

BETA Function

Advanced Applied Mathematical



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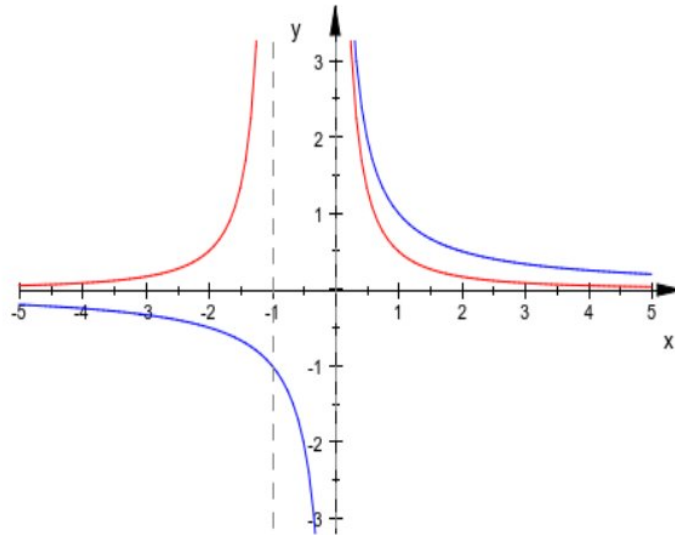
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Beta function is defined by the integral

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0 \quad (1)$$



Some properties of Beta function

1. $B(x, y)$ is symmetric function in x and y variables, i.e.,

$$B(x, y) = B(y, x) \quad (2)$$

Proof:

Since $B(x, y) = \int_{t=0}^1 t^{x-1} (1-t)^{y-1} dt$

Let $1-t = u \rightarrow t = 1-u \rightarrow dt = -du$

If $t = 0 \rightarrow u = 1$

If $t = 1 \rightarrow u = 0$

$$\begin{aligned} B(x, y) &= \int_{u=1}^0 (1-u)^{x-1} u^{y-1} (-du) && \text{rearranging} \\ &= -\int_1^0 u^{y-1} (1-u)^{x-1} du && \text{changing integral limit} \\ &= \int_0^1 u^{y-1} (1-u)^{x-1} du && \text{comparing with Eq.(1)} \\ &= B(y, x) \end{aligned}$$

(1)

2. Beta function takes several integral forms as follows:

$$B(x, y) = 2 \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{y-1} \theta d\theta \quad (3)$$

Proof:

Since $B(x, y) = \int_{t=0}^1 t^{x-1} (1-t)^{y-1} dt$

Let $t = \sin^2 \theta \quad 1-t = 1 - \sin^2 \theta = \cos^2 \theta$

$$dt = 2 \sin \theta \cos \theta d\theta$$

If $t = 0 \rightarrow \sin^2 \theta = 0 \rightarrow \theta = 0$

If $t = 1 \rightarrow \sin^2 \theta = 1 \rightarrow \theta = \pi/2$

$$\begin{aligned} \therefore B(x, y) &= \int_{\theta=0}^{\pi/2} (\sin^2 \theta)^{x-1} (\cos^2 \theta)^{y-1} (2 \sin \theta \cos \theta d\theta) \\ &= 2 \int_0^{\pi/2} \sin^{2x-2} \theta \cos^{2y-2} \theta \sin \theta \cos \theta d\theta \\ &= 2 \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta \end{aligned}$$

3. $B(x, y) = \frac{1}{a^{x+y-1}} \int_0^a u^{x-1} (a-u)^{y-1} du \quad (4)$

Proof:

Since $B(x, y) = \int_{t=0}^1 t^{x-1} (1-t)^{y-1} dt$

Let $t = \frac{u}{a} \rightarrow dt = \frac{du}{a}$

If $t = 0 \rightarrow u = 0$

If $t = 1 \rightarrow u = a$

$$\begin{aligned} \therefore B(x, y) &= \int_{u=0}^a \left(\frac{u}{a}\right)^{x-1} \left(1 - \frac{u}{a}\right)^{y-1} \left(\frac{du}{a}\right) \\ &= \frac{1}{a^{m-1+n-1+1}} \int_0^a u^{x-1} (a-u)^{y-1} du \\ &= \frac{1}{a^{m+n-1}} \int_0^a u^{x-1} (a-u)^{y-1} du \end{aligned}$$

(2)

$$4. \quad B(x, y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} du \quad (5)$$

Proof:

$$\text{Since } B(x, y) = \int_{t=0}^1 t^{x-1} (1-t)^{y-1} dt$$

$$\text{Let } t = \frac{u}{1+u} \longrightarrow dt = \frac{1+u-u}{(1+u)^2} du = \frac{du}{(1+u)^2}$$

$$\text{If } t = 0 \longrightarrow u = 0$$

$$\text{If } t = 1 \longrightarrow u = \infty$$

$$\begin{aligned} \therefore B(x, y) &= \int_{u=0}^{\infty} \left(\frac{u}{1+u} \right)^{x-1} \left(1 - \frac{u}{1+u} \right)^{y-1} \frac{du}{(1+u)^2} \\ &= \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x-1+y-1+2}} du \\ &= \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} du \end{aligned}$$

5. The relation between Beta and Gamma function:

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad (6)$$

Proof:

$$\text{Since } \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx,$$

$$\text{and } \Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x} dx$$

$$\begin{aligned} \text{Hence } \Gamma(m) \Gamma(n) &= \int_0^{\infty} u^{m-1} e^{-u} du \int_0^{\infty} v^{n-1} e^{-v} dv \\ &= \int_{u=0}^{\infty} \int_{v=0}^{\infty} u^{m-1} v^{n-1} e^{-(u+v)} du dv \end{aligned}$$

$$\begin{aligned} \text{Let } u &= x^2 \quad \text{and} \quad v = y^2 \\ du &= 2x dx \quad \text{and} \quad dv = 2y dy \end{aligned}$$

$$\begin{aligned} \Gamma(m) \Gamma(n) &= \int_{x=0}^{\infty} \int_{y=0}^{\infty} x^{2m-2} y^{2n-2} e^{-(x^2+y^2)} 2x dx 2y dy \\ &= 4 \int_0^{\infty} \int_0^{\infty} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx dy \end{aligned}$$

(3)

Transform the integration from cartesian to polar coordinates, where

$$x = r \cos \theta \text{ and}$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

$$dx dy = r dr d\theta$$

$$\begin{aligned} \Gamma(m) \Gamma(n) &= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} (r \cos \theta)^{2m-1} (r \sin \theta)^{2n-1} e^{-r^2} r dr d\theta \\ &= 4 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta \int_0^{\infty} r^{2(m+n)-1} e^{-r^2} dr \end{aligned}$$

$$\therefore B(m, n) = 2 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta$$

And

$$\int_0^{\infty} r^k e^{-ar^p} dr = \frac{\Gamma\left(\frac{k+1}{p}\right)}{p a^{\frac{k+1}{p}}}$$

$$\therefore \Gamma(m) \Gamma(n) = 2 B(m, n) \frac{\Gamma(m+n)}{2}$$

$$\text{Therefore, } B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

(4)

Example: Show that

$$\int_0^{\pi/2} \sin^a \theta d\theta = \frac{\Gamma\left(\frac{a+1}{2}\right) \sqrt{\pi}}{2 \Gamma\left(\frac{a+2}{2}\right)}, \quad a > -1$$

Solution

With the help of Eq. (3):

$$\begin{aligned} \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta &= \frac{1}{2} B(x, y) \\ &= \frac{1}{2} \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0 \end{aligned}$$

Consider $2x - 1 = a \rightarrow x = \frac{a+1}{2} \rightarrow \because x > 0 \rightarrow \frac{a+1}{2} > 0 \rightarrow \therefore a > -1$

and $2y - 1 = 0 \rightarrow y = \frac{1}{2}$

$$\begin{aligned} \therefore \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta &= \int_0^{\pi/2} \sin^a \theta d\theta = \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{a+2}{2}\right)} \\ &= \frac{\Gamma\left(\frac{a+1}{2}\right) \sqrt{\pi}}{2 \Gamma\left(\frac{a+2}{2}\right)}, \quad a > -1 \end{aligned}$$

Useful relation for the product of gamma function and its complementary:

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad (7)$$

Such as $\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin \pi x}$

أمثلة محلولة في ص 94 (مطلوبة)

1. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ sol. $\frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{4 \Gamma\left(\frac{3}{4}\right)}$

2. Show that $\int_0^2 \sqrt[3]{4x-x^3} dx = \frac{4\pi}{3\sqrt{3}}$

(5)