



BESSEL'S FUNCTIONS

ADVANCED APPLIED MATHEMATICS

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ف 415

Bessel's functions are solutions of one of the most important linear differential equation called *Bessel's* Differential equation. The following is the *Bessel's* differential equation of order c , where c is positive real number.

$$x^2 \frac{d^2 y(x)}{dx^2} + x \frac{dy(x)}{dx} + (x^2 - c^2) y(x) = 0 \quad (1)$$

or may be written as:

$$x^2 y''(x) + x y'(x) + (x^2 - c^2) y(x) = 0 \quad (2)$$

The Series solution:

Since any analytic function can be expanded as a power series, one may consider the following solution:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r} \quad (3)$$

Where $a_{-1} = a_{-2} = \dots = 0$

The coefficients a_k 's and the arbitrary constant r must be determined.

بما ان $y(x)$ المعطى بالصيغة (3) هي حل للمعادلة التفاضلية (1)، اذن فهي تحقق معادلة بيسل التفاضلية. نجد المشتقة الأولى والثانية للدالة $y(x)$ نسبة الى x ونعوضهما بالمعادلة التفاضلية (1) وكالتالي:

$$\frac{dy}{dx} = \sum (k+r) a_k x^{k+r-1}$$

$$\frac{d^2 y}{dx^2} = \sum (k+r)(k+r-1) a_k x^{k+r-2}$$

Substitute the derivatives into eq. (1)

$$x^2 \sum_k (k+r)(k+r-1) a_k x^{k+r-2} + x \sum_k (k+r) a_k x^{k+r-1} + (x^2 - c^2) \sum_k a_k x^{k+r} = 0$$

$$\sum_k (k+r)(k+r-1) a_k x^{k+r} + \sum_k (k+r) a_k x^{k+r} + \sum_k a_k x^{k+r+2} - c^2 \sum_k a_k x^{k+r} = 0$$

$$\sum_k \{[(k+r)(k+r-1) + (k+r) - c^2] a_k x^{k+r} + a_k x^{k+r+2}\} = 0$$

(1)

بما ان متسلسلة القوى هذه تساوي صفر فان معاملات x المرفوعة لأي قوة يجب ان تساوي صفر. سوف نختار معاملات x المرفوعة للأس $k+r$ (x^{k+r}) .

عامل مشترك

$$[(k+r)(k+r-1) + (k+r) - c^2]a_k + a_{k-2} = 0$$

$$[(k+r)(k+r-1+1) - c^2]a_k + a_{k-2} = 0$$

$$[(k+r)(k+r) - c^2]a_k + a_{k-2} = 0$$

$$\therefore [(k+r)^2 - c^2]a_k + a_{k-2} = 0 \text{ for } k=0, 1, 2, \dots \quad (4)$$

This is a recursive relation. It combines the odd coefficients with each other and also combine the even coefficients with each other.

$$\text{If } k=0 \rightarrow (r^2 - c^2)a_0 = 0 \quad \text{since } a_{-2} = 0$$

$$\therefore a_0 \neq 0 \rightarrow r^2 - c^2 = 0 \rightarrow r = \pm c$$

$$\text{If } k=1 \rightarrow [(r+1)^2 - c^2]a_1 = 0 \quad \text{since } a_{-1} = 0$$

$$[(r+1)^2 - r^2]a_1 = 0$$

بما ان المقدار داخل الاقواس المربعة لا يمكن ان يساوي صفر فان a_1 يجب ان يساوي صفر وبالتالي وحسب العلاقة (4) فان جميع المعاملات الفردية يجب ان تساوي صفر. وبذلك يتبقى من المتسلسلة المعاملات الزوجية فقط.

Assume the solution of $r = c$. Eq. (4) becomes

$$[(k+c)^2 - c^2]a_k + a_{k-2} = 0$$

$$[k^2 + 2kc + c^2 - c^2]a_k + a_{k-2} = 0$$

$$[k^2 + 2kc]a_k + a_{k-2} = 0$$

$$k(k+2c)a_k + a_{k-2} = 0$$

(2)

$$\therefore a_k = \frac{-1}{k(k+2c)} a_{k-2}, \quad k = 2, 4, 6, \dots \quad (5)$$

Only the Even coefficients

Therefore, the solution $y(x)$ of the *Bessel's diff. eq.* is:

$$y(x) = \sum_{m=0}^{\infty} a_{2m} x^{2m+c} \quad (6)$$

Finding the even coefficients, a_{2m} from Eq. (5)

$$\text{For } k=2 \quad a_2 = \frac{-a_0}{2(2+2c)} = \frac{-a_0}{2^2(1+c)}$$

$$\text{For } k=4 \quad a_4 = \frac{-a_2}{4(4+2c)} = \frac{-a_2}{2^2 \cdot 2(2+c)} = \frac{a_0}{2^4 \cdot 2(1+c)(2+c)}$$

$$\text{For } k=6 \quad a_6 = \frac{-a_4}{6(6+2c)} = \frac{-a_4}{2^2 \cdot 3(3+c)} = \frac{-a_0}{2^6 \cdot 3! (1+c)(2+c)(3+c)}$$

.....

$$\text{For } k=2m \quad a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (1+c)(2+c)(3+c) \dots (m+c)}, \quad m \geq 1$$

$$\text{If } c \equiv \text{integer} \longrightarrow (1+c)(2+c)(3+c) \dots (m+c) = \frac{(m+c)!}{c!}$$

$$\text{In general } c \text{ is a real positive} \longrightarrow = \frac{\Gamma(m+c+1)}{\Gamma(c+1)}$$



using $\Gamma(x+1) = x!$

$$\therefore a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!} \frac{\Gamma(c+1)}{\Gamma(m+c+1)}$$

the only remaining unknown coefficient is a_0 which can be determined by normalization:

$$a_0 = \frac{1}{2^c \Gamma(c+1)}$$

$$\begin{aligned} \therefore a_{2m} &= \frac{(-1)^m}{2^{2m} m!} \frac{1}{2^c \Gamma(c+1)} \frac{\Gamma(c+1)}{\Gamma(m+c+1)} \\ &= \frac{(-1)^m}{2^{2m+c} m! \Gamma(m+c+1)} \end{aligned}$$

(3)

Therefore, the solution $y(x)$ of *Bessel's diff. eq.* for $r=c$ is called *First kind Bessel's function of order c*, $J_c(x)$

$$y(x) \equiv J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c} \quad (7a)$$

If $c=n$ (integer),

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{x}{2}\right)^{2m+c} \quad (7b)$$

The other independent solution is obtained for $r = -c$

$$J_{-c}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m-c+1)} \left(\frac{x}{2}\right)^{2m-c} \quad (8)$$

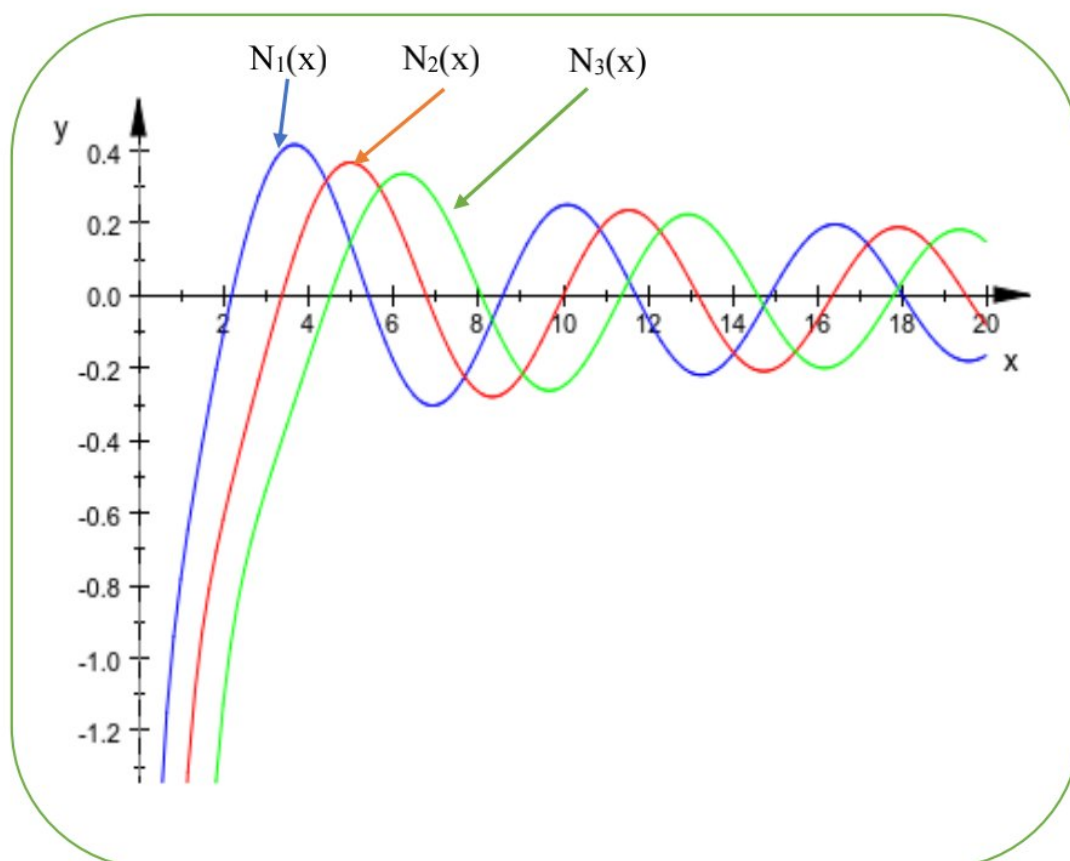
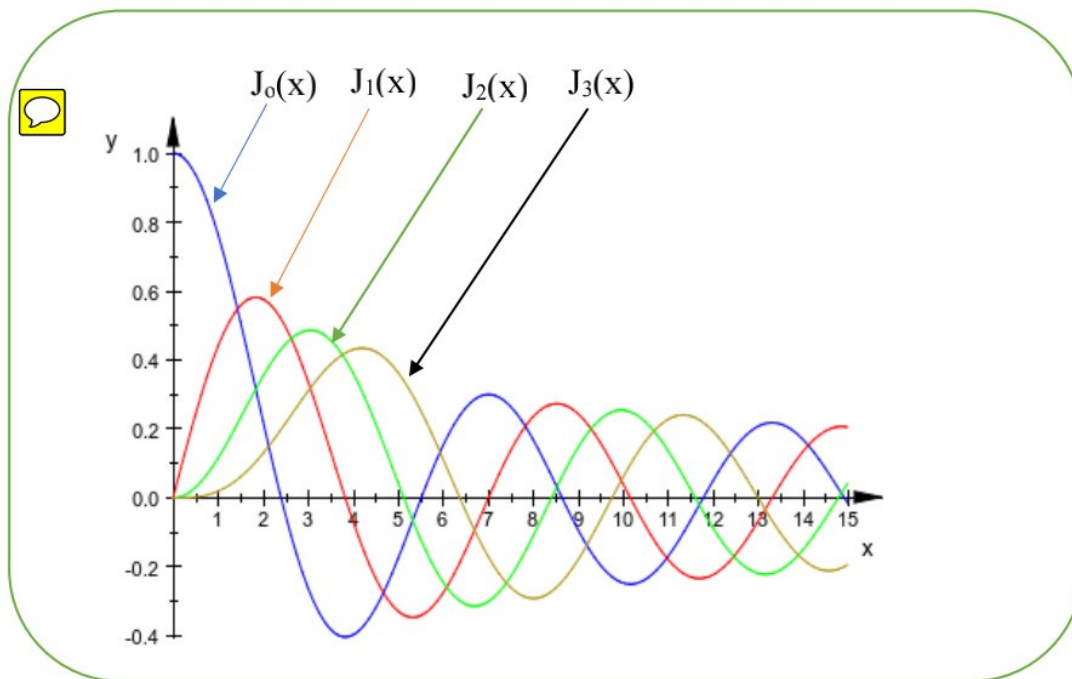
The superposition of these two independent solutions gives a new independent solution called *second kind Bessel's function of order c* "Neuman function", $N_c(x)$

$$N_c(x) = \frac{\cos(c\pi) J_c(x) - J_{-c}(x)}{\sin(c\pi)} \quad (9)$$

Therefore, the general solution of *Bessel's* differential equation is:

$$y(x) = A J_c(x) + B N_c(x), \quad \text{if } c \equiv \text{integer} \quad \leftarrow$$

$$y(x) = A J_c(x) + B J_{-c}(x), \quad \text{if } c \equiv \text{not integer}$$



(5)

Problem: Prove the following relation which connect the positive and negative order of 1st kind Bessel's functions:

$$J_{-n}(x) = (-1)^n J_n(x) \quad (10)$$

Solution

$$\begin{aligned} \therefore J_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} \\ \therefore J_{-n}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n} \end{aligned}$$

$$m - n + 1 > 0 \longrightarrow m > n - 1 \longrightarrow m = n, n + 1, n + 2, \dots \infty$$

Therefore, the series must be started from $m = n$

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m}{m! \Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n}$$

Making the following transformation:

$$\text{Let } m - n = k \longrightarrow m = k + n$$

This means that the summation index must be transformed from m to k

$$\begin{aligned} J_{-n}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)! \Gamma(k+n-n+1)} \left(\frac{x}{2}\right)^{2(k+n)-n} \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)! \Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+2n-n} \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)! k!} \left(\frac{x}{2}\right)^{2k+n} \\ &= (-1)^n J_n(x) \end{aligned}$$

$$= J_n(x)$$

حسب العلاقة (7b)

(6)

The Generating function

Bessel's generating function is:

$$g(x, t) = e^{\frac{x}{2}(t-t^{-1})}$$

Bessel's functions can be determined by using the generating function where $J_n(x)$ are the coefficients of t^n in the expansion of the generating function.

$$g(x, t) = e^{\frac{x}{2}(t-t^{-1})} = \sum_n J_n(x) t^n$$

Problem: From the expansion of the generating function, derive the series form of Bessel's function (eq. 7b)

$$\because e^y = \sum_{r=0}^{\infty} \frac{y^r}{r!}$$

$$\therefore g(x, t) = e^{\frac{xt}{2}} e^{\frac{-x}{2t}} = \sum_{r=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^r}{r!} \sum_{s=0}^{\infty} \frac{\left(\frac{-x}{2t}\right)^s}{s!} = \sum_n J_n(x) t^n$$

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{r+s}}{r! s!} t^{r-s} = \sum_n J_n(x) t^n$$

Let $r - s = n \rightarrow r = s + n$ Transformation from r to n

$$\sum_n \sum_s \frac{(-1)^s \left(\frac{x}{2}\right)^{(s+n)+s}}{(s+n)! s!} t^{(s+n)-s} = \sum_n J_n(x) t^n$$

$$\sum_n \sum_s \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+n}}{(s+n)! s!} t^n = \sum_n J_n(x) t^n$$

Equating the coefficients of t^n for both sides yields:

$$\therefore J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+n)! s!} \left(\frac{x}{2}\right)^{2s+n}$$

(7)

Recurrence Relations

$$1. \quad J'_c(x) = -J_{c+1}(x) + \frac{c}{x} J_c(x) \quad (11)$$

Proof:

$$J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c}$$

Differentiate $J_c(x)$ w.r.t. x

$$\begin{aligned} J'_c(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} (2m+c) x^{2m+c-1} \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} (2m) x^{2m+c-1} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} (c) x^{2m+c-1} \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c-1} + \underbrace{\frac{c}{x} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c}}_{J_c(x)} \end{aligned}$$

Set $m-1 = k$
 $\therefore m = k+1$

$J_c(x)$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (k+1)}{(k+1)! \Gamma(k+1+c+1)} \left(\frac{x}{2}\right)^{2(k+1)+c-1} + \frac{c}{x} J_c(x) \\ &= - \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{(k+1)! k! \Gamma(k+1+c+1)} \left(\frac{x}{2}\right)^{2k+2+c-1} + \frac{c}{x} J_c(x) \\ &= - \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+(c+1)+1)} \left(\frac{x}{2}\right)^{2k+(c+1)}}_{J_{c+1}(x)} + \frac{c}{x} J_c(x) \\ &= -J_{c+1}(x) + \frac{c}{x} J_c(x) \end{aligned}$$

Notice that, for $c = 0 \rightarrow J_1(x) = -J'_0(x)$

(8)

$$2. \quad J'_c(x) = \frac{1}{2} (J_{c-1} - J_{c+1}) \quad (12)$$

Proof:

$$J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c}$$

Differentiate $J_c(x)$ w.r.t. x

$$J'_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} (2m+c) x^{2m+c-1}$$

$$J'_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} 2^{2m+c-1} \left(\frac{2m+c}{2}\right) x^{2m+c-1}$$

$$J'_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{2m+c}{2}\right) \left(\frac{x}{2}\right)^{2m+c-1}$$

$$J'_c(x) = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} (2m+c) \left(\frac{x}{2}\right)^{2m+c-1}$$

$$J'_c(x) = \frac{1}{2} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m (m+c)}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c-1} + \sum_{m=1}^{\infty} \frac{(-1)^m m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c-1} \right\}$$

$$\begin{aligned} \Gamma(x+1) &= x \Gamma(x) \\ \Gamma(m+c+1) &= (m+c) \Gamma(m+c) \end{aligned}$$

$$\begin{aligned} \text{Set } m-1 &= k \\ m &= k+1 \end{aligned}$$

$$\begin{aligned} J'_c(x) &= \frac{1}{2} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m (m+c)}{m! (m+c) \Gamma(m+c)} \left(\frac{x}{2}\right)^{2m+c-1} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (k+1)}{(k+1)! \Gamma(k+1+c+1)} \left(\frac{x}{2}\right)^{2(k+1)+c-1} \right\} \\ &= \frac{1}{2} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c)} \left(\frac{x}{2}\right)^{2m+c-1} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (k+1)}{(k+1) k! \Gamma(k+(c+1)+1)} \left(\frac{x}{2}\right)^{2k+c+1} \right\} \\ &= \frac{1}{2} \left\{ J_{c-1}(x) - \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+(c+1)+1)} \left(\frac{x}{2}\right)^{2k+c+1} \right\} \\ &= \frac{1}{2} \{J_{c-1}(x) - J_{c+1}(x)\} \end{aligned}$$

(9)

$$3. \quad J'_c(x) = J_{c-1}(x) - \frac{c}{x} J_c \quad (13)$$

Proof:

Can be derived by subtracting eq. (11) from eq. (12)

$$4. \quad J_{c-1}(x) + J_{c+1}(x) = \frac{2c}{x} J_c \quad (14)$$

Proof:

Can be derived by subtracting 2*eq. (11) from eq. (12). Or by using the generating function as follows:

$$\because g(x, t) = e^{\frac{x}{2}(t-t^{-1})} = \sum_n J_n(x) t^n$$

Differentiate both sides w.r.t. t

$$\frac{\partial g}{\partial t} = \frac{x}{2} (1 + t^{-2}) e^{\frac{x}{2}(t-t^{-1})} = \sum_n n J_n(x) t^{n-1}$$

$$\frac{x}{2} (1 + t^{-2}) g(x, t) = \sum_n n J_n(x) t^{n-1}$$

$$\frac{x}{2} (1 + t^{-2}) \sum_n J_n(x) t^n = \sum_n n J_n(x) t^{n-1}$$

$$\frac{x}{2} \left\{ \sum_n J_n(x) t^n + \sum_n J_n(x) t^{n-2} \right\} = \sum_n n J_n(x) t^{n-1}$$

Equating the coefficients of t^{n-1} for both sides yields:

$$\frac{x}{2} \{J_{n-1}(x) + J_{n+1}(x)\} = n J_n(x)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$J_{c-1}(x) + J_{c+1}(x) = \frac{2c}{x} J_c(x)$$

(10)

$$5. \quad \frac{d}{dx}(x^c J_c) = x^c J_{c-1} \quad (15)$$

Proof:

$$\because J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c}$$

Multiply J_c by x

$$x^c J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1) 2^{2m+c}} x^{2m+2c}$$

Then differentiate w.r.t x

$$\begin{aligned} \frac{d}{dx}[x^c J_c] &= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2c)}{m! \Gamma(m+c+1) 2^{2m+c}} x^{2m+2c-1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+c)}{m! (m+c) \Gamma(m+c) 2^{2m+c}} x^{2m+2c-1} \\ &= x^c \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c) 2^{2m+c-1}} x^{2m+c-1} \\ &= x^c \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c)} \left(\frac{x}{2}\right)^{2m+c-1} \\ &= x^c J_{c-1}(x) \end{aligned}$$

Integrating both sides yield the following integral

$$\int x^c J_{c-1}(x) dx = x^c J_c(x) + \text{const.} \quad (16)$$

$$6. \quad \frac{d}{dx}(x^{-c}J_c) = -x^{-c}J_{c+1}(x) \quad (17)$$

Proof:

$$\because J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c}$$

Multiply J_c by x^{-c}

$$x^{-c}J_c(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+c+1) 2^{2m+c}} x^{2m}$$

Then differentiate w.r.t x

$$\frac{d}{dx}[x^{-c}J_c] = \sum_{m=1}^{\infty} \frac{(-1)^m 2m}{m! \Gamma(m+c+1) 2^{2m+c}} x^{2m-1}$$

$$= x^{-c} \sum_{m=1}^{\infty} \frac{(-1)^m m}{m! \Gamma(m+c+1) 2^{2m+c-1}} x^{2m+c-1}$$

$$= x^{-c} \sum_{m=1}^{\infty} \frac{(-1)^m m}{m! \Gamma(m+c+1)} \left(\frac{x}{2}\right)^{2m+c-1}$$

Set $m-1 = k$
 $m = k+1$

$$= x^{-c} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (k+1)}{(k+1)! \Gamma(k+1+c+1)} \left(\frac{x}{2}\right)^{2k+c+1}$$

$$= -x^{-c} \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{(k+1) k! \Gamma(k+1+c+1)} \left(\frac{x}{2}\right)^{2k+c+1}$$

$$= -x^{-c} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+c+2)} \left(\frac{x}{2}\right)^{2k+c+1}$$

$$= -x^{-c}J_{c+1}(x)$$

Integrating both sides yield the following integral

(12)

$$\int x^{-c} J_{c+1}(x) dx = x^{-c} J_c(x) + \text{const.} \quad (18)$$

All of the above recurrence relations are still valid for the second kind *Bessel's* functions and also for the third kind. For example, one can show that the recurrence relation, eq. (15) is true for *Neuman* function.

Example: Show that

$$\frac{d}{dx}(x^c N_c) = x^c N_{c-1}(x) \quad (19)$$

and

$$\frac{d}{dx}(x^{-c} N_c) = -x^{-c} N_{c+1}(x) \quad (20)$$

Solution:

$$\because N_c(x) = \frac{\cos(c\pi) J_c - J_{-c}}{\sin(c\pi)}$$

Multiply N_c by x^c

$$x^c N_c(x) = \frac{\cos(c\pi) x^c J_c - x^c J_{-c}}{\sin(c\pi)}$$

$x^c J_{c-1}$

$-x^c J_{-c+1}$

Differentiate w.r.t x

$$\frac{d}{dx}[x^c N_c(x)] = \frac{\cos(c\pi) \frac{d}{dx}[x^c J_c] - \frac{d}{dx}[x^c J_{-c}]}{\sin(c\pi)}$$

$$\begin{aligned} &= x^c \frac{\cos(c\pi) J_{c-1} - J_{-c+1}}{\sin(c\pi)} \\ &= x^c \frac{-\cos(c-1)\pi J_{c-1} + J_{-(c-1)}}{-\sin(c-1)\pi} \\ &= x^c \frac{\cos(c-1)\pi J_{c-1} - J_{-(c-1)}}{\sin(c-1)\pi} \\ &= x^c N_{c-1}(x) \end{aligned}$$

(13)

$\cos(c\pi) = -\cos(c-1)\pi$
 $\sin(c\pi) = -\sin(c-1)\pi$

بنفس الطريقة يمكن اثبات العلاقة (20)

Integral forms of Bessel's functions

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin \theta) d\theta, \quad \text{for } n = 0, 1, 2, \dots \quad (21)$$

Proof:

The generating function:

$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

$$J_{-n} = (-1)^n J_n$$

$$\begin{aligned} e^{\frac{x}{2}(t-t^{-1})} &= \dots + J_{-3}t^{-3} + J_{-2}t^{-2} + J_{-1}t^{-1} + J_0 + J_1t^1 + J_2t^2 + J_3t^3 + \dots \\ &= J_0 + J_1(t - t^{-1}) + J_2(t^2 + t^{-2}) + J_3(t^3 - t^{-3}) + J_4(t^4 + t^{-4}) + \dots \end{aligned}$$

$$\text{Let } t = e^{i\theta} \longrightarrow \frac{x}{2}(t - t^{-1}) = \frac{x}{2}(e^{i\theta} - e^{-i\theta}) = \frac{x}{2} 2i \sin \theta = i x \sin \theta$$

$$\text{And } t - t^{-1} = 2i \sin \theta, \quad t^2 + t^{-2} = 2 \cos \theta$$

$$t^3 - t^{-3} = 2i \sin 3\theta, \quad t^4 + t^{-4} = 2 \cos 4\theta$$

$$e^{i x \sin \theta} = J_0 + 2i[J_1 \sin \theta + J_3 \sin 3\theta + \dots] + 2[J_2 \cos 2\theta + J_4 \cos 4\theta + \dots]$$

$$\therefore e^{i x \sin \theta} = \cos(x \sin \theta) + i \sin(x \sin \theta)$$

Equating real parts for both sides yields:

$$\cos(x \sin \theta) = J_0 + 2[J_2 \cos 2\theta + J_4 \cos 4\theta + \dots] = J_0 + 2 \sum_{\substack{m=\text{even} \\ m \neq 0}} J_m \cos m\theta \quad (22)$$

Equating imaginary parts for both sides $\cos(x \sin \theta)$ yields:

$$\sin(x \sin \theta) = 2[J_1 \sin \theta + J_3 \sin 3\theta + \dots] = 2 \sum_{m=\text{odd}} J_m \sin m\theta \quad (23)$$

(14)

Using the following orthogonality properties for *cosinusoidal* functions

$$\int_0^\pi \cos n\theta \cos m\theta \, d\theta = \int_0^\pi \sin n\theta \sin m\theta \, d\theta = \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & n = m \neq 0 \end{cases}$$

Multiply Eq. (22) $\times \cos n\theta$ and integrate over θ from 0 to π yields

$$\int_0^\pi \cos n\theta \cos(x \sin \theta) \, d\theta = J_0 \int_0^\pi \cos n\theta \, d\theta + 2 \sum_{\substack{m=\text{even} \\ m \neq 0}} J_m \int_0^\pi \cos n\theta \cos m\theta \, d\theta$$



هذا التكامل = صفر

Since the summation is over m , therefore, all these integrals are equal to zero due to the orthogonality properties except the integral when $m=n$ which is equal to $\pi/2$

$$\int_0^\pi \cos n\theta \cos(x \sin \theta) \, d\theta = 2 J_n \frac{\pi}{2}, \text{ if } n \equiv \text{even}, n = 0, 2, 4, \dots$$

$$\therefore J_n = \frac{1}{\pi} \int_0^\pi \cos n\theta \cos(x \sin \theta) \, d\theta, \quad n \equiv \text{even}, n = 0, 2, 4, \dots \quad (24)$$

And similarly,

$$J_n = \frac{1}{\pi} \int_0^\pi \sin n\theta \sin(x \sin \theta) \, d\theta, \quad n \equiv \text{odd}, n = 1, 3, 5, \dots \quad (25)$$



Adding Eq. (24) and Eq. (25) and using $\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 = \cos(\theta_1 - \theta_2)$ yields:



$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin \theta) \, d\theta$$

(15)