

Gamma Function

Advanced Applied
Mathematical



Dr. Sabah M.M. Ameen

Physics Department

College of Science

University of Basrah

Advanced Applied Mathematical

Gamma function is defined by the integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0 \quad (1)$$

Some useful remarks:

$$\int f(x) dx = G(x)$$

$$\int_a^b f(x) dx = \text{Constant}$$

$$\int_a^x f(y) dy = Q(x)$$

$$\int_0^{\infty} e^{-t} t^{n-1} dt = \Gamma(n), \quad n > 0$$

$$\int_0^{\infty} e^{-y} y^{n-1} dy = \Gamma(n)$$

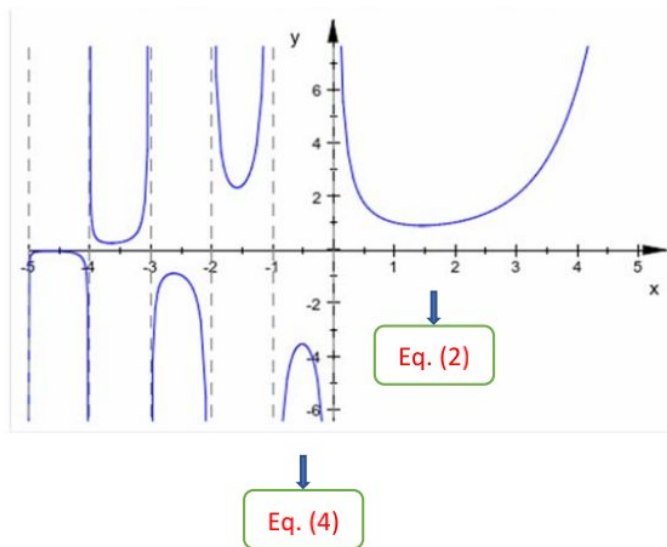
$$\int_0^{\infty} e^{-t} t^{x-1} dt = \Gamma(x)$$

$$\int_0^{\infty} e^{-t} t^{-1} dt = \Gamma(0), \quad \times$$

$$\int_0^{\infty} e^{-t} t^{-1/2} dt = \Gamma(1/2)$$

$$\int_0^{\infty} e^{-t} t^{2.2} dt = \Gamma(3.2)$$

$$\int_0^{\infty} e^{-t} t^x dt = \Gamma(x+1), \quad x+1 > 0 \quad \text{or} \quad x > -1$$



Some properties of Gamma function

1. The recursive relation

$$\Gamma(x+1) = x \Gamma(x), \quad x > 0 \quad (2)$$

من هذه العلاقة يمكن إيجاد دالة كما لجميع قيم x الموجبة.

Proof:

$$\text{Since } \Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt$$

integrate by part

$$\int u dv = uv - \int v du$$

$$\text{Let } u = t^x, dv = e^{-t} dt$$

$$du = x t^{x-1} dt, v = -e^{-t}$$

$$\begin{aligned} \text{Therefore, } \Gamma(x+1) &= [t^x (-e^{-t})]_0^{\infty} - \int_0^{\infty} (-e^{-t}) x t^{x-1} dt \\ &= -[t^x e^{-t}]_0^{\infty} + x \int_0^{\infty} t^{x-1} e^{-t} dt \\ &= -[\infty e^{-\infty} - 0 e^{-0}] + x \Gamma(x) \\ &= -[0 - 0] + x \Gamma(x) \\ &= x \Gamma(x) \end{aligned}$$

$$e^{-\infty} = 0$$

2. $\Gamma(1) = 1$

$$\text{Since } \Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt$$

$$\text{Therefore, } \Gamma(1) = \int_0^{\infty} t^0 e^{-t} dt$$

$$= \int_0^{\infty} e^{-t} dt$$

$$= \lim_{R \rightarrow \infty} [-e^{-t}]_0^R$$

$$= -\left[\lim_{R \rightarrow \infty} e^{-R} - e^0 \right]$$

$$= -[0 - 1] = 1$$

(2)

$$\lim_{R \rightarrow \infty} e^{-R} = 0$$

3. Gamma function is a generalized form for the factorial function.

If x is integer (n), gamma function became factorial function

i.e.,

$$\Gamma(n+1) = n! , \quad n > 0 \quad (3)$$

Proof:

Using eq.(2) for integer values of x

$$\begin{aligned} \Gamma(n+1) &= n \Gamma(n) && \text{using } \Gamma(n) = (n-1)\Gamma(n-1) \\ &= n(n-1)\Gamma(n-1) && \text{using } \Gamma(n-1) = (n-2)\Gamma(n-2) \\ &= n(n-1)(n-2)\Gamma(n-2) \\ &= n(n-1)(n-2)(n-3)\Gamma(n-3) \\ &= n(n-1)(n-2)(n-3) \dots \dots 1. \Gamma(1) && \text{where } \Gamma(1) = 1 \\ &= n(n-1)(n-2)(n-3) \dots \dots 1 \\ &= n! \end{aligned}$$

4. Eq. (2) can be used to evaluate gamma function for negative values of x .

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} , \quad x \neq 0 , \quad x+1 > 0 \quad (4)$$

$\text{or } -1 < x < 0$

يمكن تكرار العلاقة (4) لإيجاد دالة كاما في بقية مدى x السالبة وكالتالي:

$$\Gamma(-0.5) = \frac{1}{-0.5} \Gamma(0.5) \quad \text{where } \Gamma(0.5) = \sqrt{\pi}$$

$$= -2\sqrt{\pi}$$

$$\Gamma(-2.5) = \frac{1}{-2.5} \Gamma(-1.5)$$

$$= \frac{1}{-2.5} \frac{1}{-1.5} \Gamma(-0.5)$$

$$= \frac{1}{-2.5} \frac{1}{-1.5} \frac{1}{-0.5} \Gamma(0.5)$$

$$= \frac{\sqrt{\pi}}{(-2.5)(-1.5)(-0.5)}$$

(3)

$$\Gamma(-3.2) = \frac{\Gamma(0.8)}{(-3.2)(-2.2)(-1.2)(-0.2)}$$

These relations can be generalized as follows:

$$\begin{aligned}\Gamma(x) &= \frac{1}{x} \Gamma(x+1) && \text{using } \Gamma(x+1) = \frac{1}{x+1} \Gamma(x+2) \\ &= \frac{1}{x} \frac{1}{x+1} \Gamma(x+2) && \text{using } \Gamma(x+2) = \frac{1}{x+2} \Gamma(x+3) \\ &= \frac{1}{x} \frac{1}{x+1} \frac{1}{x+2} \Gamma(x+3)\end{aligned}$$

وهكذا نستمر بالزيادة الى ان تصل الى الجزء الموجب من x .

$$\Gamma(x) = \frac{\Gamma(x+k)}{x(x+1)(x+2)\dots(x+k-1)}, \quad n \neq 0, \neq 1 \neq 2 \dots, \quad 0 < x+k < 1 \text{ and } k \text{ is integer}$$

$$\Gamma(-3.2) = \frac{\Gamma(0.8)}{(-3.2)(-2.2)(-1.2)(-0.2)}$$

أمثلة محلولة في ص 90 (مطلوبة)

Example 1: Evaluate $\int_0^{\infty} t^4 e^{-2t} dt$ sol. $\frac{3}{4}$

Example 2: show that $\int_0^1 \frac{dy}{\sqrt{\ln(1/y)}} = \sqrt{\pi}$

Example 3: show that $\int_0^{\pi/2} (\tan^3 \theta + \tan^5 \theta) e^{-\tan^2 \theta} d\theta = 1/2$

Problem 13

Show that $\int_0^{\infty} x^n e^{-ax^m} dx = \frac{\Gamma(\frac{n+1}{m})}{ma^{\frac{n+1}{m}}}, \quad (5)$

where n, m, a are real positive constants

Solution:

Since $\int_0^{\infty} y^{n-1} e^{-y} dy = \Gamma(n), n > 0$

لذلك نقوم بتحويل التكامل المطلوب اثباته (5) الى صيغة تشبه التكامل المعروف اعلاه وذلك بإجراء التحويل الاتي:

Let $y = ax^m$

هذا التحويل يعني ان متغير التكامل في العلاقة (5) يجب ان يتحول من x الى y .

$$x^m = \frac{y}{a} \rightarrow x = \left(\frac{y}{a}\right)^{\frac{1}{m}} \rightarrow x^n = \left(\frac{y}{a}\right)^{\frac{n}{m}}$$

$$m x^{m-1} dx = \frac{dy}{a} \quad dy = a m x^{m-1} dx$$

$$= a m \frac{x^m}{x} dx$$

$$= a m \frac{\left(\frac{y}{a}\right)^{\frac{1}{m}}}{\left(\frac{y}{a}\right)^{\frac{1}{m}}} dx$$

$$= a m \left(\frac{y}{a}\right)^{1-\frac{1}{m}} dx$$

$$= a m \left(\frac{y}{a}\right)^{\frac{m-1}{m}} dx$$

$$\therefore dx = \frac{dy}{ma} \left(\frac{y}{a}\right)^{\frac{1-m}{m}}$$

Therefore eq.(5) becomes

$$\begin{aligned} \int_0^{\infty} x^n e^{-ax^m} dx &= \int_0^{\infty} \left(\frac{y}{a}\right)^{\frac{n}{m}} e^{-y} \frac{dy}{ma} \left(\frac{y}{a}\right)^{\frac{1-m}{m}} \\ &= \frac{1}{m a^{\frac{n+1}{m}}} \int_0^{\infty} y^{\frac{n-m+1}{m}} e^{-y} dy \\ &= \frac{1}{m a^{\frac{n+1}{m}}} \Gamma\left(\frac{n-m+1}{m} + 1\right) \\ &= \frac{1}{m a^{\frac{n+1}{m}}} \Gamma\left(\frac{n+1}{m}\right) \end{aligned}$$

(5)

Example: Show that $\Gamma(1/2) = \sqrt{\pi}$

Solution:

$$\therefore \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, x > 0$$

For $x=1/2$

$$\Gamma(1/2) = \int_0^{\infty} t^{-1/2} e^{-t} dt$$

To solve this integration, let

$$t = y^2 \rightarrow dt = 2y dy$$

$$\Gamma(1/2) = \int_0^{\infty} y^{-1} e^{-y^2} 2y dy$$

$$= 2 \int_0^{\infty} e^{-y^2} dy$$

$$= 2 I$$

Where $I = \int_0^{\infty} e^{-y^2} dy$

$$I^2 = \int_0^{\infty} e^{-y^2} dy \cdot \int_0^{\infty} e^{-x^2} dx$$

$$= \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} dx dy$$

Transform the coordinates from cartesian (x, y) to polar (r, θ) coordinates:

$$dA = dx dy = r dr d\theta$$

$$x^2 + y^2 = r^2$$

$$I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r dr d\theta$$

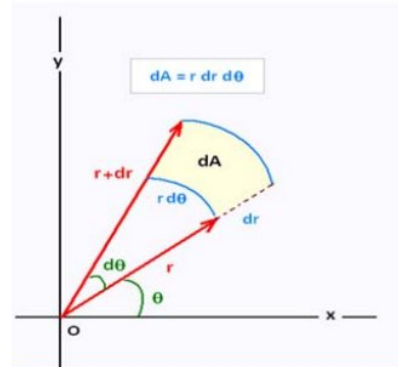
$$= \int_{\theta=0}^{\pi/2} d\theta \int_{r=0}^{\infty} r e^{-r^2} dr$$

$$= \frac{\pi}{2} \int_0^{\infty} r e^{-r^2} dr$$

$$= \frac{\pi}{2} \frac{\Gamma(1)}{2}$$

$$= \frac{\pi}{4}$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = 2 I = 2 * \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$



compare with

$$\int_0^{\infty} x^n e^{-ax^m} dx = \frac{\Gamma\left(\frac{n+1}{m}\right)}{ma^{\frac{n+1}{m}}}$$

$$n=1, a=1, m=2$$

(6)