Stirling’s Formula

Derivation and approximation

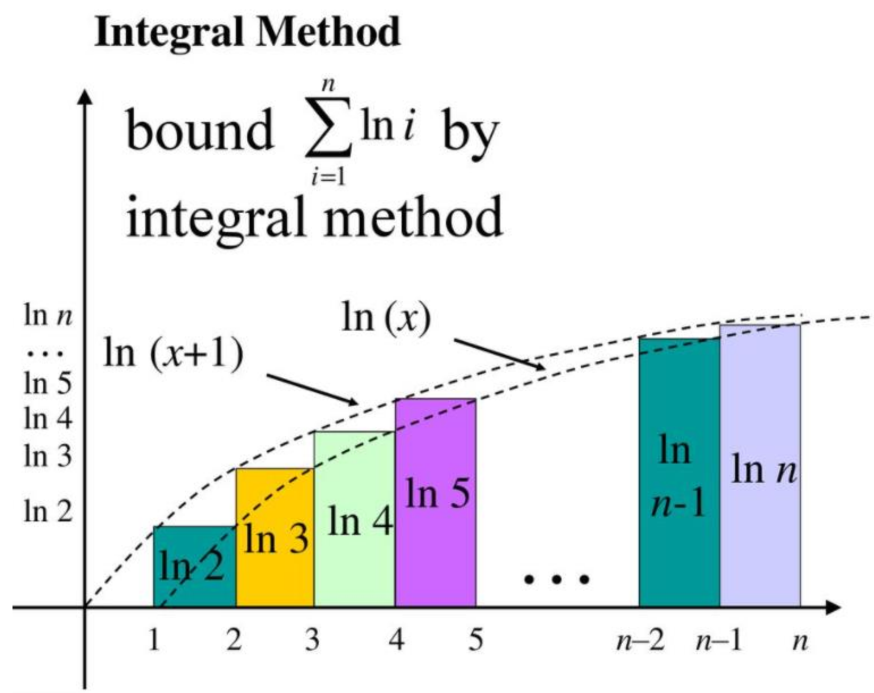
Factorial value of any number is defined by:	*Values of factorials*
$n! = 1 \cdot 2 \cdot 3 \cdots n$ $n! = n \cdot (n-1)!$ $\frac{n!}{(n-1)!} = n$ $\frac{n!}{m!} = (m+1) \cdot (m+2) \cdots n$ *If* $n > m$ $\frac{n!}{m!} = \frac{1}{(m+1) \cdot (m+2) \cdots n}$ *If* $m > n$	0! = 1 1! = 1 2! = 2 3! = 6 4! = 24 5! = 120 6! = 720 7! = 5 040 8! = 40 320 9! = 362 880 10! = 3 628 800 11! = 39 916 800 12! = 479 001 600 13! = 6 227 020 800 14! = 87 178 291 200 15! = 1 307 674 368 000 16! = 20 922 789 888 000 17! = 355 687 428 096 000 18! = 6 402 373 705 728 000 19! = 121 645 100 408 832 000 20! = 2 432 902 008 176 640 000

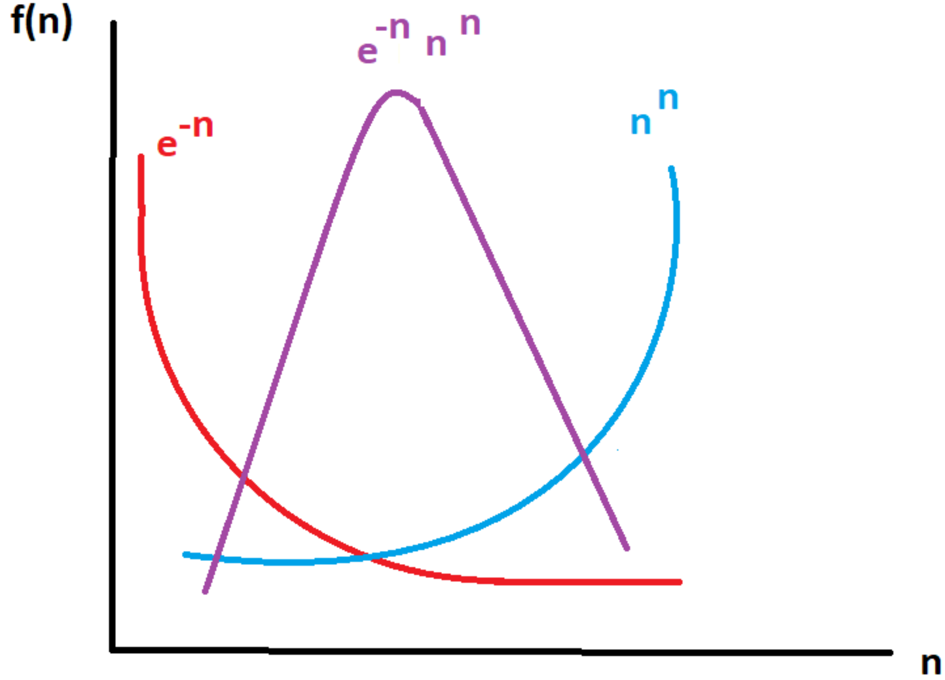
Stirling's Approximation
(to $n!$)

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\begin{aligned}
\ln(n!) &= \ln(1 * 2 * 3 * \cdots * n) \\
&= \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n) \\
&= \sum_{i=1}^{n} \ln(i)
\end{aligned}$$

$$\begin{aligned}
\ln(n!) &= \sum_{i=1}^{n} \ln(i) \\
&\approx \int_1^n \ln(x)dx \\
&= n\ln(n) - n + 1 \\
n! &\approx e^{n\ln(n)-n+1} \\
&= \left(e^{\ln(n)}\right)^n e^{-n} e \\
&= n^n e^{-n} e \\
&= e * \left(\frac{n}{e}\right)^n
\end{aligned}$$





$$n! \approx \left(\frac{n}{e}\right)^n$$

This is good enough for a variety of uses . . .

Deriving Stirling's Formula $\ln(n!) \approx (n+\frac{1}{2})\ln n - n + \frac{1}{2}\ln 2\pi$

We begin by calculating the integral $\int_0^\infty e^{-x} x^n dx$ (where $n \geq 0$) using integration by parts.

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

$$u(x) = x^n \qquad u'(x) = nx^{n-1}$$
$$v(x) = -e^{-x} \qquad v'(x) = e^{-x}$$

$$\int_0^\infty x^n e^{-x} dx = [x^n \cdot -e^{-x}]_0^\infty + n\int_0^\infty x^{n-1} \cdot e^{-x} dx$$

$[x^n \cdot -e^{-x}]_0^\infty$ always zero.

$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (1) = n!$.

$$\int_0^\infty x^n e^{-x} dx = n!$$

$$\ln e^{-x} x^n = -x + n\ln x =: f(x)$$

$$f'(x) \stackrel{!}{=} 0 = -1 + \frac{n}{x}$$

$$\Rightarrow x_0 = n$$

So we find it has a maximum at $x_0 = n$. Expanding $f(x)$ around n:

$$f(x) = -n + n\ln n + (-1 + \frac{n}{x})|_{x=n}(x-n) + \frac{1}{2!}(\frac{-n}{x^2})|_{x=n}(x-n)^2 + \dots$$

$$= -n + n\ln n - \frac{1}{2n}(x-n)^2$$

Therefore, $f(x) = \ln e^{-x}x^n \approx -n + n\ln n - \frac{1}{2n}(x-n)^2$.

If we take the exponential of $f(x)$ and integrate, $\int \exp(f(x))dx = \int e^{-x}x^n dx$

$$\simeq \int \exp(-n + n\ln n - \frac{1}{2n}(x-n)^2)\,dx$$

$$= \exp(-n + n\ln n)\int_0^\infty \exp(-\frac{(x-n)^2}{2n})dx \overset{(2)}{=} n!$$

This is calculable by analogy with the Gaussian distribution, where

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-\frac{(x-\bar{x})^2}{2\sigma^2}).$$

Given the sum of all probabilities $\int_{-\infty}^{\infty} P(x)dx = 1$, it follows

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp(-\frac{(x-\bar{x})^2}{2\sigma^2})dx.$$

we note σ^2 and $\bar{x}$ both translate to n in our analogy:

$$n! \simeq \exp(-n + n\ln n) \int_0^{\infty} \exp(-\frac{(x-n)^2}{2n})\,dx$$

$$= \exp(-n + n\ln n)\sqrt{2\pi n}$$

$$\ln n! \simeq -n + n\ln n + \frac{1}{2}\ln 2\pi n$$

$$= (n + \frac{1}{2})\ln n - n + \frac{1}{2}\ln 2\pi,$$