Lecture 6

Vibrations and Waves

If any object is displaced slightly from equilibrium it will oscillate about its equilibrium position in what is called simple harmonic motion.
The most common examples are a mass on a spring, and a simple pendulum.
In this lecture we examine in detail the motion of simple harmonic oscillators.

- Simple Harmonic Motion
- Elastic Potential Energy
- Comparison with Circular Motion
- The Simple Pendulum
- Problems

Simple Harmonic Motion

Idea: Any object that is initially displaced slightly from a stable equilibrium point will oscillate about its equilibrium position. It will, in general, experience a restoring force that depends linearly on the displacement $\boldsymbol{x}$ from equilibrium:

Hooke's Law:

$$F_s = -kx \qquad (1)$$

where the equilibrium position is chosen to have $\boldsymbol{x}$ -coordinate $x = 0$ and k is a constant that depends on the system under consideration. The units of k are:

$$[k] = \frac{\text{Newtons}}{\text{metre}} . \qquad (2)$$

Definitions:

Amplitude (A):The maximum distance that an object moves from its equilibrium position.

A simple harmonic oscillator moves back and forth between the two positions of maximum displacement, at $x = A$ and $x = -A$.

Period (T):The time that it takes for an oscillator to execute one complete cycle of its motion.

If it starts at $t = 0$ at $x = A$, then it gets back to $x = A$ after one full period at $t = T$.

Frequency (f):The number of cycles (or oscillations) the object completes per unit time.

$$f = \frac{1}{T} . \qquad (3)$$

The unit of frequency is usually taken to be 1 Hz = 1 cycle per second.

Simple Harmonic Oscillator: Any object that oscillates about a stable equilibrium position and experiences a restoring force approximately described by Hooke's law. Examples of simple harmonic oscillators include: a mass attached to a spring, a molecule inside a solid, a car stuck in a ditch being ``rocked out'' and a pendulum.

Note:

- The negative sign in Hooke's law ensures that the force is always opposite to the direction of the displacement and therefore back towards the equilibrium position (i.e. a restoring force).
- The constant k in Hooke's law is traditionally called the *spring constant* for the system, even when the restoring force is not provided by a simple spring.
- The motion of any simple harmonic oscillator is completely characterized by two quantities: the amplitude, and the period (or frequency).

Elastic Potential Energy

Idea: In order to stretch a spring, it is necessary to do external work on the spring. This work is stored in the spring and is called the **elastic potential energy** (PE_s). PE_s depends on the spring constant, k , and the net displacement from equilibrium x and is given by:

$$PE_s = \frac{1}{2} kx^2. \qquad (4)$$

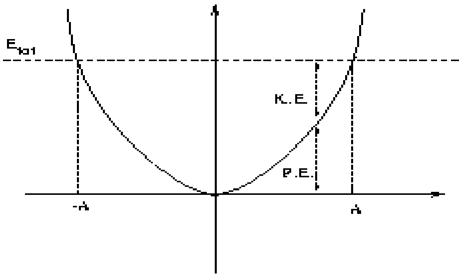
This potential energy can be changed to kinetic energy by releasing the spring and allowing it to pull (or push) back towards its equilibrium position.

The elastic potential energy contributes to the total mechanical energy of the harmonic oscillator:

$$\begin{aligned} E_{total} &= KE + PE_s + PE_{grav} \\ &= \frac{1}{2} mv^2 + \frac{1}{2} kx^2 + mgy \end{aligned} \qquad (5)$$

where y is the height of the mass as measured from some arbitrary reference point. In the absence of friction, the total mechanical energy is conserved, i.e. it is constant throughout the motion. This is expressed graphically (for zero gravitational potential, $PE_{grav} = 0$) in Fig.6.1.

Figure 6.1: Potential Energy as a Function of Position



Idea: When a harmonic oscillator reaches its maximum displacement, $x = A$, it must turn around and go back. At this turning point, the velocity is zero, and the total mechanical energy can be written in terms of the amplitude ($PE_{grav} = 0$):

$$E_{total} = \frac{1}{2} kA^2. \qquad (6)$$

Combining Eqs(6.5) and (6.6) gives an expression for the velocity as a function of the displacement:

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} \quad . \qquad (7)$$

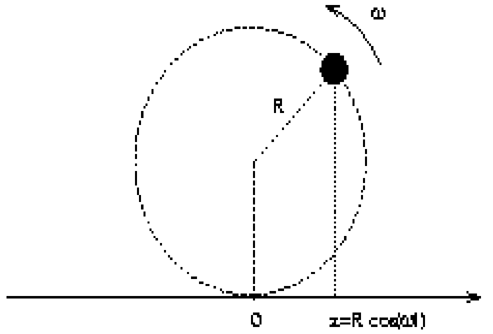
The maximum velocity is reached at the equilibrium position $x = 0$. At this point all of the energy of the system is in the form of kinetic energy:

$$v_{\max} = \sqrt{\frac{kA^2}{m}} \quad . \qquad (8)$$

Comparison with Circular Motion

Idea: There is a very strong analogy between circular motion and simple harmonic motion. Consider a particle moving with constant angular velocity ω in a circle of radius R , as shown in Fig.6.2.

Figure 6.2: Analogue Between Circular and Simple Harmonic Motion



Its x-coordinate is given as a function of time by:

$$x = R\cos(\omega t) \qquad (9)$$

and the x-component of its tangential velocity is:

$$v_x = - v_t\sin(\omega t) = - R\omega\sin(\omega t). \qquad (10)$$

From this we deduce that

$$\omega^2 x^2 + v_x^2 = R^2 \omega^2 \qquad (11)$$

which can be solved for v_x :

$$v_x = \sqrt{\omega^2(R^2 - x^2)} \quad . \qquad (12)$$

This is precisely the same as Eq.(6.7) relating the speed of an SHO to its position, providing we identify the radius with the amplitude and the angular velocity with $\sqrt{k/m}$. Since the period (and frequency) are known for circular motion ($T = 2\pi/\omega$, $f = 1/T$), this analogy allows us to deduce expressions for the period (and frequency) of the corresponding simple harmonic oscillator:

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \quad . \qquad (13)$$

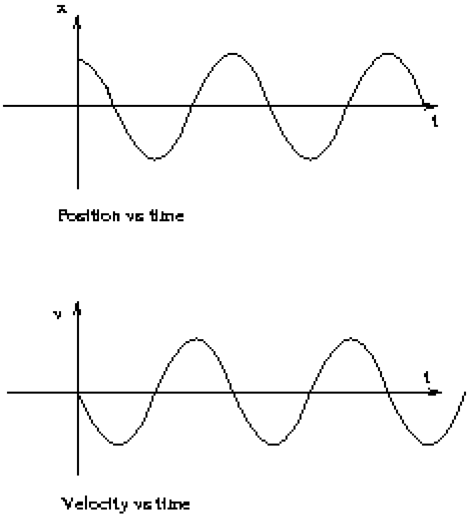
Similarly, the displacement and velocity as functions of time can also be deduced:

$$x = A\cos\left(\sqrt{\frac{k}{m}}\,t\right) = A\cos(2\pi f t) \qquad (14)$$

$$v = -\sqrt{\frac{k}{m}}\,A\sin\left(\sqrt{\frac{k}{m}}\,t\right) = -\sqrt{\frac{k}{m}}\,A\sin(2\pi f t). \qquad (15)$$

These expressions are plotted in the following figures.

Figure 6.3: Position and Velocity as Functions of Time

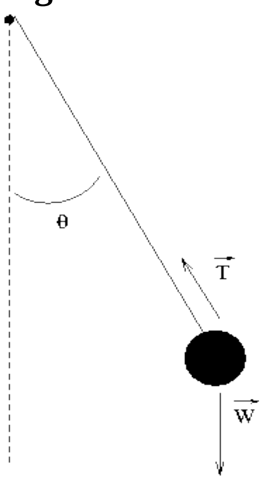


Note: ($\sqrt{k/m}$)$t = 2\pi f t$ gives the argument of the sine and cosine functions in *radians*. Make sure that your calculator is set to radians when doing these problems.

The Simple Pendulum

If a pendulum of mass *m* attached to a string of length *L* is displaced by an angle θ from the vertical (see figure below),

Figure 6.4: The Simple Pendulum



it experiences a net restoring force due to gravity:

$$F_r = - mg\sin\theta. \qquad (16)$$

For small angles, sin $\theta \approx \theta$, providing θ is expressed in radians (try it on your calculator for θ= 0.1,0.5,1.0 radians). In terms of radians,

$$\theta = \frac{s}{L}\ radians$$

where *s* is the arc length and *L* is the length of the string. Thus, for small displacements, *s* , the restoring force can be written:

$$F_r = -\left(\frac{mg}{L}\right)s.$$

Since the restoring force is proportional to the displacement, the pendulum is a simple harmonic oscillator with ``spring constant" $k = mg/L$. The period of a simple pendulum is therefore:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{L}{g}}. \qquad (17)$$

Note:

- In this small angle approximation, the amplitude of the pendulum has no effect on the period. This is what makes pendulums such good time keepers. As they inevitably lose energy due to frictional forces, their amplitude decreases, but the period remains constant.

Problems

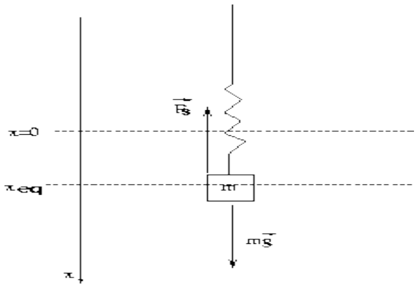
Problem 6.1

A 0.5 kg mass is hung on a vertical massless spring. The new equilibrium position of the spring is found to be 3 cm below the equilibrium position of the spring without the mass. a) What is the spring constant, *k* ? b) Show that the mass and spring system oscillates with simple harmonic motion about the new equilibrium position.

Solution:

a) Since the mass/spring system is in equilibrium, the downward force of gravity must be balanced by the upward pull of the spring. See diagram below (Note that we have for convenience defined the positive x-axis to point downwards.).

Figure 6.5: Problem 6.1



Thus at $x = x_{eq} = 0.03m$,

$$\sum F = 0 = kx_{eq} - mg \qquad (18)$$

$$\Rightarrow k = \frac{mg}{x_{eq}} = \frac{(0.5)(9.8)}{(0.03)} = 163 \ N/m .$$

b) If the spring is now displaced a distance s above the new equilibrium position, so that $x = x_{eq} - s$, the net force upward due to the spring is: $F_s = k(x_{eq} - s)$ The total force upward is therefore

$F_{tot} = F_s - mg = kx_{eq} - ks - mg$

Since the equilibrium position obeys Eq.(6.18), the equation for the acceleration of the mass reduces to:

$$F_{tot} = - ks \qquad (19)$$

i.e. The net force obeys Hook's law as a function of the displacement, s , and the mass undergoes simple harmonic motion about the new equilibrium position. Note that Eq.(6.19) is valid for both positive and negative values of $\boldsymbol{s}$.

Problem 6.2

Consider a SHO with $m = 0.5$ kg, $k = 10$ N/m and amplitude $A = 3$ cm. a) What is the total energy of the oscillator? b) What is its maximum speed? c) What is the speed when $x = 2$ cm? d) What are the kinetic and potential energies when $x = 2$ cm?

Solution:

a)

$$E_{total} = \frac{1}{2} kA^2 = \frac{1}{2}(10)(0.03) = 0.0045 \ J$$

b)

$$v_{max} = \sqrt{\frac{kA^2}{m}} = \sqrt{\frac{(10)(0.03)^2}{(0.5)}} = 0.134 \ m/s$$

c)

Using Eq.(6.7) for the velocity at $x = 0.02m$,

$$v = \sqrt{\frac{k(A^2 - x^2)}{m}} = \sqrt{\frac{(10)((0.03)^2 - (0.02)^2)}{(0.5)}} = 0.10 \ m/s$$

d)
The potential and kinetic energies at that point are:

$$PE_s = \frac{1}{2}kx^2 = \frac{1}{2}(10)((0.02)^2) = 0.002\ J$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(0.5)((0.1)^2) = 0.0025\ J$$

Problem 6.3
If the oscillator in the above problem is released from rest at $x = A$ when the clock is set to $t = 0$ seconds, a) determine the position and velocity of the oscillator at $t = 2$ s. b) At what time does the oscillator get to $x = -1$ cm?

Solution:

a)
At $t = 2$ s, we have:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$= (0.03)\cos\left(\sqrt{\frac{10}{0.5}}(2)\right) = -0.027\ m$$

and

$$v(t) = \sqrt{\frac{k}{m}}A\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$v(2) = \sqrt{\frac{10}{0.5}}(0.03)\sin\left(\sqrt{\frac{10}{0.5}}(2)\right) = 0.062\ m/s$$

b)
We must find the value of t for which the displacement

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t = -1\ \text{cm}\right)$$

$$\Rightarrow t = \sqrt{\frac{m}{k}}\cos^{-1}\left(\frac{x}{A}\right)$$

$$= \sqrt{\frac{0.5}{10}}\cos^{-1}\left(\frac{-0.01}{0.03}\right) = 0.43\ s$$

Problem 6.4

If the mass in the above problem experiences a constant frictional force of 0.1 N and is released from rest at $x = A$, a) what is the velocity of the mass when it first passes through the equilibrium position at $x = 0$. b) At what value of x does the velocity of the mass first go to zero after it has been released?

Solution:

a)

When the mass passes through the equilibrium position, it has covered a distance of d = 0.03 m. Applying the Work-Energy Theorem to the problem:

$$W_{\text{ext}} = \Delta KE + \Delta PE_s + \Delta PE_{\text{grav}}$$

$$-F_{\text{fr}} d = \left(\frac{1}{2}mv_f^2 - 0\right) + \left(0 - \frac{1}{2}kA^2\right) + 0$$

$$\Rightarrow v_f = \sqrt{\frac{-2F_{\text{fr}}d + kA^2}{m}}$$

$$= \sqrt{\frac{-2(0.1)(0.03) + (10)(0.03)^2}{0.5}} = 0.077 \ m/s$$

b)

Assume that the velocity of the mass vanishes at $x = x_0$ .The total distance moved from its starting position is therefore $d = A - x_0$ and the Work-Energy Theorem states

$$W_{\text{ext}} = \Delta KE + \Delta PE_s + \Delta PE_{\text{grav}}$$

$$-F_{\text{fr}} d = 0 + \left(\frac{1}{2}kx_0^2 - \frac{1}{2}kA^2\right) + 0$$

$$-F_{\text{fr}}(A - x_0) = \left(\frac{1}{2}kx_0^2 - \frac{1}{2}kA^2\right)$$

$$\Rightarrow 0 = kx_0^2 - 2F_{\text{fric}}x_0 - kA^2 + 2F_{\text{fric}}$$

$$= (10)x_0^2 - 2(0.1)x_0 + \left[-(10)(0.03^2) + 2(0.1)(0.03)\right]$$

$$= (10)x_0^2 - 0.2x_0 - 0.003$$

This is a quadratic equation which we can solve for the unkown, x_0 . The two possible answers are

$$x_0 = \frac{0.2 \pm \sqrt{(0.2)^2 - 4 \cdot 10 \cdot (-0.003)}}{2 \cdot 10} \ m$$

$$= -0.01 \ m \quad or \ 0.03 \ m \qquad (20)$$

The solution x_0 = 0.03 m corresponds to the original starting position, at which the mass did indeed have zero velocity. It is not the answer we are interested in, however. The physically relevant answer is x_0 = - 0.01 Thus, the mass comes to rest at 0.01 m past the original equilibrium position.

Problem 6.5

A simple pendulum is used in a physics laboratory experiment to obtain an experimental value for the gravitational acceleration, g . A student measures the length of the pendulum to be 0.510 meters, displaces it 10^{o} from the equilibrium position, and releases it. Using a stopwatch, the student determines that the period of the pendulum is 1.44 s. Determine the experimental value of the gravitational acceleration.

Solution:

Since the angle through which the pendulum is initially displaced is small, Eq.(6.17) in the Lecture Notes is valid for the period T :

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$\Rightarrow g = \frac{4\pi^2 L}{T^2}$$

$$= \frac{4\pi^2(0.510)}{1.44^2} = 9.71\ m/s^2$$

which is within 1% of the correct value 9.8 m/s^2.