Basrah University
College of Engineering
Electrical Engineering Department

Introduction to Electrical Networks

Asst. Lect: Hamzah Abdulkareem

Chapter Four

The Filters

Part 3

- ***Butterworth Filter***
- ***Analog to analog transformations***
- ***Design of Low Pass Butterworth Filters***

4.8 Butterworth Filter

The Butterworth filter is a type of signal-processing filter designed to have a frequency response that is as flat as possible in the passband as shown in Fig. (4.12). It is also referred to as a maximally flat magnitude filter. It was first described in 1930 by the British engineer and physicist **Stephen Butterworth**. A Butterworth filter is a filter with an amplitude response of:-

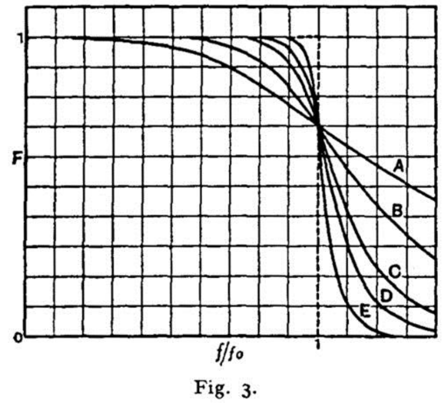

Fig 4.12 The frequency response plot from Butterworth's 1930 paper.

$$|H_n(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} \quad \ldots.(20)$$

Where ω_c is the cutoff frequency and n is the filter order.

We will design a normalized filter (taking $\omega_c = 1\ rad/sec$), and then scale filter to the desired cutoff frequency, so evaluating ($\omega_c = 1\ rad/sec$) this will lead Eq. (20) to be:-

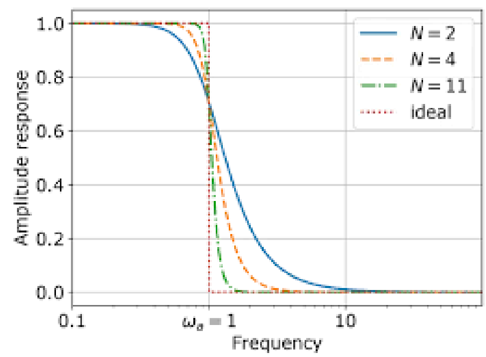

Fig 4.13 The frequency response plot of several Butterworth filter.

$$|H_n(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} \quad \ldots.(21)$$

Note that

$$|H_n(j\omega)|^2 = H_n(j\omega)H_n(-j\omega) = \frac{1}{1+\omega^{2n}} \quad \ldots.(22)$$

So, to find the transfer function of the mentioned filter we applied the below points:-

1- The transfer function of the filter is denoted as $\boldsymbol{H_n(s)}$.
2- Recall that $\boldsymbol{s} = \sigma + j\omega$.
3- The frequency response $H_n(j\omega)$ can be obtained from $\boldsymbol{H_n(s)}$ by evaluating $\boldsymbol{s} = j\omega$, ie $\omega = s/j$.
4- Applying the aforementioned points in Eq. (22), we will get

$$|H_n(j\omega)|^2 = H_n(j\omega)H_n(-j\omega) = H_n(s)H_n(-s) = \frac{1}{1+(s/j)^{2n}} \quad \ldots.(23)$$

Since $H_n(s)H_n(-s) = \frac{1}{1+(s/j)^{2n}}$, $H_n(s)H_n(-s)$ has $\boldsymbol{2n}$ poles, and they occur when:-

$$(s/j)^{2n} = -1 \quad \ldots(24)$$

Isolating $\boldsymbol{s}$ yields

$$s^{2n} = -(j)^{2n} \quad \ldots (25)$$

Since $-1 = e^{j\pi(2k-1)}$ for integer k, and $-j = e^{j\pi/2}$, Eq. (25) will be

$$s^{2n} = e^{j\pi(2k-1+n)} \quad \ldots (26)$$

Taking $1/2n$ root of the each side of Eq. (20) yields

$$s = e^{\frac{j\pi}{2n}(2k-1+n)} = \cos\left(\frac{\pi}{2n}(2k-1+n)\right) + j\sin\left(\frac{\pi}{2n}(2k-1+n)\right) \quad \ldots (27)$$

For $k = 1,2,\ldots\ldots,2n$.

From Eq. (27), it is clear that the poles will be located in a unit circle with the origin center as shown in Fig. 4.14.

From Fig. 4.14, the left n poles are corresponded to $H_n(s)$, while the right n poles are corresponded to $H_n(-s)$.

If we wish the filter $H_n(s)$ to be stable, the poles of $H_n(s)$are selected to be those in the left half plane and $H_n(s)$ can be written in the following form:

$$H_n(s) = \frac{1}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

$$= \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + 1}$$

$$= \frac{1}{B_n(s)} \quad \ldots (28)$$

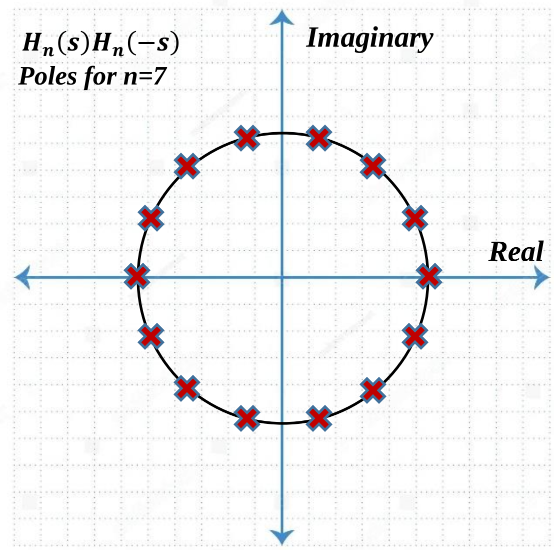

Fig 4.14 Poles distribution of seventh order Butterworth Filter.

Where $B_n(s)$ is nth order Butterworth Polynomial.

According to the above $H_n(s)$ can be computed by Hand or by evaluating the value of $B_n(s)$ in Table 4.1.

Table 4.1 ***Factors of Butterworth polynomials of order 1 through 10*.**

n	Factors of Butterworth Polynomials $B_n(s)$
1	$(s+1)$
2	$(s^2+1.414214s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765367s+1)(s^2+1.847759s+1)$
5	$(s+1)(s^2+0.618034s+1)(s^2+1.618034s+1)$
6	$(s^2+0.517638s+1)(s^2+1.414214s+1)(s^2+1.931852s+1)$
7	$(s+1)(s^2+0.445042s+1)(s^2+1.246980s+1)(s^2+1.801938s+1)$
8	$(s^2+0.390181s+1)(s^2+1.111140s+1)(s^2+1.662939s+1)(s^2+1.961571s+1)$
9	$(s+1)(s^2+0.347296s+1)(s^2+s+1)(s^2+1.532089s+1)(s^2+1.879385s+1)$
10	$(s^2+0.312869s+1)(s^2+0.907981s+1)(s^2+1.414214s+1)(s^2+1.782013s+1)(s^2+1.975377s+1)$

4.8.1 Analog to analog transformations:

In the following discussion, a normalized low-pass filter will be used as a prototype filter for illustration. For normalization purposes we selected $\boldsymbol{\omega_c} = 1\ rad/sec$. For the other values of $\boldsymbol{\omega_c}$, a scaling will be performed to $\boldsymbol{H_n(s)}$ in Table 4.1 by replacing $\boldsymbol{s}$ with $\boldsymbol{s/\omega_c}$.

$$\boldsymbol{H'(s) = H(s)|_{s\to s/w_c} = H(s/w_c)} \qquad \ldots (29)$$

For the other types of filters, we apply Table 4.2 scaling.

Table 4.2 ***Normalizing Butterworth Filters* .**

Filter Type	***Prototype response***	***Transformed filter response***
Low-pass filter	Normalized Low-pass G(s)	$H(s) = G(s)\|_{s\to s/w_c}$
High-pass filter	Normalized Low-pass G(s)	$H(s) = G(s)\|_{s\to w_c/s}$
Band-pass filter	Normalized Low-pass G(s)	$H(s) = G(s)\|_{s\to \frac{s^2+w_l w_u}{s(w_u-w_l)}}$
Band-stop filter	Normalized Low-pass G(s)	$H(s) = G(s)\|_{s\to \frac{s(w_u-w_l)}{s^2+w_l w_u}}$

4.8.2 Design of Low Pass Butterworth Filters

The filter requirements are normally given in terms of a set of critical frequency, say ω_p, ω_s and gains G_p, G_s. A common set of conditions for the low-pass response is given in Fig. 4.15.

The dB Gain for any ω_x is

$$G_x = 20\log_{10}|H_n(j\omega_x)| = 20\log_{10}\left(\frac{1}{\sqrt{1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}}}\right) = 0 - 20 = -10\log_{10}\left(1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}\right) \quad \ldots (30)$$

So, gains at frequencies ω_p, ω_s are

$$G_{p,dB} = -10\log_{10}\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right) \Rightarrow \left(\frac{\omega_p}{\omega_c}\right)^{2n} = 10^{-\frac{G_{p,dB}}{10}} - 1 \quad \ldots (31)$$

$$G_{s,dB} = -10\log_{10}\left(1+\left(\frac{\omega_s}{\omega_c}\right)^{2n}\right) \Rightarrow \left(\frac{\omega_s}{\omega_c}\right)^{2n} = 10^{-\frac{G_{s,dB}}{10}} - 1 \quad \ldots (32)$$

Dividing Eq. (32) to Eq. (31), yields

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1} \quad \ldots (33)$$

By solving Eq. (30), gets

$$n = \frac{\log_{10}\left[\frac{10^{-\frac{G_{s,dB}}{10}} - 1}{10^{-\frac{G_{p,dB}}{10}} - 1}\right]}{2\log_{10}(\omega_s/\omega_p)} \quad \ldots (34)$$

Solving Eq. (31 & 32), yields

$$\omega_c = \omega_p \Big/ \left(10^{-\frac{G_{p,dB}}{10}} - 1\right)^{\frac{1}{2n}} \quad \ldots (35)$$

$$\omega_c = \omega_s \Big/ \left(10^{-\frac{G_{s,dB}}{10}} - 1\right)^{\frac{1}{2n}} \quad \ldots (36)$$

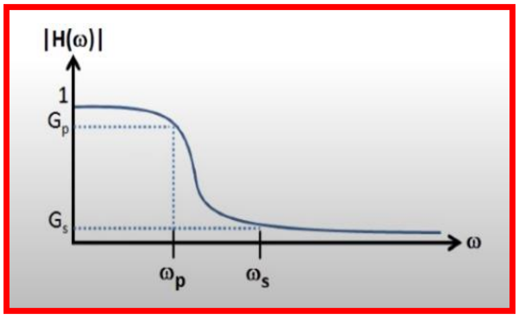

Fig 4.15 Butterworth low pass filter specifications.

Example 4.4:- Design a low-pass Butterworth filter with below specifications:-

$\boldsymbol{G_{p,dB}} = -3\ dB, \boldsymbol{G_{s,dB}} = \boldsymbol{-25\ dB}, \boldsymbol{\omega_p} = 20\ rad/\sec\ \ \&\ \boldsymbol{\omega_s} = 50\ rad/\,sec$

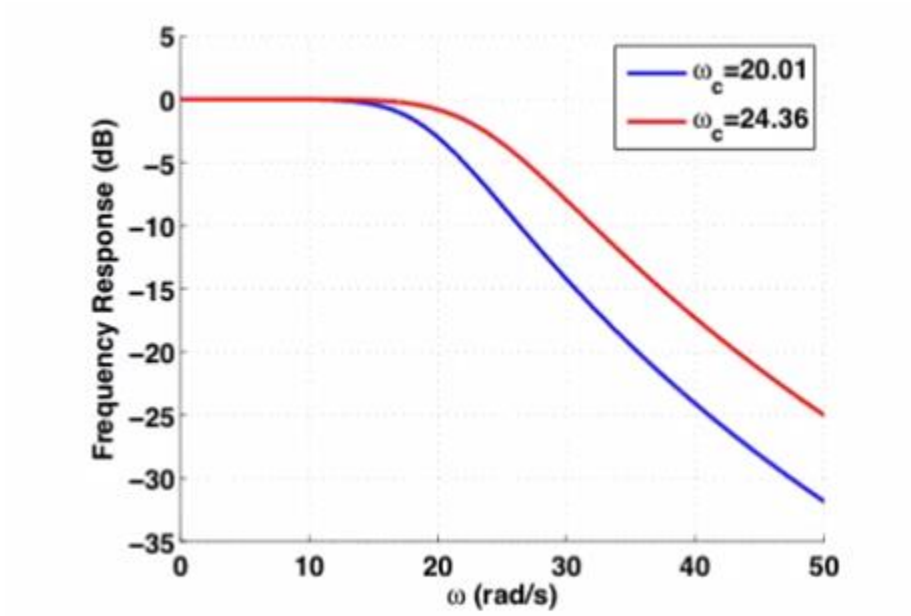

Fig 4.15 For Example 4.4.