P*
(b) Mid point
(c) 2 *m* from *P*
(d) At *Q*

Solution : (b) Potential energy of block at starting point = Kinetic energy at point P = Work done against friction in traveling a distance s from point P.

$\therefore \quad mgh = \mu mgs \Rightarrow s = \frac{h}{\mu} = \frac{1.5}{0.2} = 7.5\,m$

i.e. block come to rest at the mid point between P and Q.

***Problem* 4.** If we throw a body upwards with velocity of $4\,ms^{-1}$ at what height its kinetic energy reduces to half of the initial value ? Take $g = 10\,m/s^2$

(a) $4m$ (b) $2\,m$ (c) $1\,m$ (d) None of these

Solution : (d) We know kinetic energy $K = \frac{1}{2}mv^2 \quad \therefore v \propto \sqrt{K}$

When kinetic energy of the body reduces to half its velocity becomes $v = \frac{u}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}\,m/s$

From the equation $v^2 = u^2 - 2gh \Rightarrow (2\sqrt{2})^2 = (4)^2 - 2 \times 10\,h \quad \therefore h = \frac{16 - 8}{20} = 0.4\,m$.

***Problem* 5.** A $2kg$ block is dropped from a height of $0.4\,m$ on a spring of force constant $K = 1960\,Nm^{-1}$. The maximum compression of the spring is

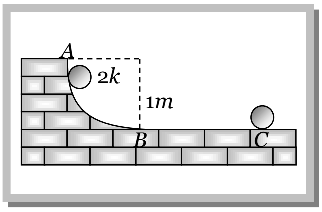
(a) $0.1\,m$ (b) $0.2\,m$ (c) $0.3\,m$ (d) $0.4\,m$

Solution : (a) When a block is dropped from a height, its potential energy gets converted into kinetic energy and finally spring get compressed due to this energy.

$\therefore$ Gravitational potential energy of block = Elastic potential energy of spring

$\Rightarrow mgh = \frac{1}{2}Kx^2 \Rightarrow x = \sqrt{\frac{2mgh}{K}} = \sqrt{\frac{2 \times 2 \times 10 \times 0.4}{1960}} = 0.09\,m \simeq 0.1\,m$.

***Problem* 6.** A block of mass $2kg$ is released from A on the track that is one quadrant of a circle of radius $1m$. It slides down the track and reaches B with a speed of $4\,ms^{-1}$ and finally stops at C at a distance of $3m$ from B. The work done against the force of friction is



(a) $10\,J$
(b) $20\,J$
(c) $2\,J$
(d) $6\,J$

Solution : (b) Block possess potential energy at point $A = mgh = 2 \times 10 \times 1 = 20\,J$

Finally block stops at point C. So its total energy goes against friction *i.e.* work done against friction is $20\,J$.

***Problem* 7.** A stone projected vertically upwards from the ground reaches a maximum height h. When it is at a height $\frac{3h}{4}$, the ratio of its kinetic and potential energies is

(a) $3:4$ (b) $1:3$ (c) $4:3$ (d) $3:1$

Solution : (b) At the maximum height, Total energy = Potential energy = mgh

At the height $\frac{3h}{4}$, Potential energy = $mg\frac{3h}{4} = \frac{3}{4}mgh$

and Kinetic energy = Total energy – Potential energy $= mgh - 3\frac{mgh}{4} = \frac{1}{4}mgh$

$\therefore \quad \frac{\text{Kinetic energy}}{\text{Potential energy}} = \frac{1}{3}$.