LAPLACE TRANSFORMATIONS

تحويلات لابلاس

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
- One of the first scientists to suggest the existence of black holes

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Laplace Transforms

Laplace transforms are invaluable for any engineer's mathematical toolbox as they make solving linear ODEs and related initial value problems, as well as systems of linear ODEs, much easier. Applications abound: electrical networks, springs, mixing problems, signal processing, and other areas of engineering and physics.

https://www.intmath.com/laplace-transformation/intro.php

1- Laplace Transform, Linearity, Shifting Theorem (*s*-Shifting)

Laplace Transform

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say $F(s)$, and is denoted by $\mathcal{L}\{f(t)\}$, thus

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \qquad (1)$$

Here we must assume that the integral exists.

The inverse transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\{F(s)\} = \int_0^\infty F(s)\, e^{st}\, ds = f(t) \qquad (2)$$

<u>Example 1</u>: Show that

$$\boldsymbol{\mathcal{L}(1) = \frac{1}{s}\,, \qquad s > 0} \tag{3}$$

<u>Solution</u>:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt\,, \quad let\ f(t) = 1$$

$$\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{-s}\left[e^{-st}\right]_0^\infty$$

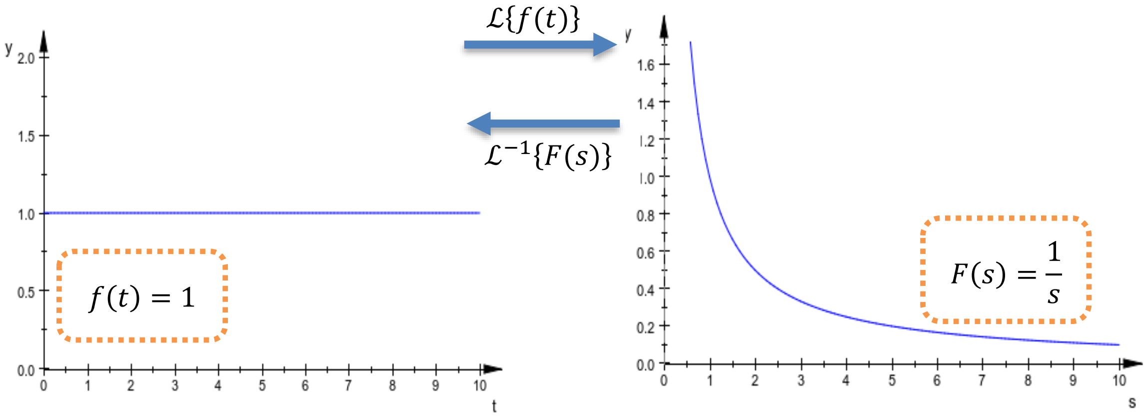
$$= -\frac{1}{s}\left\{\lim_{T\to\infty} e^{-sT} - e^0\right\}$$

$= 0\ if\ s > 0$

$$= -\frac{1}{s}\,\{0 - 1\}$$

$$= -\frac{1}{s}\,(-1)$$

$$= \frac{1}{s}$$



Example 2:

Find Laplace transform of $f(t) = e^{at},\ i.e.,\ \mathcal{L}\{e^{at}\}$

Solution:

Again by Eq. (1)

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)\,, \quad let\ f(t) = e^{at}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \frac{1}{-(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{1}{-(s-a)}\left[\lim_{T\to\infty} e^{-(s-a)T} - e^{-(s-a)0}\right] \qquad = 0\ if\ s > a$$

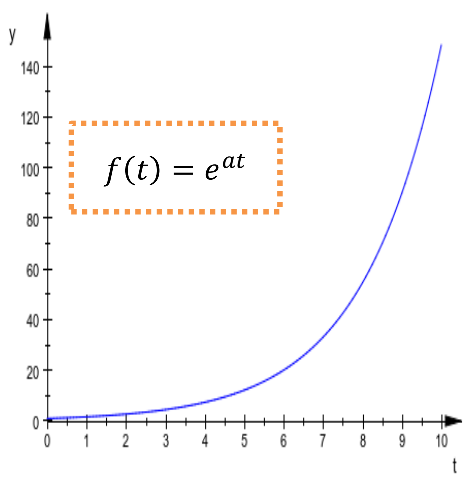
$$= \frac{1}{-(s-a)}[0-1]$$

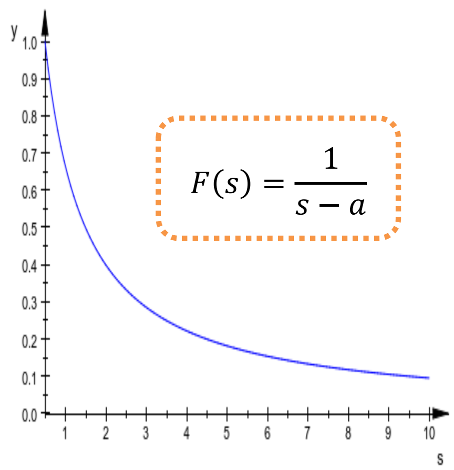
$$= \frac{1}{s-a}\,, \qquad when\ \ s > a$$

Thus

$$\therefore \mathcal{L}\{e^{at}\} = F(s) = \frac{1}{s-a}\,, for\ \ s > a \qquad (4)$$

$$\therefore \mathcal{L}\{e^{-at}\} = F(s) = \frac{1}{s+a}\,, for\ \ s > -a \qquad (5)$$





Theorem 1: Linearity of the Laplace Transform

Theorem: Laplace transform is a linear operation

$$\mathcal{L}\{a\,f(t) \pm b\,g(t)\} = a\,\mathcal{L}\{f(t)\} \pm b\,\mathcal{L}\{g(t)\} = aF(s) + bG(s) \qquad (6)$$

Proof

$$\mathcal{L}\{a\,f(t) \pm b\,g(t)\} = \int_0^{\infty} \{a\,f(t) \pm b\,g(t)\}\,e^{-st}\,dt$$

$$= a\int_0^{\infty} f(t)\,e^{-st}\,dt \pm b\int_0^{\infty} g(t)\,e^{-st}\,dt$$

$$= a\,\mathcal{L}\{f(t)\} \pm b\,\mathcal{L}\{g(t)\}$$

$$= a\,F(s) \pm b\,G(s)$$

Example 4 **Using the linearity property, derive the formulas**

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad (7)$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2} \qquad (8)$$

Solution

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a) + (s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2\,s}{s^2 - a^2}\right]$$

$$= \frac{s}{s^2 - a^2}$$

<u>Example 5</u> Derive the following formulas

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, \quad (9)$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, \quad (10)$$

<u>Solution</u>

Assuming $L_c = \mathcal{L}\{\cos at\} = \int_0^\infty \cos at \; e^{-st}\, dt$

and $L_s = \mathcal{L}\{\sin at\} = \int_0^\infty \sin at \; e^{-st}\, dt$

Therefore

$$L_c = \mathcal{L}\{\cos at\} = \int_0^\infty \cos at \; e^{-st}\, dt$$

Integrating by part by assuming
$u = \cos at$ and $dv = e^{-st}\, dt$
$\therefore du = -a \sin at \; dt$ and $v = \frac{e^{-st}}{-s}$

$$= \left[\cos at \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} (-a \sin at \; dt)$$

$$= \frac{-1}{s}[\cos at \, e^{-st}]_0^\infty - \frac{a}{s} \int_0^\infty e^{-st} \; \sin at \; dt$$

$$= \frac{-1}{s}[0 - \cos 0 \; e^0] - \frac{a}{s} L_s$$

$$\therefore L_c = \frac{1}{s} - \frac{a}{s} L_s \quad (5.1)$$

Similarly

$$L_s = \frac{a}{s} L_c \quad (5.2)$$

This is two linear equations, can be solved simultaneously:

From Eq. (5.1) → $L_c = \frac{1}{s} - \frac{a}{s}\frac{a}{s} L_c = \frac{1}{s} - \frac{a^2}{s^2} L_c$

$$L_c + \frac{a^2}{s^2} L_c = \frac{1}{s}$$

$$L_c \left(1 + \frac{a^2}{s^2}\right) = \frac{1}{s}$$

$$L_c\left(\frac{s^2+a^2}{s^2}\right)=\frac{1}{s}$$

$$L_c\,\frac{s^2+a^2}{s}=1$$

$$\therefore L_c=\frac{s}{s^2+a^2} \tag{5.3}$$

Substituting Eq. (5.3) into Eq. (5.2)

$$L_s=\frac{a}{s}\frac{s}{s^2+a^2}=\frac{a}{s^2+a^2}$$

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n=0,1,\cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Example 6: Show that

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}} \tag{11}$$

Solution:

$$\mathcal{L}\{t^{-1/2}\} = \int_0^\infty t^{-1/2}\, e^{-st}\, dt$$

Let $st = x$ → $dt = \frac{dx}{s}$

$$= \int_0^\infty \left(\frac{x}{s}\right)^{-1/2} e^{-x}\, \frac{dx}{s}$$

$$= \int_0^\infty \left(\frac{s}{x}\right)^{1/2} e^{-x}\, \frac{dx}{s}$$

$$= \int_0^\infty \left(\frac{1}{x}\right)^{1/2} e^{-x}\, \frac{dx}{s^{1/2}}$$

$$= \sqrt{\frac{1}{s}} \int_0^\infty x^{-1/2}\, e^{-x}\, dx$$

$$= \sqrt{\frac{1}{s}}\; \Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\frac{1}{s}}\; \sqrt{\pi}$$

$$= \sqrt{\frac{\pi}{s}}$$

Example 7: Show that

$$\mathcal{L}\{t^n\} = \begin{cases} \dfrac{\Gamma(n+1)}{s^{n+1}}, & n > -1 \\[2ex] \dfrac{n!}{s^{n+1}}, & n = 0, 1, 2, \ldots \end{cases} \tag{11}$$

Solution:

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n\, e^{-st}\, dt$$

Let $st = x$ $\quad dt = \frac{dx}{s}$

$$= \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\, \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n\, e^{-x}\, dx$$

$\int_0^\infty x^n\, e^{-x}\, dx = \Gamma(n+1),\ \ n+1 > 0,$ or $n > -1$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, \qquad n > -1$$

For integer values of n, i.e., n=0, 1, 2, … → $\Gamma(n+1) = n!$

$$\therefore\ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad n = 0, 1, 2, \ldots$$

eg., if $\quad n = 0, \qquad \mathcal{L}\{1\} = \frac{1}{s}$

if $\quad n = 1, \qquad \mathcal{L}\{t\} = \frac{1}{s^2}$

Shifting Theorem (s-Shifting)

If $\mathcal{L}\{f(t)\} = F(s)$,

therefore $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$,

where a is constant

Proof:

$$\because \; F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$\therefore \; F(s-a) = \int_0^{\infty} f(t)\, e^{-(s-a)t}\, dt$$

$$= \int_0^{\infty} [f(t)\, e^{at}]\, e^{-st}\, dt$$

Let $f(t)e^{at} = g(t)$

$$= \int_0^{\infty} g(t)\, e^{-st}\, dt$$

$$= \mathcal{L}\{g(t)\}$$

$$= \mathcal{L}\{f(t)\, e^{at}\}$$

Example 8: Find Laplace transform of $e^{2t}\, t$, *i.e.*, find $\mathcal{L}\{e^{2t}\, t\}$

Solution:

$$\because \; \mathcal{L}\{t\} = \frac{1}{s^2} = F(s)$$

$$\therefore \; \mathcal{L}\{e^{2t}\, t\} = F(s-2)$$

$$= \frac{1}{(s-2)^2}$$

Example 8:

Find $\mathcal{L}\{e^{-3t} \cos bt\}$

Solution:

$$\because \; \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2} = F(s)$$

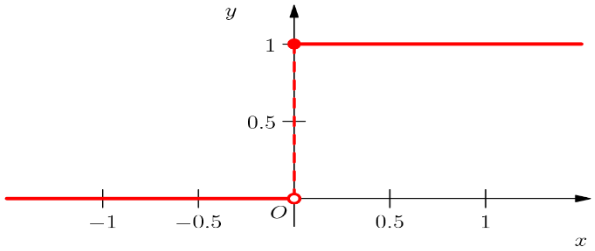
$$\therefore \; \mathcal{L}\{e^{-3t} \cos bt\} = F(s+3)$$

$$= \frac{s+3}{(s+3)^2+b^2}$$

2- Unit step Function

The Unit step function is also called Heaviside function which defined as follows:

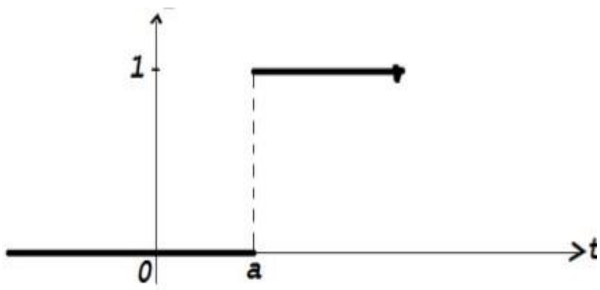
$$u(t) = \begin{cases} 0\,, & t < 0 \\ 1\,, & t > 0 \end{cases}$$



shifting the unit step function is:

$$u(t-a) = \begin{cases} 0\,, & t-a < 0 \\ 1\,, & t-a > 0 \end{cases}$$

$$= \begin{cases} 0\,, & t < a \\ 1\,, & t > a \end{cases}$$



Example 8:

Show that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}\,, \qquad s > 0$$

Solution:

$$\because\ \mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)\, e^{-st}\, dt$$

$$= \int_0^a u(t-a)\, e^{-st}\, dt + \int_a^\infty u(t-a)\, e^{-st}\, dt$$

$$= \int_0^a (0)\, e^{-st}\, dt + \int_a^\infty (1)\, e^{-st}\, dt$$

$$= \int_a^\infty e^{-st}\, dt$$

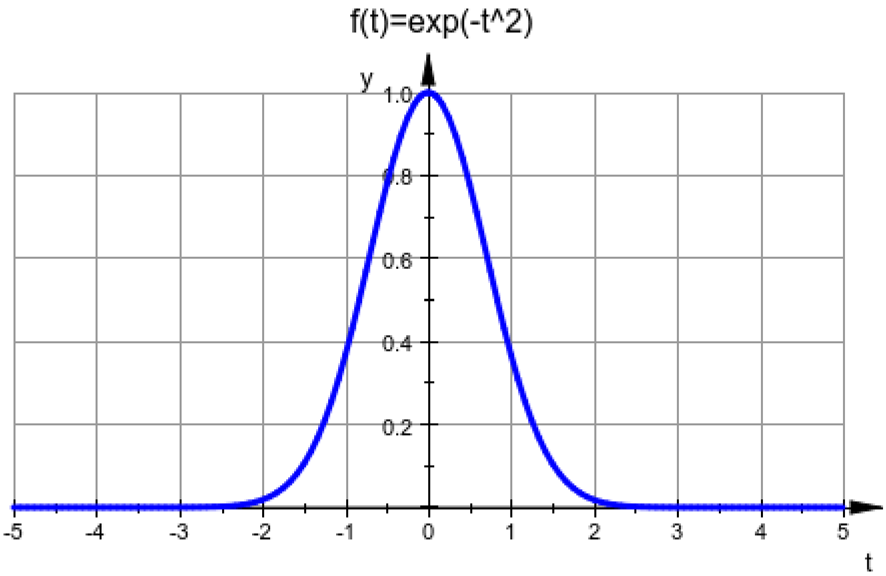
$$= \frac{1}{-s}\,[e^{-st}]_a^\infty$$

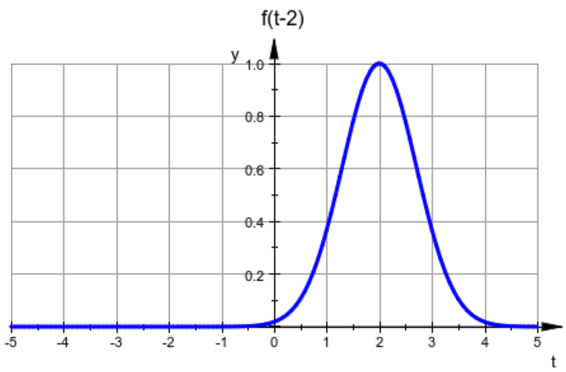
$$= \frac{1}{-s}\left[\lim_{T\to\infty} e^{-sT} - e^{-sa}\right]$$

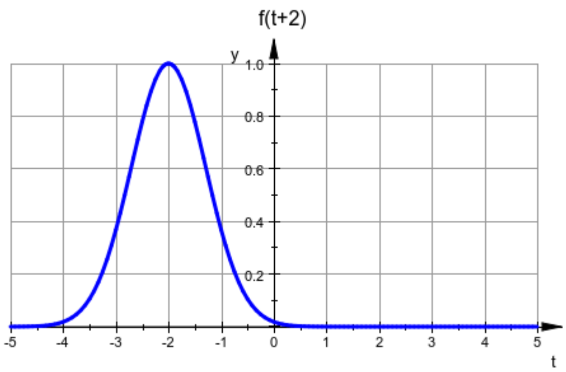
1st term is equal to zero if $s > 0$

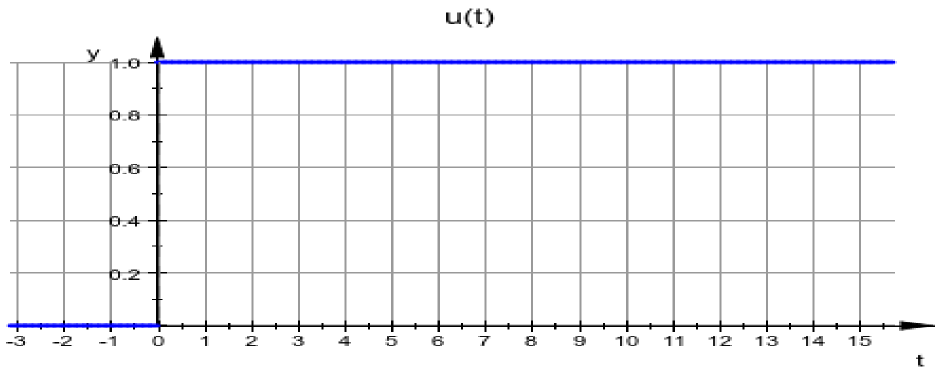
$$= \frac{1}{-s}\,[0 - e^{-sa}] = \frac{e^{-as}}{s}$$

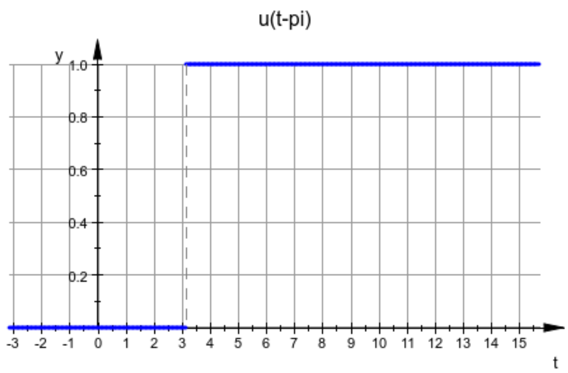
Application of the Unit step function

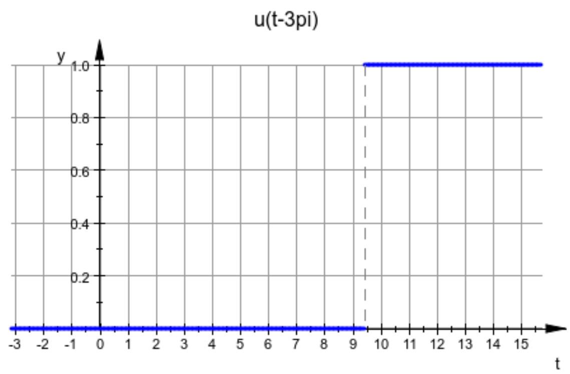


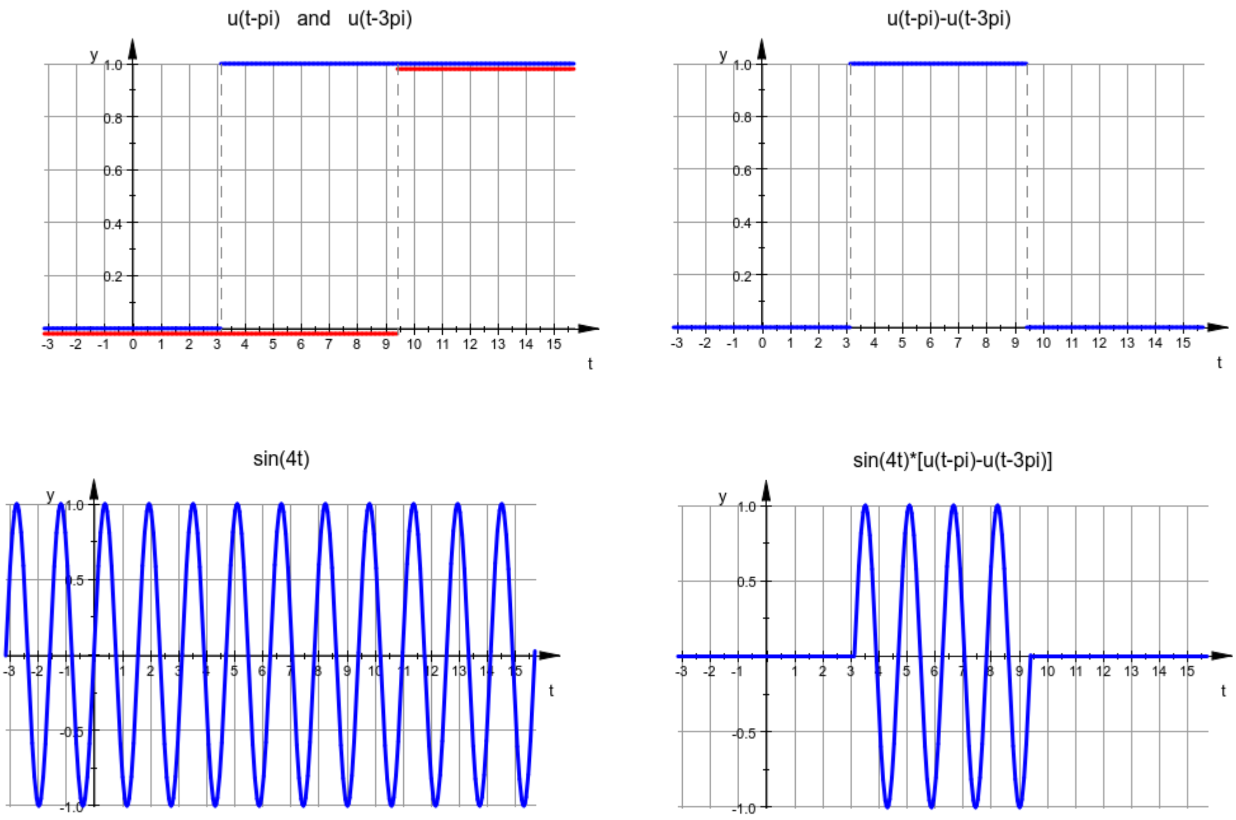












The rectangular pulses can be generated by summing Unit step functions

Example 9: Generate a rectangular pulse $f(t)$ of amplitude $\boldsymbol{A}$ and period $\boldsymbol{a}$ starting from t=0.

Solution: $f(t) = A[u(t) - u(t-a)]$

Example 10: Find Laplace transform for a rectangular pulse

$$f(t) = A[u(t) - u(t-a)]$$

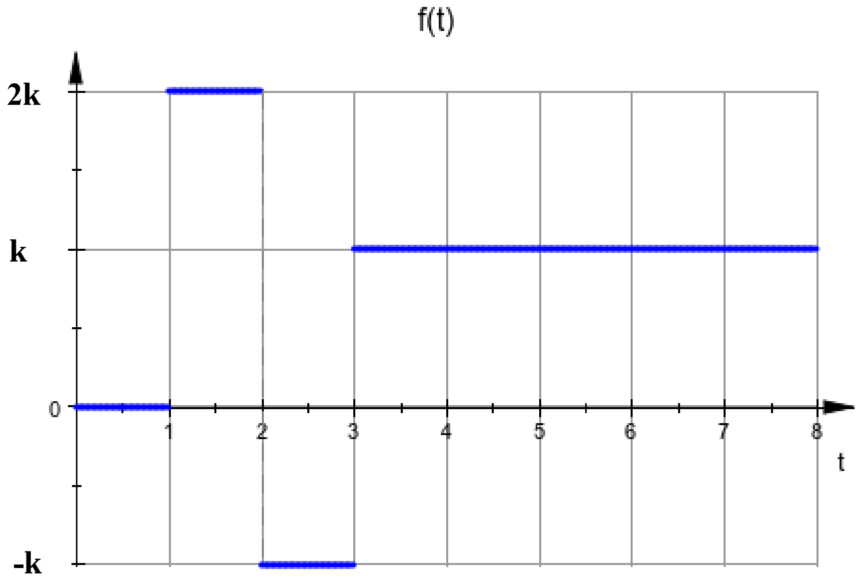
Solution:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{A[u(t) - u(t-a)]\} \\ &= A\,\mathcal{L}\{[u(t) - u(t-a)]\} \\ &= A[\mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-a)\}] \\ &= A\left[\frac{1}{s} - \frac{e^{-as}}{s}\right] \end{aligned}$$

If $f(t) = A[u(t) - u(t-a)] \longrightarrow \mathcal{L}\{f(t)\} = A\left[\frac{1}{s} - \frac{e^{-as}}{s}\right]$

Example11: Find L.T. for the pulse indicated in the figure using the formula

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$



Solution:

$f(t)$ can be expressed by using the Unit step function:

$$f(t) = 2k[u(t-1) - u(t-2)] - k[u(t-2) - u(t-3)] + ku(t-3)$$

$$= k[2u(t-1) - 2u(t-2) - u(t-2) + u(t-3) + u(t-3)]$$

$$= k[2u(t-1) - 3u(t-2) + 2u(t-3)]$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{k[2u(t-1) - 3u(t-2) + 2u(t-3)]\}$$

$$= k\,\mathcal{L}\{2u(t-1) - 3u(t-2) + 2u(t-3)\}$$

$$= k\,[2\mathcal{L}\{u(t-1)\} - 3\mathcal{L}\{u(t-2)\} + 2\mathcal{L}\{u(t-3)\}]$$

$$= k\left[\frac{2e^{-s}}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-3s}}{s}\right]$$

$$= \frac{k}{s}\,[2e^{-s} - 3e^{-2s} + 2e^{-3s}]$$

3- Theorem: Lplace transform of the derivative

If $f(t)$ and it's derivative $f'(t)$ are continuous for att $t \geq 0$, hence, $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{f'(t)\}$ are exist.

$$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)} \qquad (12)$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integrating by part by assuming
$u = e^{-st}$ and $dv = f'(t)\, dt$
$\therefore\ du = -se^{-st}\, dt$ and $v = f(t)$

$$= [f(t)\, e^{-st}]_0^\infty - \int_0^\infty f(t)\,(-s\, e^{-st}\, dt)$$

$$= [0 - f(0)] + s\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

$$= s\,\mathcal{L}\{f(t)\} - f(0)$$

Example 12: Show that

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)} \qquad (13)$$

Provided that f, f, f'' are continuous for $t \geq 0$ and their L.T. are exist.

Solution:

From Eq. (12)

$$\because\ \mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$\therefore\ \mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\,[s\,\mathcal{L}\{f(t)\} - f(0)] - f'(0)$$

$$= s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

Similarly, one can show that

$$\mathcal{L}\{f^{(3)}(t)\} = s^3\,\mathcal{L}\{f(t)\} - s^2\,f(0) - s\,f^{(1)}(0) - f^{(2)}(0) \tag{14}$$

Generally,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - s^{n-3}f^{(2)}(0) - \cdots - f^{(n-1)}(0) \tag{15}$$

Example 13: Find L.T for $f(t) = t\cos at$ for $f(0) = 0$

Solution:

$$f(t) = t\cos at$$

$$f'(t) = \cos at - at\sin at \qquad \therefore f'(0) = 1$$

$$f''(t) = -a\sin at - a\sin at - a^2t\cos at = -2a\sin at - a^2t\cos at$$

$$= -2a\sin at - a^2\,f(t)$$

Using Eq. (13)

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

$$\mathcal{L}\{-2a\sin at - a^2\,f(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\times 0 - 1$$

$$-2a\,\mathcal{L}\{\sin at\} - a^2\,\mathcal{L}\{f(t)\} = s^2\,\mathcal{L}\{f(t)\} - 1$$

$$-2a\,\mathcal{L}\{\sin at\} + 1 = s^2\,\mathcal{L}\{f(t)\} + a^2\,\mathcal{L}\{f(t)\}$$

$$-2a\left[\frac{a}{s^2+a^2}\right] + 1 = [s^2 + a^2]\,\mathcal{L}\{f(t)\}$$

$$1 - \frac{2a^2}{s^2+a^2} = [s^2 + a^2]\,\mathcal{L}\{f(t)\}$$

$$\frac{s^2+a^2-2a^2}{s^2+a^2} = [s^2 + a^2]\,\mathcal{L}\{f(t)\}$$

$$\frac{s^2-a^2}{s^2+a^2} = [s^2 + a^2]\,\mathcal{L}\{f(t)\}$$

$$\frac{s^2-a^2}{(s^2+a^2)^2} = \mathcal{L}\{f(t)\}$$

$$\therefore \mathcal{L}\{t\cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

<u>4- Theorem</u>: Differentiation of laplace's transform

If $f(t)$ satisfies the condition of the existence theorem and its $\mathcal{L}\{f(t)\} = F(s)$, hence,

$$\mathcal{L}\{t\, f(t)\} = -\frac{dF(s)}{ds} \qquad (16)$$

<u>Proof</u>:

$$\because F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, (-t\, e^{-st})\, dt$$

$$= -\int_0^\infty t\, f(t)\, e^{-st}\, dt$$

Let $t\, f(t) = g(t)$

$$= -\int_0^\infty g(t) e^{-st}\, dt$$

$$= -\mathcal{L}\{g(t)\} = -\mathcal{L}\{t\, f(t)\}$$

Therefore, one can conclude that

$$\mathcal{L}\{t^n\, f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}, n = 0, 1, 2, \dots \qquad (17)$$

Example 14:

let $g(t) = t\, sin\, t.$

Find $\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{g'(t)\}$

Solution:

Let $f(t) = \sin t$

$$\because\ \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = F(s) = \frac{1}{s^2+1}$$

Using Eq. (16)

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\, f(t)\} = \mathcal{L}\{t \sin t\} = -\frac{dF(s)}{ds}$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{0-2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}$$

$$\therefore\ \mathcal{L}\{g(t)\} = \mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2}$$

Using Eq. (12)

$$\mathcal{L}\{g'(t)\} = s\, \mathcal{L}\{g(t)\} - g(0)$$

$$= s\, \frac{2s}{(s^2+1)^2} - 0$$

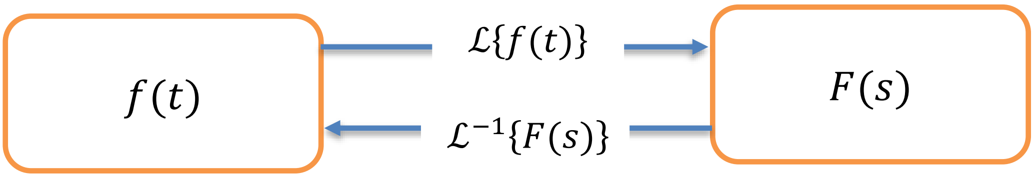
$$= \frac{2s^2}{(s^2+1)^2}$$

$$\therefore\ \mathcal{L}\{g'(t)\} = \frac{2s}{(s^2+1)^2}$$

5- Inverse Laplace Transform

If $\mathcal{L}\{f(t)\} = F(s)$

Therefore $\mathcal{L}^{-1}\{F(s)\} = f(t)$



E.g.,

If $\mathcal{L}\{e^{at})\} = \frac{1}{s-a}$

Therefore $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

Shifting property of the inverse L.T

$$\because \mathcal{L}\{e^{at}\, f(t))\} = F(s-a)$$

Therefore,

$$\therefore \mathcal{L}^{-1}\{F(s-a)\} = e^{at}\, f(t)$$

where $F(s) = \mathcal{L}\{f(t))\}$

6- Ordinary differential equation

There are many methods concern with solving linear ordinary differential equations (ODE) with initial conditions. These problems are called initial value problem (IVP). Laplace transform method is one of the most popular methods which characterized by its simplicity. It uses the initial conditions implicitly in the solution steps.

The general form of linear second order ODE is

$$a\,\frac{d^2y}{dt^2} + b\,\frac{dy}{dt} + c\;y(t) = f(t),$$

where a, b and c are constants.

The ODE can be transformed into algebraic equation by using Laplace transform

$$\mathcal{L}\{y(t)\} = Y(s),$$

where the solution of the ODE, $y(t)$, can be found by applying the inverse LT, i.e.,

$$\mathcal{L}^{-1}\{Y(s)\} = y(t)$$

Example 15:

Solve the following IVP

$$\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2\;y(t) = 2\;e^{-4t}\,,$$

for the following initial values:

$$y(0) = 0$$

$$y'(0) = 1$$

Solution:

Taking L.T for the ODE yields

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\;\mathcal{L}\{y\} = 2\;\mathcal{L}\{e^{-4t}\}$$

Let $\mathcal{L}\{y(t)\} = Y(s)$

From Eq. (12) $\longrightarrow$ $\mathcal{L}\{y'(t)\} = s\;Y(s) - y(0)$

$+ \qquad\qquad = s\;Y(s)$

From Eq. (13) ⟶ $\mathcal{L}\{y''(t)\} = s^2Y(s) - s\,y(0) - y'(0)$

$$= s^2\,Y(s) - 1$$

Substituting $\mathcal{L}\{y'(t)\}$ and $\mathcal{L}\{y''(t)\}$ into the ODE

$$[s^2Y(s) - 1] - 3[s\,Y(s)] + 2\,Y(s) = 2\left[\frac{1}{s+4}\right]$$

$$Y(s)[s^2 - 3\,s + 2] = 1 + 2\left[\frac{1}{s+4}\right]$$

$$Y(s)[s^2 - 3\,s + 2] = \frac{s+4+2}{s+4}$$

$$Y(s) = \frac{s+6}{(s^2 - 3\,s + 2)(s+4)}$$

$$\therefore Y(s) = \frac{s+6}{(s-1)(s-2)(s+4)} \qquad (15.1)$$

Using partial fractions, in order to apply the linearity property of Laplace's operator:

$$\frac{s+6}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$s + 6 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

$$s + 6 = A(s^2 + 2s - 8) + B(s^2 + 3s - 4) + C(s^2 - 3s + 2)$$

$$s + 6 = s^2(A + B + C) + s(2A + 3B - 3C) + (-8A - 4B + 2C)$$

$$\therefore\; s^2(A + B + C) + s(2A + 3B - 3C - 1) + (-8A - 4B + 2C - 6) = 0$$

Equating the coefficients of s^2 to zero $\quad A + B + C = 0$ ……...(1)

Equating the coefficients of s to zero $\quad 2A + 3B - 3C - 1 = 0$ ……...(2)

Equating the coefficients of s^0 to zero $\quad -8A - 4B + 2C - 6 = 0$……..(3)

This is a system of three linear equations, can be solved simultaneously to find the coefficients A, B, and C.

$$A = -\frac{7}{5} \qquad B = \frac{4}{3} \qquad C = \frac{1}{15}$$

Another method for calculating the coefficients

$$\frac{s+6}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4} = \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$\because\ s+6 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

Letting $s = 1$ gives → $7 = A\,(-1)(5) = -5A$ → $A = -\frac{7}{5}$

Letting $s = 2$ gives → $8 = B\,(1)(6) = 6B$ → $B = \frac{4}{3}$

Letting $s = -4$ gives → $2 = C\,(-5)(-6) = 15C$ → $C = \frac{4}{3}$

Substituting these coefficients into Eq. (15.1)

$$Y(s) = \frac{\frac{-7}{5}}{s-1} + \frac{\frac{4}{3}}{s-2} + \frac{\frac{1}{15}}{s+4}$$

Taking the inverse Laplace transform $\mathcal{L}^{-1}\{Y(s)\} = y(t)$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{-7}{5}}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{4}{3}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{15}}{s+4}\right\}$$

$$y(t) = -\frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{4}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{15}\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

Using Eq. (4) $\quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

A table of Laplace transforms is useful which can be found online here.

Therefore, the solution of the IVP is:

$$y(t) = -\frac{7}{5}\,e^{t} + \frac{4}{3}\,e^{2t} + \frac{1}{15}\,e^{-4t}$$

Example 16:

Solve the following IVP

$$y''(t) + y' - 2\,y(t) = 4\,,$$

with the following initial values:

$$y(0) = 2$$

$$y'(0) = 1$$

Solution:

We begin by applying the Laplace transform to both sides. By linearity of the Laplace transform, we have

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{4\}.$$

Therefore,

$$(s^2\mathcal{L}\{y\} - 2s - 1) + (s\mathcal{L}\{y\} - 2) - 2\mathcal{L}\{y\} = \frac{4}{s}.$$

Next, combine like terms to get

$$(s^2 + s - 2)\,\mathcal{L}\{y\} = \frac{4}{s} + 2s + 3$$

$$(s-1)(s+2)\mathcal{L}\{y\} = \frac{2s^2 + 3s + 4}{s}.$$

Notice that the coefficient in front of $\mathcal{L}\{y\}$ is the characteristic equation of the differential equation. This is not a coincidence. Putting under a common denominator, dividing and factoring we get

$$\mathcal{L}\{y\} = \frac{2s^2 + 3s + 4}{s(s-1)(s+2)}.$$

To find $y(t)$, we need to take the Inverse Laplace Transform of the right-hand side. Unfortunately, finding a function y such that the right-hand side is the Laplace transform of y is not an easy task. The technique that just about always works is partial fractions. We write

$$\frac{2s^2 + 3s + 4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$\frac{2s^2 + 3s + 4}{s(s-1)(s+2)} = \frac{A(s-1)(s+2) + B\, s(s+2) + C\, s(s-1)}{s(s-1)(s+2)}$$

which gives

$$A(s-1)(s+2) + B\, s(s+2) + C\, s(s-1) = 2s^2 + 3s + 4.$$

Letting $s = 0$	gives	$-2A=4$	and	$A=-2$
Letting $s = 1$	gives	$3B=9$	and	$B=3$
Letting $s = -2$	gives	$6C=6$	and	$C=1$

Now we solve

$$\mathcal{L}\{y\} = -\frac{2}{s} + \frac{3}{s-1} + \frac{1}{s+2}$$

or

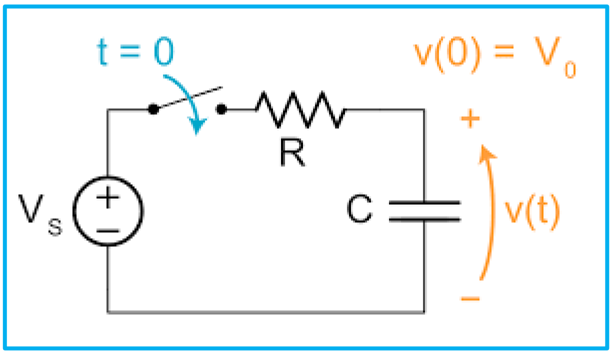
$$y(t) = -2\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

Now we can use the table to get

$$y(t) = -2 + 3\,e^{t} + e^{-2t}.$$

<u>Example 17</u>: Discharging the capacitor

Find the voltage $v(t)$ across the capacitor as a function of time.



10. Applications of Laplace (intmath.com)

<u>Solution</u>:

$$V_R + v = V_s$$

If $V_s = 0$ (when the source is removed and the circuit is closed)

$$IR + v = 0$$

$$I(t)\,R + v(t) = 0$$

$$C\frac{dv}{dt}R + v(t) = 0$$

$$\frac{dv}{dt} + \frac{1}{RC}v = 0$$

$$\frac{dv(t)}{dt} = -\frac{1}{RC}\,v(t) \qquad (17.1)$$

$$\frac{dv(t)}{v} = -\frac{dt}{RC}$$

Integrate both sides → $ln(v) = -\frac{t}{RC} + const.$

Initial condition: at $t = 0$ → $v(0) = V_o$ → $ln(V_o) = const.$

$$\therefore ln(v) = -\frac{t}{RC} + ln(V_o)$$

$$ln(v) - ln\,(V_o) = -\frac{t}{RC}$$

$$ln\left(\frac{v}{Vo}\right) = -\frac{t}{RC}$$

$$v(t) = V_o\, e^{-\frac{t}{RC}}$$

Solving Eq. (17.1) using Laplace transformation assuming that $\mathcal{L}\{v(t)\} = V(s)$ and $\mathcal{L}^{-1}\{V(s)\} = v(t)$

$$v'(t) + \frac{1}{RC}\, v(t) = 0$$

$$RC\, v'(t) + v(t) = 0$$

$$RC\, \mathcal{L}\{v'(t)\} + \mathcal{L}\{v(t)\} = 0$$

$$RC\, [s\, V(s) - v(0)] + V(s) = 0$$

$$RC\, [s\, V(s) - V_o] + V(s) = 0$$

$$V(s)[RC\, s + 1] - RC\, V_o = 0$$

$$V(s)[RC\, s + 1] = RC\, V_o$$

$$V(s) = \frac{RCV_o}{RC\, s + 1} = \cancel{RC}\, V_o \frac{1}{\cancel{RC}\left\{s + \frac{1}{RC}\right\}} = V_o \frac{1}{\left\{s + \frac{1}{RC}\right\}}$$

Taking the I.L.T

$$\mathcal{L}^{-1}\{V(s)\} = v(t) = V_o\, \mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{RC}}\right\}$$

$$\boxed{v(t) = V_o\, e^{-t/RC}}$$

Where we use Eq. (5) $\quad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$