Basrah University
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Introduction to Electrical Networks

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Chapter Three

Two-Port Networks

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Chapter Three

Two-Port Networks

3.1 Two-Port Network

A pair of terminals through which a current may enter or leave a network is known as a port. Two-terminal devices or elements (such as resistors, capacitors, and inductors) result in one-port networks. Most of the circuits we have dealt with so far are two-terminal or one-portcircuits, represented in Fig. 3.1(a).

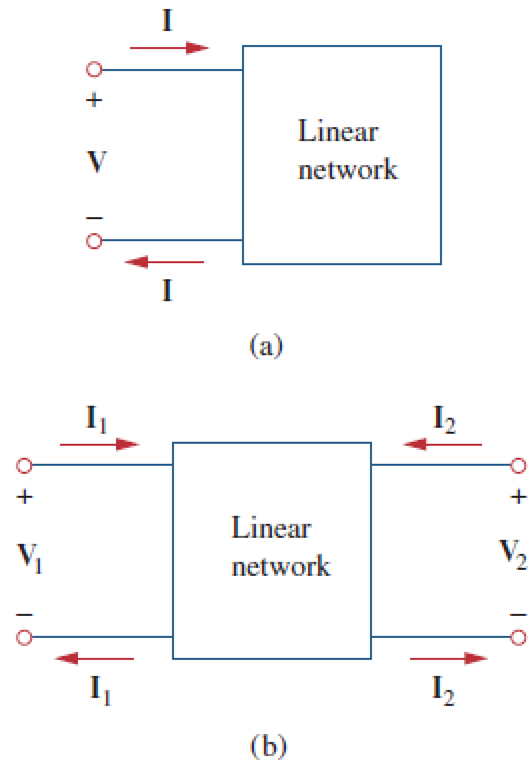

Fig 3.1 (a) One-port network, (b) two-port network.

We have considered the voltage across or current through a single pair of terminals—such as the twoterminals of a resistor, a capacitor, or an inductor. We have also studied four-terminal or two-port circuits involving op amps, transistors, and transformers, as shown in Fig. 3.1(b). In general, a network may have n ports. A port is an access to the network and consists of a pair of terminals; the current entering one terminal leaves through the other terminal so that the net current entering the port equals zero.

So, ***A two-port network*** is an electrical network with two separate ports for input and output.

Our study of two-port networks is for at least two reasons. First, such networks are useful in communications, control systems, power systems, and electronics. For example, they are used in electronics to model transistors and to facilitate cascaded design. Second, knowing the parameters of a two-port network enables us to treat it as a "black box" when embedded within a larger network. To characterize a two-port network requires that we relate the terminalquantities $\mathbf{I_1}, \mathbf{I_2}, \mathbf{V_1}$ & $\mathbf{V_2}$, and in Fig. 3.1(b), out of which two areindependent. The various terms that relate these voltages and currents are called parameters. Our goal in this chapter is to derive six sets of these parameters. We will show the relationship between these parameters andhow two-port networks can be connected in series, parallel, or cascade.

3.2 Impedance Parameters

Impedance and admittance parameters are commonly used in the synthesis of filters. They are also useful in the design and analysis of impedance-matching networks and power distribution networks. A two-port network can be voltage-driven as in Fig. 3.2. Thus, the terminal voltages can be related to the terminal currents as

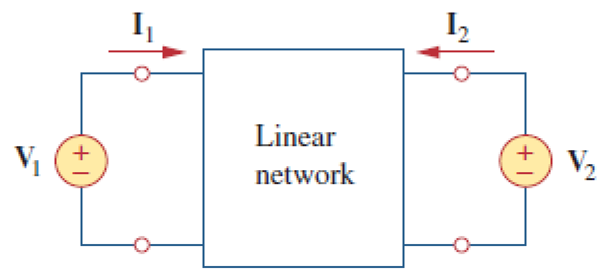

Fig 3.2 Linear two-port network driven by voltage sources.

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2 \quad \dots (1)$$

or in matrix form as

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = [\mathbf{z}] \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \quad \dots (2)$$

where the **z** terms are called the ***impedance parameters***, or simply ***z parameters***, and have units of **ohms**.

The values of the parameters can be evaluated by setting $\mathbf{I_1} = \mathbf{0}$ (input port open-circuited) or $\mathbf{I_2} = \mathbf{0}$ (output port open-circuited) as shown in Fig. 3.3. Thus,

$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0}, \qquad \mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0}$$
$$\mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0}, \qquad \mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} \quad \dots (3)$$

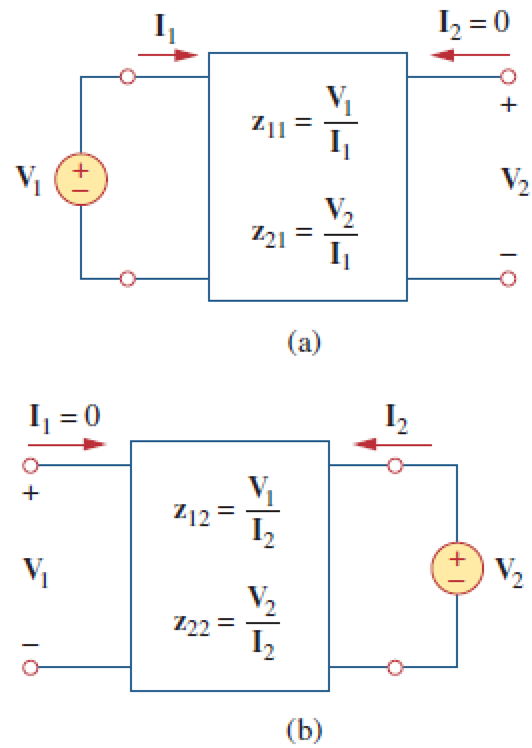

Fig 3.3 Determination of the z parameters: (a) finding z_{11} and $\boldsymbol{z}_{21}$, (b) finding $\boldsymbol{z}_{12}$ and $\boldsymbol{z}_{22}$.

When $\boldsymbol{z}_{11} = \boldsymbol{z}_{22}$, the two-port network is said to be ***symmetrical***. In addition, $\boldsymbol{z}_{12} = \boldsymbol{z}_{21}$, the two-port is said to be ***reciprocal***.

A reciprocal network can be replaced by the ***T-equivalent circuit*** in Fig. 3.4(a). If the network is not ***reciprocal***, a more general equivalent network is shown in Fig. 3.4(b); notice that this figure follows directly from Eq. (1).

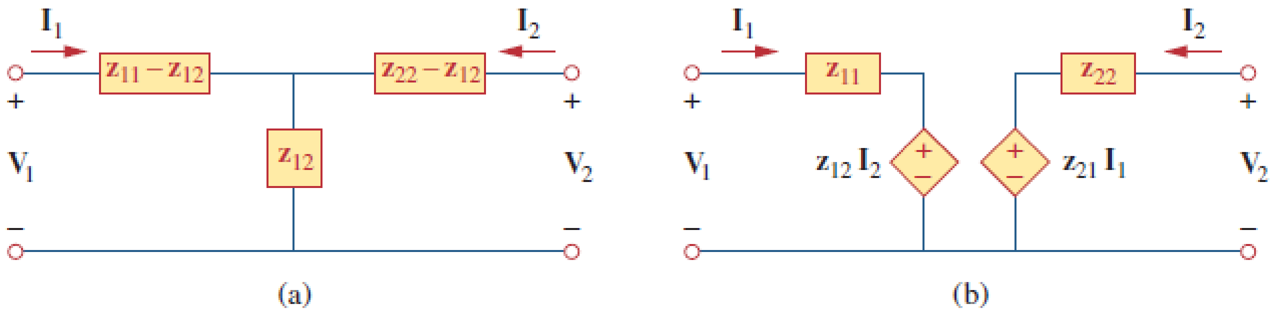

Fig 3.4 (a) T-equivalent circuit (for reciprocal case only), (b) general equivalent circuit.

It should be mentioned that for some two-port networks, the z parameters do not exist because they cannot be described by Eq. (1). As an example, consider the ideal transformer of Fig. 3.5. The defining equations for the two-port network are:

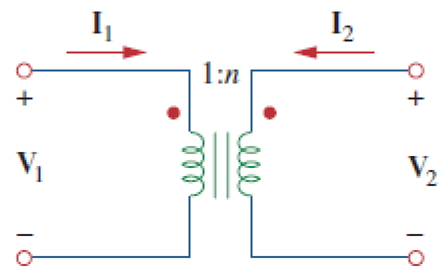

Fig 3.5 An ideal transformer has no z parameters.

$$V_1 = \frac{1}{n} V_2, \qquad I_1 = -n I_2 \qquad \ldots (4)$$

Observe that it is impossible to express the voltages in terms of the currents, and vice versa, as Eq. (1) requires. Thus, the ideal transformer has no z parameters. However, it does have hybrid parameters, as we shall see in net section.

Exaple 3.1:- Determine the z parameters for the circuit in Fig. 3.6.

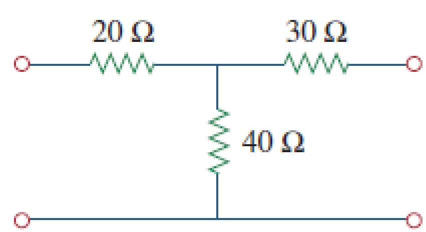

Fig 3.6 For Example 3.1.

Example 3.2:- Find $\mathbf{I_1}$ and $\mathbf{I_2}$ in the circuit in Fig. 3.7.

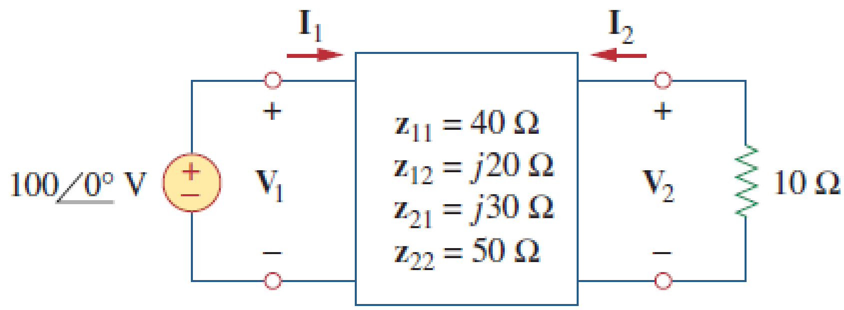

Fig 3.7 For Example 3.2.

3.3 Admittance Parameters

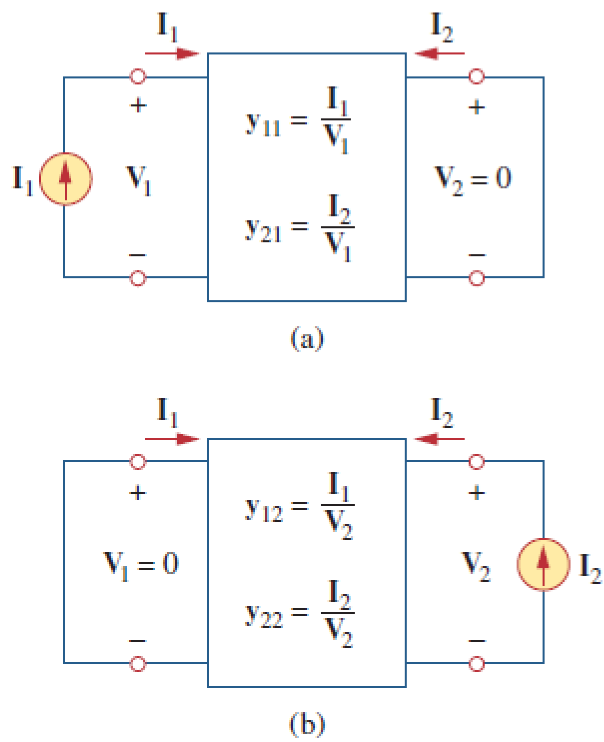

Fig 3.8 Determination of the $\boldsymbol{y}$ *parameters: (a) finding* $\boldsymbol{y}_{11}$ *and* $\boldsymbol{y}_{21}$*, (b) finding* $\boldsymbol{y}_{12}$ *and* $\boldsymbol{y}_{22}$*.*

In the previous section we saw that impedance parameters may not exist for a two-port network. So there is a need for an alternative means of describing such a network. This need may be met by the second set of parameters, which we obtain by expressing the terminal currents in terms of the terminal voltages. In either Fig. 3.8(a) or (b), the terminal currents can be expressed in terms of the terminal voltages as

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2 \quad \ldots (4)$$

or in matrix form as

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = [\mathbf{y}] \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \quad \ldots (5)$$

The values of the parameters can be determined by setting $\mathbf{V_1 = 0}$ (input port short-circuited) or $\mathbf{V_2 = 0}$ (output port short circuited). Thus

$$\mathbf{y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0}, \qquad \mathbf{y}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0}$$
$$\mathbf{y}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0}, \qquad \mathbf{y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} \quad \ldots (6)$$

For a two-port network that is linear and has no dependent sources, the transfer admittances are equal ($\boldsymbol{y}_{12} = \boldsymbol{y}_{21}$). This can be proved in the same way as for the $\boldsymbol{z}$ parameters. Areciprocal network ($\boldsymbol{y}_{12} = \boldsymbol{y}_{21}$) can be modeled by the π -equivalent circuit in Fig. 3.9(a). If the network is not reciprocal, a more general equivalent network is shown in Fig. 3.9(b).

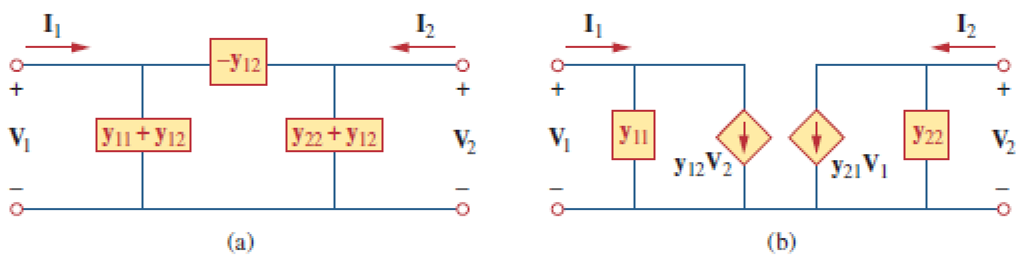

Fig 3.9 a) π *-equivalent circuit (for reciprocal case only), (b) general equivalent circuit.*

Example 3.3:- Obtain the y parameters for the π network shown in Fig. 3.10.

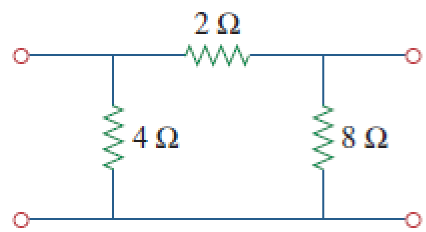

Fig 3.10 For Example 3.3.

Example 3.4:- Determinee the y parameters for the two-port shown in Fig. 3.11.

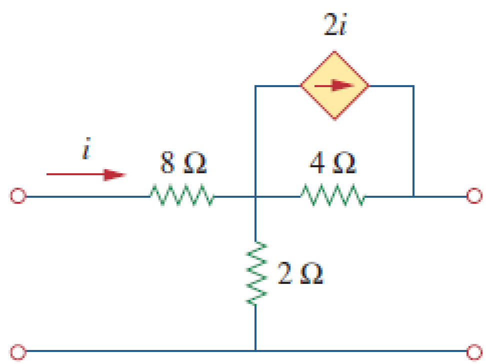

Fig 3.11 For Example 3.4.

3.4 Hybrid Parameters

The z and y parameters of a two-port network do not always exist. So there is a need for developing another set of parameters. This third set of parameters is based on making $\mathbf{V_1}$ and $\mathbf{I_2}$ the dependent variables. Thus, we obtain

$$\begin{aligned} \mathbf{V}_1 &= \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \\ \mathbf{I}_2 &= \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2 \end{aligned} \qquad \ldots (7)$$

or in matrix form,

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & \mathbf{h}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix} = [\mathbf{h}] \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix} \qquad \ldots (8)$$

The **h** terms are known as the hybrid parameters (or, simply, h parameters) because they are a hybrid combination of ratios. They are very useful for describing electronic devices such as transistors; it is much easier to measure experimentally the h parameters of such devices than to measure their z or y parameters. In fact, we have seen that the ideal transformer in Fig. 3.5, described by Eq. (4), does not have z parameters. The ideal transformer can be described by the hybrid parameters, because Eq. (4) conforms with Eq. (7). The values of the parameters are determined as

$$\begin{aligned} \mathbf{h}_{11} &= \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}, & \mathbf{h}_{12} &= \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} \\ \mathbf{h}_{21} &= \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}, & \mathbf{h}_{22} &= \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} \end{aligned} \qquad \ldots (9)$$

It is evident from Eq. (9) that the parameters h_{11}, h_{12}, h_{21} , and h_{22} represent an impedance, a voltage gain, a current gain, and an admittance, respectively. This is why they are called the hybrid parameters. To be specific,

$$\begin{aligned} \mathbf{h}_{11} &= \text{Short-circuit input impedance} \\ \mathbf{h}_{12} &= \text{Open-circuit reverse voltage gain} \\ \mathbf{h}_{21} &= \text{Short-circuit forward current gain} \\ \mathbf{h}_{22} &= \text{Open-circuit output admittance} \end{aligned} \qquad \ldots (10)$$

The procedure for calculating the h parameters is similar to that used for the z or y parameters. For ***reciprocal networks*** $h_{12} = -h_{21}$. Figure 3.12 shows the hybrid model of a two-port network.

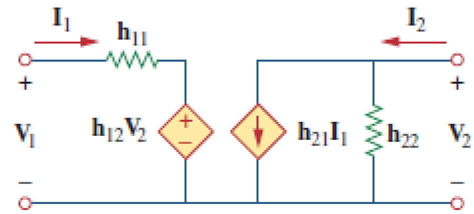

Fig 3.12 The h-parameter equivalent network of a two-port network.

A set of parameters closely related to the h parameters are the g parameters or inverse hybrid parameters. These are used to describe the terminal currents and voltages as

$$\mathbf{I}_1 = \mathbf{g}_{11}\mathbf{V}_1 + \mathbf{g}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{g}_{21}\mathbf{V}_1 + \mathbf{g}_{22}\mathbf{I}_2 \qquad \dots (11)$$

Or

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{11} & \mathbf{g}_{12} \\ \mathbf{g}_{21} & \mathbf{g}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} = [\mathbf{g}] \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} \qquad \dots (12)$$

The values of the g parameters are determined as

$$\mathbf{g}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{I}_2=0}, \qquad \mathbf{g}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{I}_2}\right|_{\mathbf{V}_1=0}$$
$$\mathbf{g}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{V}_1}\right|_{\mathbf{I}_2=0}, \qquad \mathbf{g}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{V}_1=0} \qquad \dots (13)$$

Thus, the inverse hybrid parameters are specifically called

$\mathbf{g}_{11}$ = Open-circuit input admittance
$\mathbf{g}_{12}$ = Short-circuit reverse current gain
$\mathbf{g}_{21}$ = Open-circuit forward voltage gain ... (14)
$\mathbf{g}_{22}$ = Short-circuit output impedance

Figure 3.13 shows the inverse hybrid model of a two-port network. The g parameters are frequently used to model field-effect transistors.

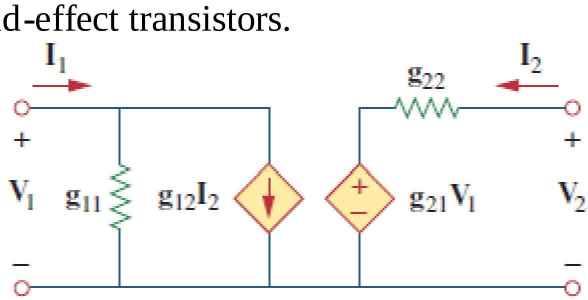

Fig 3.13 The g-parameter equivalent network of a two-port network.

Example 3.5:- Determine the Thevenin equivalent at the output port of the circuit in Fig. 3.14.

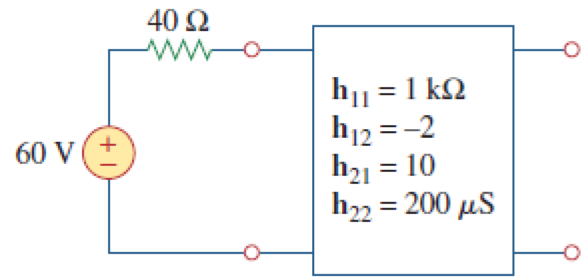

Fig 3.14 For Example 3.5.

Example 3.6:- Find the g parameters as functions of s for the circuit in Fig. 3.15.

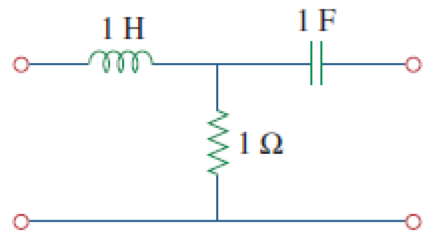

Fig 3.15 For Example 3.6.

3.5 Transmission Parameters

Another set of parameters relates the variables at the input port to those at the output port. Thus,

$$\begin{aligned} \mathbf{V}_1 &= \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2 \\ \mathbf{I}_1 &= \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2 \end{aligned} \qquad \ldots (15)$$

or

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix} \qquad \ldots (16)$$

Equations (15) and (16) relate the input variables ($\mathbf{V_1}$ and $\mathbf{I_1}$) to the output variables ($\mathbf{V_2}$ and $-\mathbf{I_2}$). Notice that in computing the transmission parameters, $-\mathbf{I_2}$ is used rather than $\mathbf{I_2}$, because the current is considered to be leaving the network, as shown in Fig. 3.16, as opposed to entering the network as in Fig. 3.1(b). This is done merely for conventional reasons; when you cascade two-ports (output to input), it is most logical to think of $\mathbf{I_2}$ as leaving the two-port. It is also customary in the power industry to consider $\mathbf{I_2}$ as leaving the two-port.

The two-port parameters in Eqs. (15) and (16) provide a measure of how a circuit transmits voltage and current from a source to a load. They are useful in the analysis of transmission lines (suchas cable and fiber) because they express sending-end variables ($\mathbf{V_1}$ and $\mathbf{I_1}$) in terms of the receiving-end variables ($\mathbf{V_2}$ and $-\mathbf{I_2}$). For this reason, they are called transmission parameters. They are also known as **ABCD** parameters. They are used in the design of telephone systems,microwave networks, and radars. The transmission parameters are determined as

$$\mathbf{A} = \frac{\mathbf{V}_1}{\mathbf{V}_2}\bigg|_{\mathbf{I}_2=0}, \qquad \mathbf{B} = -\frac{\mathbf{V}_1}{\mathbf{I}_2}\bigg|_{\mathbf{V}_2=0}$$

$$\mathbf{C} = \frac{\mathbf{I}_1}{\mathbf{V}_2}\bigg|_{\mathbf{I}_2=0}, \qquad \mathbf{D} = -\frac{\mathbf{I}_1}{\mathbf{I}_2}\bigg|_{\mathbf{V}_2=0} \qquad \ldots (17)$$

Thus, the transmission parameters are called, specifically,

$$\begin{aligned} \mathbf{A} &= \text{Open-circuit voltage ratio} \\ \mathbf{B} &= \text{Negative short-circuit transfer impedance} \\ \mathbf{C} &= \text{Open-circuit transfer admittance} \\ \mathbf{D} &= \text{Negative short-circuit current ratio} \end{aligned} \qquad \ldots (18)$$

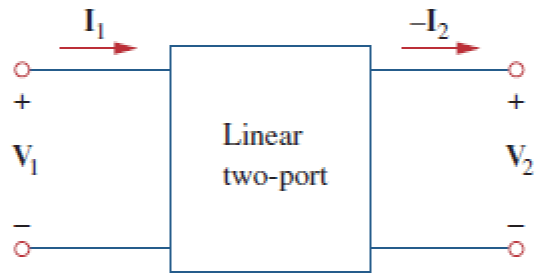

Fig 3.16 Terminal variables used to define the ADCB parameters.

Our last set of parameters may be defined by expressing the variables at the output port in terms of the variables at the input port. We obtain

$$\mathbf{V}_2 = \mathbf{a}\mathbf{V}_1 - \mathbf{b}\mathbf{I}_1$$
$$\mathbf{I}_2 = \mathbf{c}\mathbf{V}_1 - \mathbf{d}\mathbf{I}_1 \qquad \ldots (19)$$

Or

$$\begin{bmatrix} \mathbf{V}_2 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ -\mathbf{I}_1 \end{bmatrix} = [\mathbf{t}] \begin{bmatrix} \mathbf{V}_1 \\ -\mathbf{I}_1 \end{bmatrix} \qquad \ldots (20)$$

The parameters a, b, c, and d are called the inverse transmission, or t, parameters. They are determined as follows:

$$\mathbf{a} = \left.\frac{\mathbf{V}_2}{\mathbf{V}_1}\right|_{\mathbf{I}_1=0}, \qquad \mathbf{b} = -\left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_1=0}$$
$$\mathbf{c} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{I}_1=0}, \qquad \mathbf{d} = -\left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_1=0} \qquad \ldots (21)$$

From Eq. (21) and from our experience so far, it is evident that these parameters are known individually as

$\mathbf{a}$ = Open-circuit voltage gain
$\mathbf{b}$ = Negative short-circuit transfer impedance
$\mathbf{c}$ = Open-circuit transfer admittance … (22)
$\mathbf{d}$ = Negative short-circuit current gain

In terms of the transmission or inverse transmission parameters, a network is reciprocal if

$$\mathbf{AD} - \mathbf{BC} = 1, \qquad \mathbf{ad} - \mathbf{bc} = 1 \qquad \ldots (23)$$

Example 3.7:- Find the transmission parameters for the two-port network in Fig. 3.17.

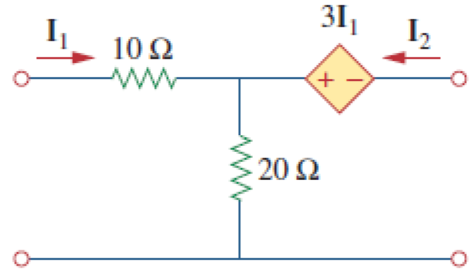

Fig 3.17 For Example 3.7.

Example 3.8:- The **ABCD** parameters of the two-port network in Fig. 3.18 are

$$\begin{bmatrix} 4 & 20\ \Omega \\ 0.1\ \text{S} & 2 \end{bmatrix}$$

The output port is connected to a variable load for maximum power transfer. Find R_L and the maximum power transferred.

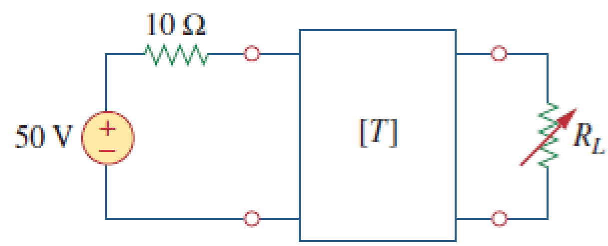

Fig 3.18 For Example 3.8.