LEGENDRE'S POLYNOMIALS

ADVANCED APPLED MATHEMATICS

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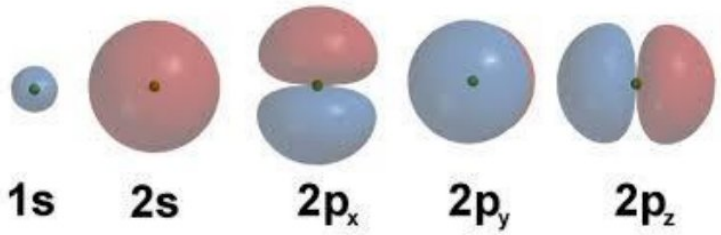
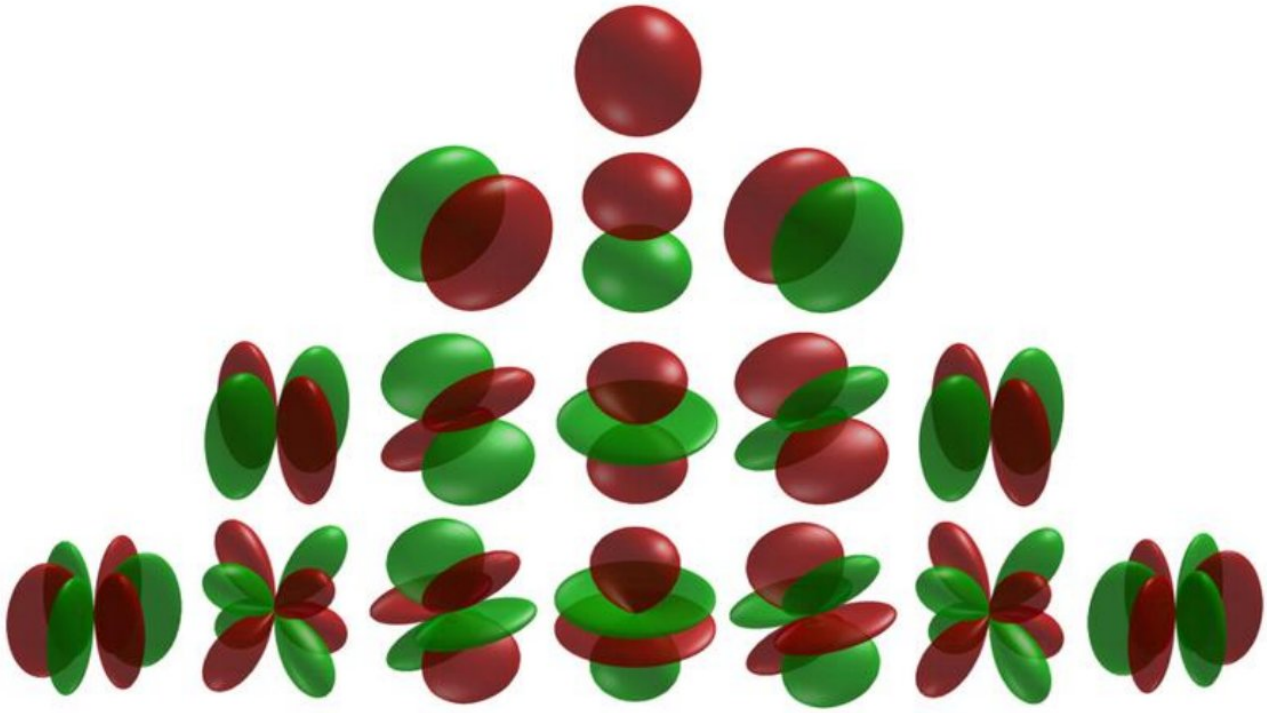
Orbitals



Spherical Harmonics



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Legengre's Polynomials

1- Legendre's differential equation

Legendre's differential equation and the corresponding functions appears as a solution of *Laplace*, *Schroedinger* and *Helmhtz* wave equation in spherical coordinates.

Legendre's differential equation takes the following form:

$$(1-x^2)\,y''(x) - 2x\,y'(x) + n(n+1)\,y(x) = 0$$

Or

$$\frac{d}{dx}[(1-x^2)\,y'(x)] + n(n+1)\,y(x) = 0 \qquad (1)$$

In most problems, usually we concern in studying a special case, where n=0 or any positive integer ($n \geq 0$).

1.1 The Series solution:

Let the solution of Eq. (1) is:

$$y(x) = \sum_{k=0}^{\infty} a_k\, x^{k+r} \qquad (2)$$

Where $a_{-1} = a_{-2} = \cdots . = 0$

The coefficients $a_k's$ and the arbitrary constant r must be determined.

بما ان $y(x)$ المعرفة كمتسلسلة قوى بالصيغة (2) هي حل للمعادلة التفاضلية (1)، اذن فهي تحقق معادلة ليجندر التفاضلية. نجد المشتقة الأولى والثانية للدالة $y(x)$ نسبة الى x ونعوضهما بالمعادلة التفاضلية (1) وكالتالي:

$$y' = \sum (k+r) a_k\, x^{k+r-1}$$

$$y'' = \sum (k+r)(k+r-1)\, a_k\, x^{k+r-2}$$

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Substitute these derivatives into eq. (1)

$$(1-x^2)\sum_k (k+r)(k+r-1)\,a_k\,x^{k+r-2} - 2x\sum_k (k+r)a_k\,x^{k+r-1} + n(n+1)\sum_k a_k\,x^{k+r} = 0$$

$$\sum_k (k+r)(k+r-1)\,a_k\,x^{k+r-2} - \sum_k (k+r)(k+r-1)a_k\,x^{k+r} - 2\sum_k (k+r)a_k\,x^{k+r} + n(n+1)\sum_k a_k\,x^{k+r} = 0$$

Collecting terms of x^{k+r}

$$\sum_k \{(k+r)(k+r-1)a_k\,x^{k+r-2} - a_k\,[(k+r)(k+r-1)+2(k+r)-n(n+1)]x^{k+r}\} = 0$$

$$(k+r)(k+r-1+2)-n(n+1)$$
$$(k+r)(k+r+1)-n(n+1)$$

Equating the coefficients of x^{k+r-2} to zero yields:

$$[(k+r)(k+r-1)\,]a_k - [(k-2+r)(k-2+r+1)-n(n+1)]a_{k-2} = 0$$

$$[(k+r)(k+r-1)\,]a_k - [(k+r-2)(k+r-1)-n(n+1)]a_{k-2} = 0$$

$$\therefore a_k = \frac{(k+r-2)(k+r-1)-n(n+1)}{(k+r)(k+r-1)}\,a_{k-2}\,,\ k \geq 2 \qquad (3)$$

This is a recurrence relation. It combines the odd coefficients with each other and also combine the even coefficients with each other.

Let k=0 $\longrightarrow$ $r(r-1)a_o - [\ldots.]a_{-2} = 0$ since $a_{-2} = 0$

$$r(r-1)a_o = 0$$

$$\because a_o \neq 0 \longrightarrow r(r-1) = 0 \longrightarrow r = 0 \ \ or \ \ r = 1$$

If $r = 0$ $\longrightarrow$

$$\therefore a_k = \frac{(k-2)(k-1)-n(n+1)}{k(k-1)}\,a_{k-2} \qquad (4)$$

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$$\therefore y(x) = \sum_{k=0} a_k\, x^k \qquad (5)$$

لغرض تحويل بسط العلاقة (4) من حاصل طرح حدين الى حاصل ضرب نجري التحويل التالي:

Rearranging the nominator of eq. (4) by assuming

$$k - 2 = m$$

and $k - 1 = m + 1$

$$\begin{aligned} Nominator &= (k-2)(k-1) - n(n+1) \\ &= m(m+1) - n(n+1) \\ &= m^2 + m - n^2 - n \\ &= m^2 - n^2 + m - n \\ &= (m-n)(m+n) + m - n \\ &= (m-n)(m+n+1) \\ &= (k-2-n)(k-2+n+1) \\ &= -(n-k+2)(n+k-1) \end{aligned}$$

$$\therefore a_k = \frac{-(n-k+2)(n+k-1)}{k(k-1)}\, a_{k-2} \qquad (5)$$

All even coefficients can be expressed in terms of a_o, whereas the odd coefficients are expressed in terms of a_1.

Even coefficients:

For k=2 $\longrightarrow$ $a_2 = \frac{-n(n+1)}{2!} a_o$

For k=4 $\longrightarrow$ $a_4 = \frac{-(n-2)(n+3)}{4\times3}\, a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}\, a_o$

For k=6 $\longrightarrow$ $a_6 = \frac{-a_4}{6(6+2c)} = \frac{-a_4}{2^2.3(3+c)} = \frac{-a_0}{2^6.3!\,(1+c)(2+c)(3+c)}$

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Odd coefficients:

For k=3 $\longrightarrow$ $a_3 = \frac{-(n-1)(n+2)}{3!} a_1$

For k=5 $\longrightarrow$ $a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1$

$$\therefore y(x) = [a_o + a_2 x^2 + a_4 x^4 + \cdots] + [a_1 x + a_3 x^3 + a_5 x^5 + \cdots]$$

$$= a_o \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 - \cdots\right] +$$

$$a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 - \cdots\right] +$$

This series is convergent for $|x| \leq 1$ and called Legendre's polynomial.

The first series becomes polynomial if n is even.

The second series becomes polynomial if n is odd.

If n is even

$$y(x) = P_n(x) = a_o \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 - \frac{n(n-4)(n-2)(n+1)(n+3)(n+5)}{6!} x^6 + \cdots\right]$$

If n is odd

$$y(x) = P_n(x) = a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} x^7 + \ldots\right]$$

Determination of a_o and a_1

These coefficients can be calculated by using the condition:

$P_n(1) = 1$ (6)

For $n = 0$ $\longrightarrow$ $P_o(x) = a_o$

at $x = 1$ $\longrightarrow$ $P_o(1) = a_o = 1$ $\longrightarrow$ $\therefore a_o = 1$

Therefore $P_o(x) = 1$ .....(7a)

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For $n = 1$ $\longrightarrow$ $P_1(x) = a_1 x$

From condition (6)

at $x = 1$ $\longrightarrow$ $P_1(1) = a_1 = 1$ $\longrightarrow$ $\therefore\ a_1 = 1$

Therefore $\boldsymbol{P_1(x) = x}$ **.....(7b)**

For $n = 2$ $\longrightarrow$ $P_2(x) = a_0\left[1 - \frac{2\times3}{2!}x^2\right] = a_0[1 - 3x^2]$

From condition (6)

at $x = 1$ $\longrightarrow$ $P_2(1) = a_0[-2] = 1$ $\longrightarrow$ $\therefore\ a_1 = -\frac{1}{2}$

Therefore $\boldsymbol{P_2(x) = \frac{1}{2}(3x^2 - 1)}$ **.....(7c)**

For $n = 3$ $\longrightarrow$ $P_3(x) = a_1\left[x - \frac{2\times5}{3!}x^3\right] = a_1\left[x - \frac{5}{3}x^3\right]$

From condition (6)

at $x = 1$ $\longrightarrow$ $P_3(1) = a_1[-\frac{2}{3}] = 1$ $\longrightarrow$ $\therefore\ a_1 = -\frac{3}{2}$

Therefore $\boldsymbol{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$ **.....(7d)**

Legendre's polynomial can be expressed as a general for any value on ***n*** (even or odd) as follows:

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k\,(2n-2k)!}{2^n\,k!\,(n-k)!\,(n-2k)!}\,x^{n-2k} \qquad (8)$$

where

$$N = \begin{cases} \frac{n}{2} & for\ n \equiv even \\ \frac{(n-1)}{2} & for\ n \equiv odd \end{cases}$$

There are two useful conditions:

$$P_n(1) = 1 \qquad (9a)$$

$$P_n(-1) = (-1)^n \qquad (9b)$$

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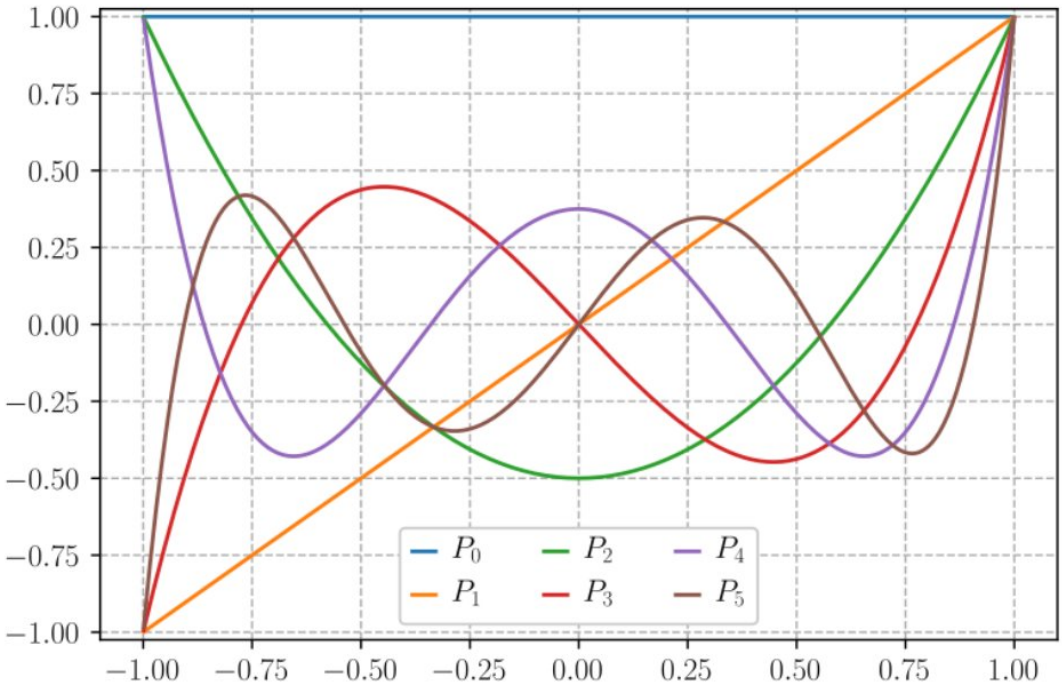
1.2 Rodrigue's formula

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2 - 1)^n \qquad (10)$$



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Polynomial Degree	Legendre Polynomial
0	$P_0(x) = \frac{1}{\sqrt{2}}$
1	$P_1(x) = \sqrt{\frac{3}{2}}\,x$
2	$P_2(x) = \frac{1}{2}\sqrt{\frac{5}{2}}\,(3x^2 - 1)$
3	$P_3(x) = \frac{1}{2}\sqrt{\frac{7}{2}}\,(5x^3 - 3x)$
4	$P_4(x) = \frac{3}{8\sqrt{2}}\,(35x^4 - 30x^2 + 3)$
5	$P_5(x) = \frac{1}{8}\sqrt{\frac{11}{2}}\,(63x^5 - 70x^3 + 15x)$
6	$P_6(x) = \frac{1}{16}\sqrt{\frac{13}{2}}\,(231x^6 - 315x^4 + 105x^2 - 5)$

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1.3 The generating function

$P_n(x)$ is the coefficients of t^n in the expansion of the generating function:

$$g(x,t) = \sum_{n=0} P_n(x)\, t^n$$

Where the generating function of Legendre's polynomial is:

$$g(x,t) = (1 - 2xt + t^2)^{-1/2}$$

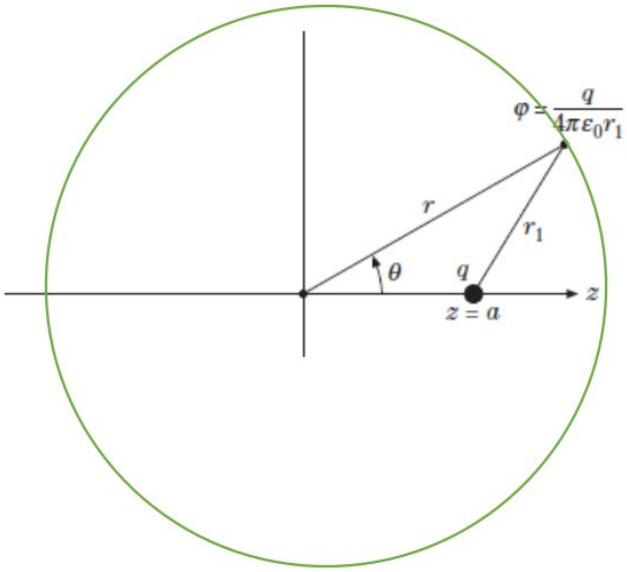
Therefore

$$g(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0} P_n(x) t^n \qquad (11)$$

Which is valid when $|x| \leq 1$ and $|t| < 1$ due to the region of convergence. Eq. (11) is of great importance in potential problems as indicated in the following example.

Example

Find the potential "ϕ" resultant from a point charge "q", located at the point $z = a$ (displaced from origin), at a point far away from the charge, i.e., Find ϕ as a function of r and θ.



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Solution

$\phi(r_1) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r_1}$ where $r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$ using the law of cosine

$$r_1 = r\sqrt{1 + \left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta}$$

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r\sqrt{1+\left(\frac{a}{r}\right)^2-2\left(\frac{a}{r}\right)\cos\theta}}$$

$$= \frac{q}{4\pi\varepsilon_o\, r}\left[1 + \left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right]^{-1/2}$$

Recall that

$$g(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0} P_n(x)t^n$$

$$\therefore g\left(\cos\theta, \frac{a}{r}\right) = \left(1 - 2\cos\theta\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^2\right)^{-1/2} = \sum_{n=0} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

This expansion is valid since $|\cos\theta| \leq 1$ and $\frac{a}{r} < 1$.

$$\therefore\ \phi(r,\theta) = \frac{q}{4\pi\varepsilon_o\, r}\sum_{n=0} P_n(\cos\theta)\ \left(\frac{a}{r}\right)^n$$

$$= \frac{q}{4\pi\varepsilon_o\, r}\left[P_0(\cos\theta) + P_1(\cos\theta)\frac{a}{r} + P_2(\cos\theta)\ \left(\frac{a}{r}\right)^2 + \cdots..\right]$$

$$= \frac{q}{4\pi\varepsilon_o\, r}\left[1 + \cos\theta\frac{a}{r} + P_2(\cos\theta)\ \left(\frac{a}{r}\right)^2 + \cdots..\right]$$

Since the observation point is far away from the point charge, one can neglect all terms of $\frac{a}{r}$ of power greater than 1. The resultant potential is:

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_o\, r}\left[1 + \cos\theta\frac{a}{r}\right]$$

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Show that

$$P_{2n}(0) = \frac{(-1)^n\,(2n)!}{(2^n\,n!)^2}$$

And

$$P_{2n+1}(0) = 0$$

n=0, 1, 2, ….

Solution:

$$\because\ g(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0} P_n(x)\,t^n$$

$$g(0,t) = (1 + t^2)^{-1/2} = \sum_{n=0} P_n(0)\,t^n$$

Using the Binomial theorem to expand the term $(1 - t^2)^{-1/2}$

$$(1-t)^{-1/2} = 1 + \frac{1}{2}\,t + \frac{1\times 3}{2^2\,2!}\,t^2 + \frac{1\times 3\times 5}{2^3\,3!}\,t^3 + \cdots + \frac{1\times 3\times 5\,\ldots\times(2n-1)}{2^n\,n!}\,t^n + \ldots$$

$$(1+t)^{-\frac{1}{2}} = (1-(-t))^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\,t + \frac{1\times 3}{2^2\,2!}\,t^2 - \frac{1\times 3\times 5}{2^3\,3!}\,t^3 + \cdots + \frac{(-1)^n 1\times 3\times 5\,\ldots\times(2n-1)}{2^n\,n!}\,t^n + \ldots$$

$$(1+t^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}\,t^2 + \frac{1\times 3}{2^2\,2!}\,t^4 - \frac{1\times 3\times 5}{2^3\,3!}\,t^6 + \cdots + \frac{(-1)^n 1\times 3\times 5\,\ldots\times(2n-1)}{2^n\,n!}\,t^{2n} + \ldots$$

$$= \sum_{n=0} P_n(0)t^n$$

Equating the coefficients of t^{2n} for both sides yields:

$$P_{2n}(0) = \frac{(-1)^n 1\times 3\times 5\,\ldots\times(2n-1)}{2^n\,n!}$$

$$P_{2n}(0) = \frac{(-1)^n(2n)!}{(2^n\,n!)^2}$$

And

$$P_{2n+1}(0) = 0$$

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1.5 The recurrence relations

Prove the recurrence relation

$$(2n+1)x\,P_n(x) = (n+1)P_{n+1} + n\,p_{n-1}(x) \qquad (12)$$

Solution:

$$\because g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0} P_n(x)\,t^n$$

Differentiate both sides with respect to *t*:

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0} n\,P_n(x)\,t^{n-1}$$

$$(1-2xt+t^2)^{-1}\,(1-2xt+t^2)^{-\frac{1}{2}}\,(x-t) = \sum_{n=0} n\,P_n(x)\,t^{n-1}$$

$$(1-2xt+t^2)^{-1}\,g(x,t)(x-t) = \sum_{n=0} n\,P_n(x)\,t^{n-1}$$

$$(1-2xt+t^2)^{-1}\sum_{n=0} P_n(x)\,t^n\;(x-t) = \sum_{n=0} n\,P_n(x)\,t^{n-1}$$

$$\sum_{n=0} P_n(x)\,t^n\;(x-t) = (1-2xt+t^2)\sum_{n=0} n\,P_n(x)\,t^{n-1}$$

$$x\sum_{n=0} P_n(x)\,t^n - \sum_{n=0} P_n(x)\,t^{n+1} = \sum_{n=0} n\,P_n(x)\,t^{n-1} - 2x\sum_{n=0} nP_n(x)\,t^n - \sum_{n=0} nP_n(x)\,t^{n+1}$$

Equating the coefficients of t^n for both sides:

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n - (n-1)P_{n-1}$$

$$x(1+2n)P_n - (1+n-1)P_{n-1} = (n+1)P_{n+1}$$

$$x(1+2n)P_n = nP_{n-1} + (n+1)P_{n+1}$$

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Problem: Use the generating function to show that

$$P_n(1) = 1$$
$$P_n(-1) = (-1)^n$$

Solution:

$$\because (1 - 2xt + t^2)^{-1/2} = \sum_{n=0} P_n(x)\, t^n$$

For $x = 1$ $\longrightarrow$ $(1 - 2t + t^2)^{-1/2} = \sum P_n(1)\, t^n$

$$(1 - t)^{-1} = \sum P_n(1)\, t^n$$

$$\sum t^n = \sum P_n(1)\, t^n \longrightarrow P_n(1) = 1$$

For $x = -1$ $\longrightarrow$ $(1 + 2t + t^2)^{-1/2} = \sum P_n(-1)\, t^n$

$$(1 + t)^{-1} = \sum P_n(-1)\, t^n$$

$$(1 - (-t))^{-1} = \sum P_n(-1)\, t^n$$

$$\sum (-t)^n = \sum P_n(-1)\, t^n$$

$$\sum (-1)^n t^n = \sum P_n(-1)\, t^n \longrightarrow P_n(-1) = (-1)^n$$

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Problem: Use the generating function to show that

$$P_n(-x) = (-1)^n P_n(x)$$

Solution:

$$\because g(x,t) = (1 - 2xt + t^2)^{-1/2}$$

$$\therefore g(-x,-t) = (1 - 2(-x)(-t) + (-t)^2)^{-1/2} = (1 - 2xt + t^2)^{-1/2} = g(x,t)$$

$$\therefore g(-x,-t) = g(x,t)$$

$$\sum_n P_n(-x)\,(-t)^n = \sum_n P_n(x)\,t^n$$

$$\sum_n P_n(-x)\,(-1)^n\,t^n = \sum_n P_n(x)\,t^n$$

$$(-1)^n P_n(-x) = P_n(x)$$

Or

$$P_n(-x) = (-1)^n P_n(x)$$

This means that Legendre's polynomials of even order have even symmetry.

And that of odd order have odd symmetry.

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1.6 The Orthogonality of Legendre's Polynomials

Legendre's polynomials are orthogonal functions in the closed interval $[-1,1]$.

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0\,, \quad if \;\; n \neq m \qquad (13)$$

Proof:

Using Eq. (1)

$$\frac{d}{dx}[(1-x^2)\, P_n'(x)] + n(n+1)\, P_n(x) = 0$$

Multiplying by $P_m(x)$ and integrate over x from -1 to 1

$$\int_{-1}^{1} \underbrace{P_m(x)}_{u} \underbrace{\frac{d}{dx}[(1-x^2)\, P_n'(x)]}_{dv}\, dx + n(n+1) \int_{-1}^{1} P_m(x)\, P_n(x) = 0$$

The first integral is solved by part $\int u\, dv = uv - \int v\, du$

By assuming $u = P_m(x)$ and $dv = \frac{d}{dx}[(1-x^2)\, P_n'(x)]\, dx$

Therefore $du = P_m'(x)\, dx$ and $v = (1-x^2)P_n'(x)$

$$[P_m(x)(1-x^2)P_n'(x)]_{-1}^{1} - \int_{-1}^{1} (1-x^2)P_n'(x)P_m'(x)\, dx + n(n+1) \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$$

The first term is equal to zero

$$[P_m(x)(1-x^2)P_n'(x)]_{-1}^{1} =$$
$$P_m(1)(1-1^2)P_n'(1) - P_m(-1)(1-(-1)^2)P_n'(-1)$$
$$= P_m(1)(0)P_n'(1) - P_m(-1)(0)P_n'(-1)$$
$$= 0$$

$$-\int_{-1}^{1} (1-x^2)P_n'(x)P_m'(x)\, dx + n(n+1) \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0 \qquad (a)$$

Interchanging n by m and vice versa

$$-\int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)\, dx + m(m+1) \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0 \qquad (b)$$

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Subtracting Eq. (b) from (a)

$$n(n+1)\int_{-1}^{1} P_m(x)\,P_n(x) - m(m+1)\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0$$

$$[n(n+1) - m(m+1)]\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

$$\begin{aligned} n(n+1) - m(m+1) &= n^2 + n - m^2 - m \\ &= n^2 - m^2 + n - m \\ &= (n-m)(n+m) + (n-m) \\ &= (n-m)(n+m+1) \end{aligned}$$

$$(n-m)(n+m+1)\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

For $n \neq m$ $\longrightarrow$ $(n-m)(n+m+1) \neq 0$

$$\therefore \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

هذا يعني ان متعددات ليجندر ذات الرتب المختلفة تكون متعامدة ضمن الفترة المغلقة [-1,1].

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1.7 The Normalization:

The normalization of Legendre's polynomials are

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1} \,, \quad for\ n \geq 0 \qquad (14)$$

Proof:

$$\because (1 - 2xt + t^2)^{-1/2} = \sum_{n=0} P_n(x)\, t^n$$

يتربيع الطرفين ينتج:

$$(1 - 2xt + t^2)^{-1} = \left[\sum_{n=0} P_n(x)\, t^n\right]^2 = \sum_{n=0} P_n(x)\, t^n \sum_{m=0} P_m(x)\, t^m$$

$$(1 - 2xt + t^2)^{-1} = \sum_{n}\sum_{m} t^{m+n}\, P_n(x)\, P_m(x)$$

Integrate both sides over x from -1 to 1

$$\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{n}\sum_{m} t^{m+n} \int_{-1}^{1} P_m(x)\, P_n(x)\, dx$$

Using the property of orthogonality

$$\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{n} t^{2n} \int_{-1}^{1} [P_n(x)]^2 \, dx$$

$$\text{L. H. S.} = \int_{-1}^{1} \frac{dx}{1 - 2xt + t^2}$$

Let $y = 1 - 2xt + t^2$

$dy = -2t\, dt$

$if\ x = -1 \rightarrow y = 1 + 2t + t^2 = (1 + t)^2$

$if\ x = 1 \rightarrow y = 1 - 2t + t^2 = (1 - t)^2$

$$= -\frac{1}{2t}\int_{(1+t)^2}^{(1-t)^2} \frac{dy}{y} = \frac{1}{2t}\int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = \frac{1}{2t}[\ln(y)]_{(1+t)^2}^{(1-t)^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

$$= \frac{1}{t}\left[2\sum_{n=0} \frac{t^{2n+1}}{2n+1}\right] = 2\sum_{n=0} \frac{t^{2n}}{2n+1}$$

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$$\therefore\ 2\sum_{n=0}\frac{t^{2n}}{2n+1}=\sum_{n}t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx$$

Equating the coefficients of t^{2n} for both sides

$$\int_{-1}^{1}[P_n(x)]^2\,dx=\frac{2}{2n+1}$$

In general, Eqs. (13) and (14) can be unified in one relation called the orthonormality relation:

$$\int_{-1}^{1}P_n(x)\,P_m(x)\,dx=\begin{cases}0\,, & if\ n\neq m\\ \dfrac{2}{2n+1}\,, & if\ n=m\end{cases}$$

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Home work #1

From Taylor expansion of the logarithm function

$$ln(1+x) = \sum_n \frac{(-1)^n x^{n+1}}{n+1}$$

Show that

$$ln\left(\frac{1+t}{1-t}\right) = 2\sum_n \frac{t^{2n+1}}{2n+1}$$

Solution:

$$ln(1-x) = ln[1+(-x)] = \sum_n \frac{(-1)^n(-x)^{n+1}}{n+1}$$

$$= \sum_n \frac{(-1)^{n+n+1} x^{n+1}}{n+1}$$

$$= \sum_n \frac{(-1)^{2n+1} x^{n+1}}{n+1}$$

$$= \sum_n \frac{(-1)\, x^{n+1}}{n+1}$$

$$ln\left(\frac{1+t}{1-t}\right) = \ln(1+t) - \ln(1-t)$$

$$= \sum_n \frac{(-1)^n t^{n+1}}{n+1} - \sum_n \frac{(-1)\, t^{n+1}}{n+1}$$

$$= \sum_n \frac{(-1)^n t^{n+1}}{n+1} + \sum_n \frac{t^{n+1}}{n+1}$$

$$= \sum_n \frac{t^{n+1}}{n+1}[1+(-1)^n]$$

هذا المقدار دائما يساوي صفر عندما تكون n عدد فردي
اما اذا كانت n عدد زوجي فانه يساوي 2

$$ln\left(\frac{1+t}{1-t}\right) = 2\sum_{n=even} \frac{t^{n+1}}{n+1}$$

$$\therefore\ ln\left(\frac{1+t}{1-t}\right) = 2\sum_{n=0} \frac{t^{2n+1}}{2n+1}$$

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1.8 Associated Legedre's Polynomials متعددات حدود ليجندر المرافقة

The associated Legendre's differential equation is:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{(1-x^2)}\right] y(x) = 0$$

And it's solutions are called the associated Legendre's polynomials that takes the following form:

$$y(x) \equiv P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \quad 0 \le m \le n$$

The associated Legendre's differential equation can be obtained after differentiating Legendre's differential equation for ***m*** times

$$(1-x^2)\, y''(x) - 2x\, y'(x) + n(n+1)\, y(x) = 0$$
$$y(x) = P_n(x)$$

The first differentiation m=1

$$(1-x^2)\, y^{(3)}(x) - 2xy^{(2)} - 2x\, y^{(2)}(x) - 2y^{(1)}(x) + n(n+1)\, y^{(1)}(x) = 0$$
$$(1-x^2)\, y^{(3)}(x) - 2x(2)y^{(2)} + [n(n+1) - 2]\, y^{(1)}(x) = 0$$

The second differentiation m=2

$$(1-x^2)\, y^{(4)} - 2xy^{(3)} - 2x(2)y^{(3)} - 4y^{(2)} + [n(n+1) - 2]\, y^{(2)} = 0$$
$$(1-x^2)\, y^{(4)} - 2x(3)y^{(3)} + [n(n+1) - 6]\, y^{(2)} = 0$$

……

After differentiating ***m*** times

$$(1-x^2)\, y^{(m+2)} - 2x(m+1)y^{(m+1)} + [n(n+1) - m(m+1)]\, y^{(m)} = 0$$

Let $u = y^m = \frac{d^m y}{dx^m} = \frac{d^m}{dx^m} P_n$

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$$(1-x^2)\,u^{(2)} - 2x(m+1)u^{(1)} + [n(n+1) - m(m+1)]\,u = 0$$

$$(1-x^2)\frac{d^2u}{dx^2} - 2x(m+1)\frac{du}{dx} + [n(n+1) - m(m+1)]\,u = 0$$

Let $V(x) = (1-x^2)^{m/2}u = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n$

or $u = (1-x^2)^{-m/2}V$

$$\frac{du}{dx} = -\frac{m}{2}\,(1-x^2)^{-\frac{m}{2}-1}(-2x)\,V + (1-x^2)^{-\frac{m}{2}}\,\frac{dV}{dx}$$

$$= m\,x(1-x^2)^{-\frac{m}{2}-1}\,V + (1-x^2)^{-\frac{m}{2}}\,\frac{dV}{dx}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left\{m\,x(1-x^2)^{-\frac{m}{2}-1}\,V\right\} + \frac{d}{dx}\left\{(1-x^2)^{-\frac{m}{2}}\,\frac{dV}{dx}\right\}$$

$$\frac{d^2u}{dx^2} = m\,\left\{x(1-x^2)^{-\frac{m}{2}-1}\frac{dV}{dx} + V\frac{d}{dx}\left[x(1-x^2)^{-\frac{m}{2}-1}\right]\right\} + \frac{d}{dx}\left[(1-x^2)^{-\frac{m}{2}}\,\frac{dV}{dx}\right]$$

$$= m\,\left\{x(1-x^2)^{-\frac{m}{2}-1}\frac{dV}{dx} + V\left[x\left(-\frac{m}{2}-1\right)(1-x^2)^{-\frac{m}{2}-2}(-2x) + (1-x^2)^{-\frac{m}{2}-1}\right]\right\}$$

$$+\left[m\,x(1-x^2)^{-\frac{m}{2}-1}\frac{dV}{dx} + (1-x^2)^{-\frac{m}{2}}\,\frac{d^2V}{dx^2}\right]$$

$$\therefore \frac{d^2u}{dx^2} = (1-x^2)^{-\frac{m}{2}}\,\frac{d^2V}{dx^2} + 2m\,x(1-x^2)^{-\frac{m}{2}-1}\frac{dV}{dx}$$

$$+ mV(1-x^2)^{-\frac{m}{2}-1}\left[2x^2\left(\frac{m}{2}+1\right)(1-x^2)^{-1} + 1\right]$$

Substituting the first and second derivatives

$$(1-x^2)\frac{d^2u}{dx^2} - 2x(m+1)\frac{du}{dx} + [n(n+1) - m(m+1)]\,u = 0$$

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$$(1-x^2)\left\{(1-x^2)^{-\frac{m}{2}}\frac{d^2V}{dx^2}+2m\,x(1-x^2)^{-\frac{m}{2}-1}\frac{dV}{dx}\right.$$
$$\left.+mV(1-x^2)^{-\frac{m}{2}-1}\left[2x^2\left(\frac{m}{2}+1\right)(1-x^2)^{-1}+1\right]\right\}$$
$$-2x(m+1)\left\{m\,x(1-x^2)^{-\frac{m}{2}-1}V+(1-x^2)^{-\frac{m}{2}}\frac{dV}{dx}\right\}$$
$$+[n(n+1)-m(m+1)]\,(1-x^2)^{-m/2}V=0$$

$$(1-x^2)^{1-\frac{m}{2}}\frac{d^2V}{dx^2}+2m\,x(1-x^2)^{-\frac{m}{2}}\frac{dV}{dx}$$
$$+mV(1-x^2)^{-\frac{m}{2}}\left[2x^2\left(\frac{m}{2}+1\right)(1-x^2)^{-1}+1\right]$$
$$-2m(m+1)x^2(1-x^2)^{-\frac{m}{2}-1}V-2x(m+1)(1-x^2)^{-\frac{m}{2}}\frac{dV}{dx}$$
$$+[n(n+1)-m(m+1)]\,(1-x^2)^{-m/2}V=0$$

$$(1-x^2)\ \frac{d^2V}{dx^2}+2m\,x\frac{dV}{dx}+mV\left[2x^2\left(\frac{m}{2}+1\right)(1-x^2)^{-1}+1\right]$$
$$-2m(m+1)x^2(1-x^2)^{-1}V-2x(m+1)\frac{dV}{dx}$$
$$+[n(n+1)-m(m+1)]\,V=0$$

$$(1-x^2)\ \frac{d^2V}{dx^2}+[2m\,x-2x(m+1)]\frac{dV}{dx}$$
$$+\left[2x^2m\left(\frac{m}{2}+1\right)(1-x^2)^{-1}+m-2m(m+1)x^2(1-x^2)^{-1}+n(n+1)\right.$$
$$\left.-m(m+1)\right]V=0$$

$$(1-x^2)\frac{d^2V}{dx^2}-2x\frac{dV}{dx}$$
$$+\left[2x^2m(1-x^2)^{-1}\left(\frac{m}{2}+1-m-1\right)+m+n(n+1)-m^2+m\right]V=0$$

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$$(1-x^2)\frac{d^2V}{dx^2}-2x\frac{dV}{dx}+\left[2x^2m(1-x^2)^{-1}\left(-\frac{m}{2}\right)+n(n+1)-m^2\right]V=0$$

$$(1-x^2)\frac{d^2V}{dx^2}-2x\frac{dV}{dx}+\left[n(n+1)-m^2-\frac{x^2m^2}{(1-x^2)}\right]V=0$$

$$(1-x^2)\frac{d^2V}{dx^2}-2x\frac{dV}{dx}+\left[n(n+1)-\frac{m^2}{(1-x^2)}\right]V=0$$

This equation is called the associated Legendre's differential equation, which is different from the conventional in containing an additional term contain the constant m.

Therefore, its solution is:

$$V(x)=P_n^m(x)=(1-x^2)^{m/2}\,u(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x),\quad 0\le m\le n$$

If $m>n$ ⟶ $P_n^m(x)=0$

If $m=0$ ⟶ $P_n^0(x)=P_n(x)$

1.9 The recurrence relations

(1) $\boxed{P_n^m(-x)=(-1)^{n+m}P_n^m(x)}$

Proof:

$$\because\ P_n^m(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$P_n^m(-x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(-x)$$

$$=(1-x^2)^{\frac{m}{2}}(-1)^m\frac{d^m}{dx^m}[(-1)^nP_n(x)]$$

$$=(1-x^2)^{\frac{m}{2}}(-1)^{n+m}\frac{d^m}{dx^m}P_n(x)$$

$$=(-1)^{n+m}(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$=(-1)^{n+m}P_n^m(x)$$

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(2) $$P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x) = (1-x^2)^{\frac{1}{2}}(2n+1)P_n^m(x)$$

Proof

$$\because P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x) = (1-x^2)^{\frac{m+1}{2}}\frac{d^{m+1}}{dx^{m+1}}P_{n+1}(x) - (1-x^2)^{\frac{m+1}{2}}\frac{d^{m+1}}{dx^{m+1}}P_{n-1}(x)$$

$$= (1-x^2)^{\frac{m+1}{2}}\frac{d^m}{dx^m}\left[\frac{d}{dx}P_{n+1}(x) - \frac{d}{dx}P_{n-1}(x)\right]$$

$$= (1-x^2)^{\frac{m+1}{2}}\frac{d^m}{dx^m}[P'_{n+1}(x) - P'_{n-1}(x)]$$

$$= (1-x^2)^{\frac{m+1}{2}}\frac{d^m}{dx^m}[(2n+1)P_n(x)]$$

$$= (1-x^2)^{\frac{1}{2}}(2n+1)(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)$$

$$= (1-x^2)^{\frac{1}{2}}(2n+1)P_n^m(x)$$

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