Basrah University
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Introduction to Electrical Networks

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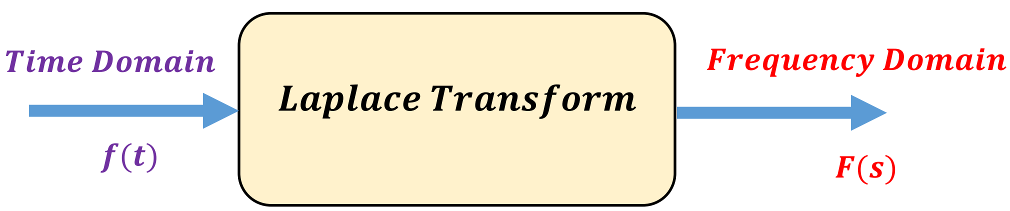
Review for Laplace Transform

- ***Laplace Transform***
- ***Properties of Laplace Transform***
- ***Inverse of Laplace Transform***

Laplace Transform

In order to compute the time response of a dynamic system, it is necessary to solve the differential equations (system mathematical model) for given inputs. Laplace transform is one of the favored ways by control engineers to do this. This technique transforms the problem from the time (or t) domain to the Laplace (or s) domain. The advantage in doing this is that complex time domain differential equations become relatively simple s domain algebraic equations. When a suitable solution is arrived at, it is inverse transformed back to the time domain.

The Laplace transform is an integral transformation of a function $\boldsymbol{f}\ (\boldsymbol{t}\ )$ from the time domain into the complex frequency domain, giving $\boldsymbol{F}\ (\boldsymbol{s})$.



Given a function $f\ (t)$, its Laplace transform, denoted by $F(s)$ is defined by

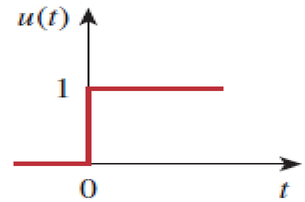
$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\, dt$$

A companion to the direct Laplace transform in the above equation is the inverse Laplace transform given by

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

Ex:- Determine the Laplace transform of each of the following functions: (a) $u(t)$, (b) $e^{-at}u(t)$, $a \geq 0$, and (c) $\delta(t)$.

a- For the unit step function $u(t)$, shown in below figure , the Laplace transform is

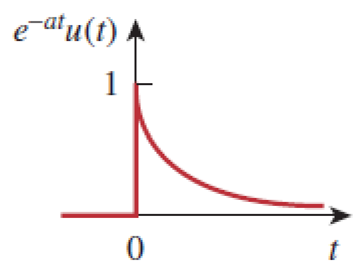


$$f(t) = \begin{cases} \mathbf{1} \ \boldsymbol{if} \ \boldsymbol{t} > \mathbf{0} \\ \mathbf{0} \ \boldsymbol{if} \ \boldsymbol{t} \leq \mathbf{0} \end{cases}$$

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}$$

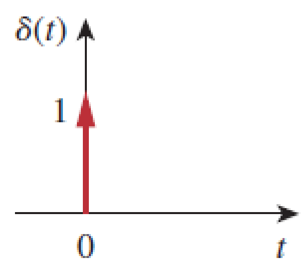
b- For the exponential function, shown in below figure, the Laplace transform is



$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

c- For the unit impulse function, shown in below figure



$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1$$

$$f(t) = \begin{cases} \mathbf{1} & \boldsymbol{if} \ \boldsymbol{t} = \mathbf{0} \\ \mathbf{0} & \boldsymbol{otherwise} \end{cases}$$

HW/ Find the Laplace transforms of these functions: $r(t) = tu(t)$, that is, the ramp function; $Ae^{-at}u(t)$; and $Be^{-j\omega t}u(t)$.

EX:- Determine the Laplace transform of $f(t) = \sin \omega t \; u(t)$.

$$F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t)e^{-st}\,dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}\,dt$$

$$= \frac{1}{2j}\int_0^\infty (e^{-(s-j\omega)t} - e^{-(s+j\omega)t})\,dt$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \boxed{\frac{\omega}{s^2+\omega^2}}$$

HW/ Determine the Laplace transform of $f(t) = 50\ \cos \omega t \; u(t)$

Properties of the Laplace Transform

The properties of the Laplace transform help us to obtain transform pairs without directly using the original Eq.

- **Linearity**

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f_2(t)$, then

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

Ex:- Determine the Laplace transform of $f(t) = u(t) + \delta(t) + t\,u(t)$

Sol/

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[t\,u(t)] = \frac{1}{s^2}$$

$$\boldsymbol{F(s) = F_1(s) + F_2(s) + F_3(s) = \frac{1}{s} + 1 + \frac{1}{s^2} = \frac{s^2+s+1}{s^2}}$$

- **Frequency Shift**

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[e^{-at} f(t) u(t)] = \int_0^\infty e^{-at} f(t) e^{-st}\,dt$$

$$= \int_0^\infty f(t) e^{-(s+a)t}\,dt = F(s+a)$$

or

$$\boxed{\mathcal{L}[e^{-at} f(t) u(t)] = F(s+a)}$$

Ex:- Evaluate the Laplace transform of $f(t) = e^{-at} \sin \omega t \ u(t)$

Sol/

$$\sin \omega t u(t) \quad \Leftrightarrow \quad \frac{\omega}{s^2 + \omega^2}$$

$$\Rightarrow \mathcal{L}[e^{-at} \sin \omega t u(t)] = \frac{\omega}{(s+a)^2 + \omega^2}$$

- **Laplace Transforms of Derivatives**

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[\dot{f}(t)] = sF(s) - f(0)$$

$$\mathcal{L}[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}[\dddot{f}(t)] = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)$$

.

.

$$\mathcal{L}[f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{n-1}(0)$$

Ex:- Use the Laplace transform of the second derivative to derive

$$\mathcal{L}[\sin \omega t \ u(t)] = \frac{\omega}{s^2 + \omega^2}$$

Sol/

$f(t) = \sin \omega t$

$\dot{f}(t) = \omega \cos \omega t$

$\ddot{f}(t) = -\omega^2 \sin \omega t$

$f(t) = 0$

$\dot{f}(0) = \omega$

By using Laplace Transforms of Derivatives property

$\mathcal{L}[\ddot{f}(t)] = s^2 F(s) - sf(0) - \dot{f}(0)$

$\Rightarrow$ $\mathcal{L}[-\omega^2 \sin \omega t] = s^2 \mathcal{L}[\sin \omega t] - s * 0 - \omega$

$\Rightarrow$ $-\omega^2 * \mathcal{L}[\sin \omega t] - s^2 \mathcal{L}[\sin \omega t] = -\omega$

$\Rightarrow$ $\omega^2 \mathcal{L}[\sin \omega t] + s^2 \mathcal{L}[\sin \omega t] = \omega$

$\Rightarrow$ $\mathcal{L}[\sin \omega t](s^2 + \omega^2) = \omega$

$\Rightarrow$ $$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

The Inverse Laplace Transform

The Laplace inverse of $F(s)$ means transform it back to the time domain and obtain the corresponding $f\ (t)$.

Suppose $F(s)$ has the general form of

$$F(s) = \frac{N(s)}{D(s)}$$

Where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial. The roots of $N(s) = 0$ are called the **zeros** of F(s), while the roots of $D(s)$=0 are the **poles** of $F(s)$. We use **partial fraction expansion** to break $F(s)$ down into simple terms whose inverse transform in the table on the previous lecture. Thus, finding the inverse Laplace transform of $F(s)$ involves two steps.

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Table on the previous lecture.

Let us consider the three possible forms $F(s)$ may take and how to apply the two steps to each form.

A-Simple Poles (Unrepeated Poles)

A simple pole means a first-order pole. If $F(s)$ has only simple poles, then $D(s)$ becomes a product of factors, so that

$$F(s) = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

Where $s = -p_1, -p_2, \ldots, -p_n$are the simple poles, and$p_i \neq p_j$ for all $i \neq j$

Assuming that the degree of $N(s)$ is less than the degree of $D(s)$, we use partial fraction expansion to decompose $F(s)$

$$F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n}$$

We can evaluate the coefficient k from

$$\boxed{k_i = (s + p_i)F(s)\,|_{s=-p_i}}$$

Ex:- Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + 12}{s(s+2)(s+3)}$$

Sol/ The partial fraction of $F(s)$ is

$$F(s) = \frac{s^2 + 12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

Where $A, B, and\ C$ are the constants to be determined. We can find it from

$$A = sF(s)\,|_{s=0} = \left.\frac{s^2 + 12}{(s + 2)(s + 3)}\right|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s + 2)F(s)\,|_{s=-2} = \left.\frac{s^2 + 12}{s(s + 3)}\right|_{s=-2} = \frac{4 + 12}{(-2)(1)} = -8$$

$$C = (s + 3)F(s)\,|_{s=-3} = \left.\frac{s^2 + 12}{s(s + 2)}\right|_{s=-3} = \frac{9 + 12}{(-3)(-1)} = 7$$

Evaluate $A = 2, B = -8$ & $C = 7$ in the general equation of $F(S)$, we will have

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

Take the inverse of the above equation (evaluate inverse of each term)

$$f(t) = (2 - 8e^{-2t} + 7e^{-3t})u(t)$$

B- Repeated Poles

Suppose $F(s)$ has n repeated poles at $s = -p$. Then we may represent $F(s)$ as

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s)$$

Where $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$ .We determine the expansion coefficient k_n as

$$k_n = (s+p)^n F(s)\,|_{s=-p}$$

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]\,|_{s=-p}$$

Repeating this gives

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]\,|_{s=-p}$$

The mth term becomes

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]\,|_{s=-p}$$

Ex:- Calculate $v(t)$ given that

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

Sol/ The partial fraction is

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$
$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$$

The constant are

$$A = sV(s)\,|_{s=0} = \frac{10s^2+4}{(s+1)(s+2)^2}\bigg|_{s=0} = \frac{4}{(1)(2)^2} = 1$$
$$B = (s+1)V(s)\,|_{s=-1} = \frac{10s^2+4}{s(s+2)^2}\bigg|_{s=-1} = \frac{14}{(-1)(1)^2} = -14$$
$$C = (s+2)^2V(s)\,|_{s=-2} = \frac{10s^2+4}{s(s+1)}\bigg|_{s=-2} = \frac{44}{(-2)(-1)} = 22$$
$$D = \frac{d}{ds}[(s+2)^2V(s)]\bigg|_{s=-2} = \frac{d}{ds}\left(\frac{10s^2+4}{s^2+s}\right)\bigg|_{s=-2}$$
$$= \frac{(s^2+s)(20s) - (10s^2+4)(2s+1)}{(s^2+s)^2}\bigg|_{s=-2} = \frac{52}{4} = 13$$

Evaluate the values of A, B, C & D on $F(s)$, we will have

$$V(s) = \frac{1}{s} - \frac{14}{s+1} + \frac{13}{s+2} + \frac{22}{(s+2)^2}$$

Taking the inverse transform of each term, we get

$$v(t) = (1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t})u(t)$$

C-Complex Poles

The complex poles mean that there are ($\sin\, or\, \cos$) at the corresponding time function. An easier approach to simplify this form is a method known as completing the square. In this case $F(s)$ may have the general form

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)$$

Where $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles. If we complete the square by letting

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s + \alpha)^2 + \beta^2$$

and we also let

$$A_1 s + A_2 = A_1(s + \alpha) + B_1\beta$$

Then the general equation becomes

$$F(s) = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1\beta}{(s + \alpha)^2 + \beta^2} + F_1(s)$$

By using the table on the previous lecture (taking the Laplace inverse)

$$\boxed{f(t) = (A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t)u(t) + f_1(t)}$$

Ex:- Find the inverse transform of the frequency-domain function in below form

$$H(s) = \frac{20}{(s + 3)(s^2 + 8s + 25)}$$

Sol/ In this example, $H(s)$ has a pair of complex poles at $s^2 + 8s + 25 = 0$ or $s = -4 \pm j3$. We let

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} = \frac{A}{s+3} + \frac{Bs+C}{(s^2+8s+25)} \qquad \ldots (1)$$

$$A = (s+3)H(s)\,\Big|_{s=-3} = \frac{20}{s^2+8s+25}\,\Big|_{s=-3} = \frac{20}{10} = 2$$

To find B & C we will

1- Put s=0 on Eq1, this will result

$$\frac{20}{75} = \frac{A}{3} + \frac{C}{25}$$

or

$$20 = 25A + 3C$$

As $A = 2$ $\Longrightarrow$ $C = -10$

2- Put $s = 1$ on Eq1, we will have

$$\frac{20}{(4)(34)} = \frac{A}{4} + \frac{B+C}{34}$$

or

$$20 = 34A + 4B + 4C$$

As $A = 2, C = -10$ $\Longrightarrow$ $B = -2$

So $H(s)$ is

$$H(s) = \frac{2}{s+3} - \frac{2s+10}{(s^2+8s+25)} = \frac{2}{s+3} - \frac{2(s+4)+2}{(s+4)^2+9}$$

$$= \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2+9} - \frac{2}{3}\,\frac{3}{(s+4)^2+9}$$

Taking the inverse of each term, we obtain

$$h(t) = \left(2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin 3t\right)u(t)$$

Laplace Transform Applications

The main use of Laplace transform in linear system studies is to **solve the linear system equations**. To solve an equation by Laplace transform, we go through four distinct stages

(a) Rewrite the equation in terms of Laplace transforms.

(b) Insert the given initial conditions.

(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the solution.

Ex:- Solve the following differential equation using Laplace transform.

$\dot{x} + 4x = 10e^{3t} \qquad x(0) = 6$

Sol/

(a) Convert the equation to Laplace transform, i.e.

$$[sX(s) + x(0)] + 4X(s) = \frac{10}{s-3}$$

(b) Insert the initial condition, $x(0) = 6$

$$[sX(s) + 6] + 4X(s) = \frac{10}{s-3}$$

$$X(s)[s+4] = \frac{10}{s-3} - 6$$

$$X(s)[s+4] = \frac{6s-8}{s-3}$$

(c) Rearrange to obtain

$$X(s) = \frac{6s-8}{(s-3)(s+4)}$$

$$X(s) = \frac{6s-8}{(s-3)(s+4)} = \frac{A}{s-3} + \frac{B}{s+4}$$

$$A = (s-3)X(s)|_{s=3} = \frac{10}{7} = 1.42$$

$$A = (s+4)X(s)|_{s=-4} = \frac{-32}{-7} = 4.57$$

So $X(s)$ is

$$X(s) = \frac{1.42}{s-3} + \frac{4.57}{s+4}$$

(d) Taking inverse Laplace transform to obtain $x(t)$.

$$x(t) = [1.42e^{3t} + 4.57e^{-4t}]u(t)$$

HW/ Using Laplace transform to solve.

$$\frac{d^2y}{dt} + 5\frac{dy}{dt} + 4y = 8 \qquad y(0) = 2 \quad \dot{y}(0) = -1$$