Basrah University
College of Engineering
Electrical Engineering Department

Introduction to Electrical Networks

Asst. Lect: Hamzah Abdulkareem

Chapter Two

s-Domain Circuit Analysis

- ***Circuit Element Models***
- ***Circuit Analysis in s-Domain***
- ***Transfer Functions***
- ***Natural Response and s-Plane***

Chapter Two

s-Domain Circuit Analysis

2.1Circuit Element Models

Having mastered how to obtain the Laplace transform and its inverse, we are now prepared to employ the Laplace transform to analyze circuits. This usually involves three steps.

1. Transform the circuit from the time domain to the s-domain.

2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.

3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Only the first step is new and will be discussed here. As we did in phasor analysis, we transform a circuit in the time domain to the frequency or s-domain by Laplace transforming each term in the circuit. Below we will evaluate the laplace transform of all passive elemnets ($\boldsymbol{R}, \boldsymbol{L}$ & $\boldsymbol{C}$), and than we will evaluate the impedance for all according to “the impedance in the ***s-domain*** as the ratio of the voltage transform to the current transform under ***zero initial conditions***.

A- Laplace Transform of a Resistance:-

The voltage-current relationship in the time domain is

$$v(t) = Ri(t) \xrightarrow{\text{Taking the Laplace transform}} V(s) = RI(s) \qquad \ldots (1)$$

So, the impdance of a resistannce (***assume zero intial conditions***) is

$$\boldsymbol{Z(s) = \frac{V(s)}{I(s)} = R} \qquad \ldots (2)$$

B- Laplace Transform of an Inductor:-

As the voltage-current relationship in the time domain is

$$v(t) = L\frac{di(t)}{dt} \xrightarrow{\text{Taking the Laplace transform}} V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-) \qquad \ldots (3)$$

Or,

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s} \qquad \ldots (4)$$

The s-domain equivalents are shown in Fig. 2.1, where the initial condition is modeled as a voltage or current source.

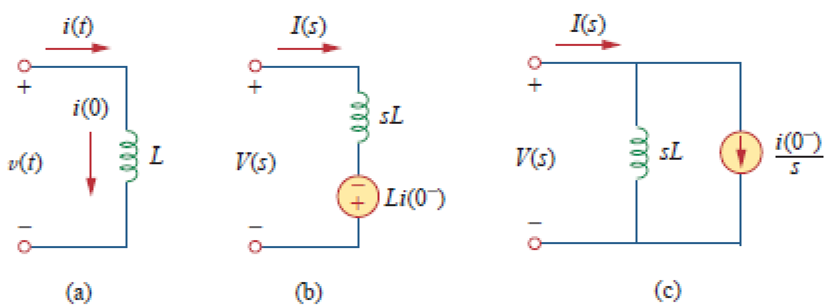

Fig 2.1 Representation of an inductor: (a)time domain, (b,c) s-domain equivalents.

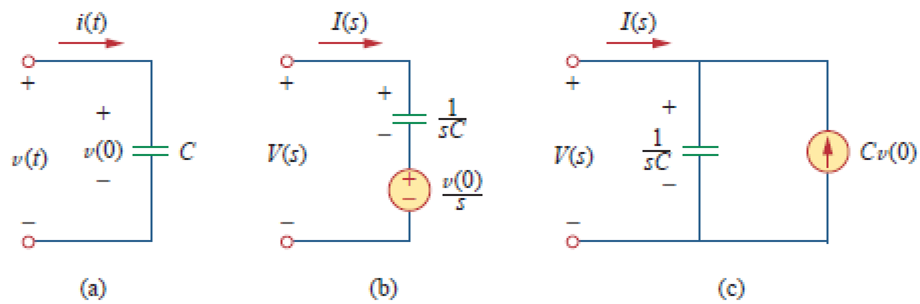

Fig 2.2 Representation of a capacitor: (a)time domain, (b,c) s-domain equivalents.

So, the impdance of an inductor (***assume zero intial conditions***) is

$$Z(s) = \frac{V(s)}{I(s)} = sL \quad \ldots(5)$$

C- Laplace Transform of a Capacitor:-

For a capacitor,

$$i(t) = C\frac{dv(t)}{dt} \xrightarrow{\text{Taking the Laplace transform}} I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-) \quad \ldots(6)$$

Or,

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s} \quad \ldots(7)$$

So, the impdance of a capacitor (***assume zero intial conditions***) is

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{sC} \quad \ldots(8)$$

The s-domain equivalents are shown in Fig. 2.2.

The admittance in the s-domain is the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \quad \ldots(9)$$

The s-domain impedance equivalents for all elements are shown in Fig. 2.3.

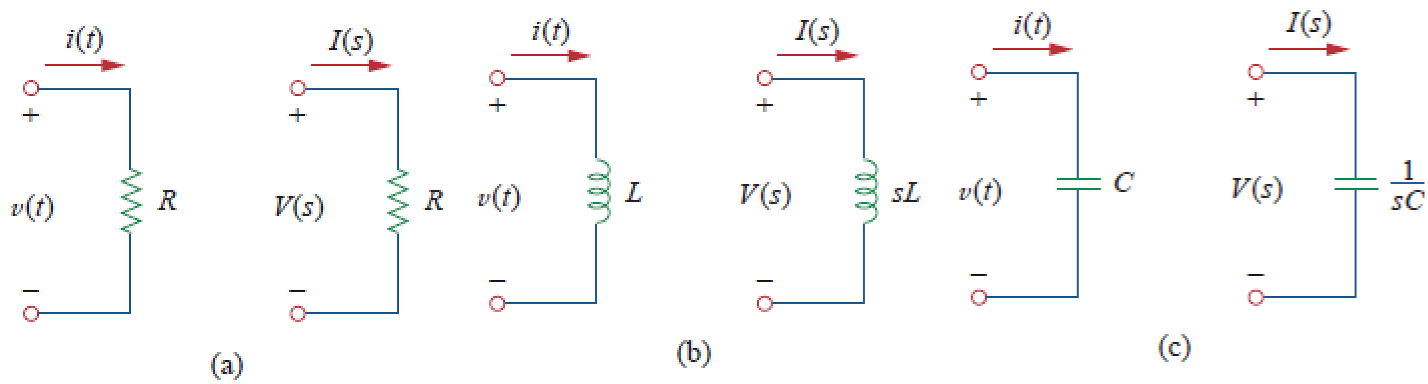

Fig 2.3 Time-domain and s-domain representations of passive elements under zero initial conditions.

Example 2.1:- Find $v_0(t)$ in the circuit of Fig. 2.4, assuming zero initial conditions.

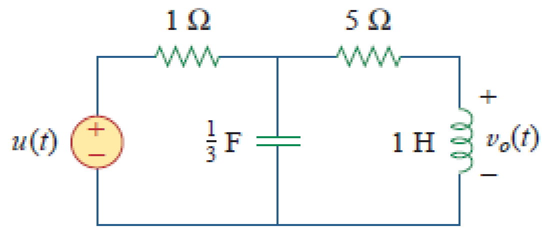

Fig 2.4 For Example 2.1.

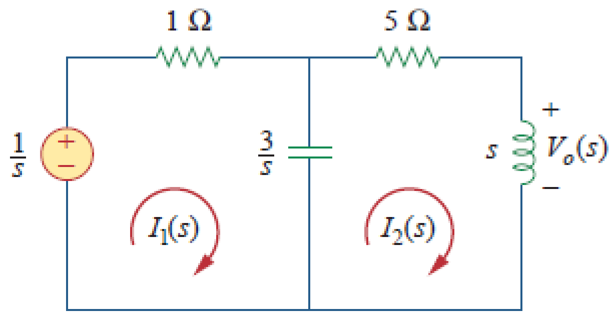

Fig 2.5 Mesh analysis of the frequency-domain equivalent of the same circuit.

Example 2.2:- In the circuit of Fig. 2.6(a), the switch moves from position a to position b at $t = 0$, Find $i(t)$ for $t > 0$.

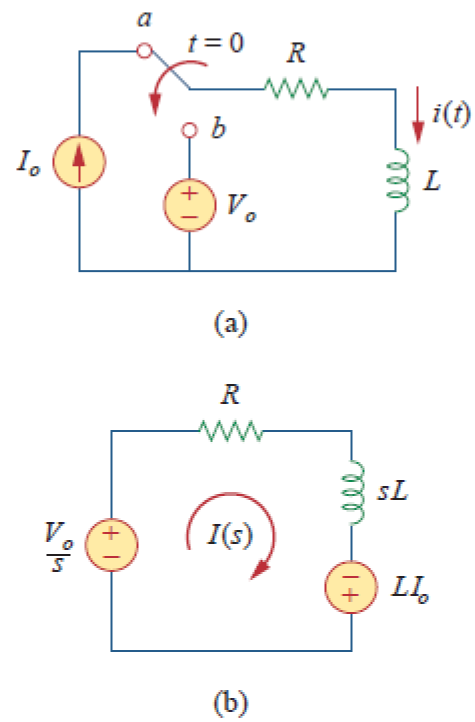

Fig 2.6 For Example 2.2.

2.2 Circuit Analysis in s-Domain

Circuit analysis is again relatively easy to do when we are in the s-domain. We merely need to transform a complicated set of mathematical relationships in the time domain into the s-domain where we convert operators (derivatives and integrals) into simple multipliers of s and $\frac{1}{s}$. This now allows us to use algebra to set up and solve our circuit equations. The exciting thing about this is that all of the circuit theorems and relationships we developed for dc circuits are perfectly valid in the s-domain.

Example 2.3:- Consider the circuit in Fig. 2.7(a). Find the value of the voltage across the capacitor assuming that the value of $v_s(t) = 10u(t)V$ and assume that at $t = 0$, $-1\ A$ flows through the inductor and $+5\ V$ is across the capacitor.

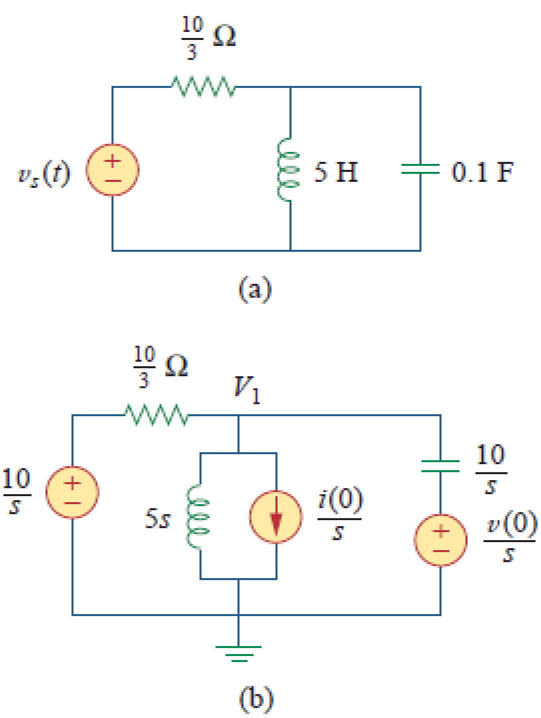

Fig 2.7 For Example 2.3.

Example 2.4:- Assume that there is no initial energy stored in the circuit of Fig. 2.8 at $t = 0$ and that at $i_s = 10\ u(t) A$ (a) Find at $V_0(s)$ using Thevenin's theorem. (b) Determine at $v_0(t)$.

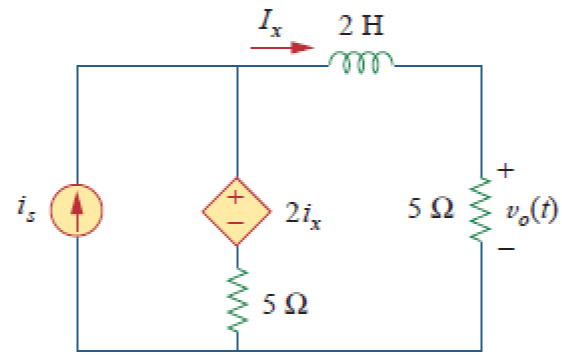

Fig 2.8 For Example 2.4.

2.3 Transfer Functions

The ***transfer function*** is a key concept in signal processing because it indicates how a signal is processed as it passes through a network. It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis. The transfer function of a network describes how the output behaves with respect to the input. It specifies the transfer from the input to the output in the ***s-domain***, assuming no initial energy. So, the ***transfer function*** $\boldsymbol{H(s)}$ is the ratio of the output response $\boldsymbol{Y(s)}$ to the input excitation $\boldsymbol{X(s)}$, ***assuming all initial conditions are zero***, thus,

$$H(s) = \frac{Y(s)}{X(s)} \qquad \ldots (10)$$

The transfer function depends on what we define as input and output. Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

$$\boldsymbol{H(s) = Voltage\ Gain = \frac{V_o(s)}{V_i(s)}}$$

$$\boldsymbol{H(s) = Cuurent\ Gain = \frac{I_o(s)}{I_i(s)}}$$

$$\boldsymbol{H(s) = Impedance = \frac{V(s)}{I(s)}}$$

$$\boldsymbol{H(s) = Admittance = \frac{I(s)}{V(s)}} \qquad \ldots (11)$$

Equation (11) assumes that both $\boldsymbol{X(s)}$ and $\boldsymbol{Y(s)}$ are known. Sometimes, we know the input $\boldsymbol{X(s)}$ and the transfer function $\boldsymbol{H(s)}$. We find the output $\boldsymbol{Y(s)}$ as

$$\boldsymbol{Y(s) = H(s)X(s)} \qquad \ldots (12)$$

and take the inverse transform to get $y(t)$.

Example 2.5:- Determine the transfer function $H(s) = V_o(s)/I_o(s)$ of the circuit in Fig. 2.9.

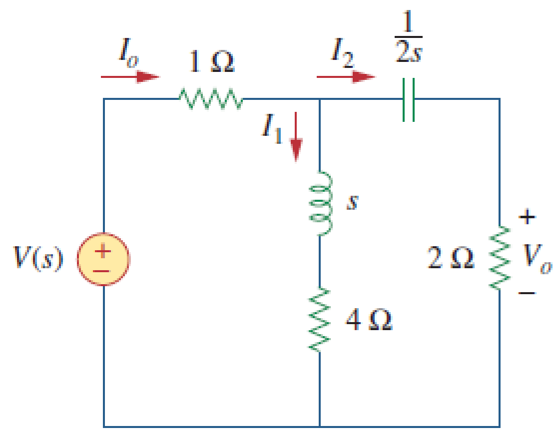

Fig 2.9 For Example 2.5.

Example 2.6:- For the s-domain circuit in Fig. 2.10, find: (a) the transfer function $H(s) = V_o/V_i$ (b) the impulse response, (c) the response when $v_i(t) = u(t)\ V$ (d) the response when $v_i(t) = 8\cos 2t\ V$.

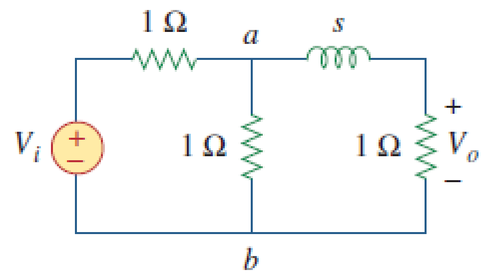

Fig 2.10 For Example 2.6.

2.4 Natural Response and s-Plane

In this section, we consider how such plots can be used to obtain the complete response of a circuit—natural plus forced—provided the initial conditions are known. The advantage of such an approach is a more intuitive linkage between the location of the critical frequencies, easily visualized through the pole-zero plot, and the desired response.

Let us introduce the method by considering the simplest example, a series RL circuit as shown in Fig. 2.11. A general voltage source $v_s(t)$ causes the current $i\ (t)$ to flow after closure of the switch at $t\ =\ 0$. The complete response $i(t)$ for $t\ >\ 0$ is composed of a natural response and a forced response:

$$i(t)\ =\ i_n(t)\ +\ i_f(t) \quad \dots (13)$$

For this circuit, we have

$$I_f(s) = \frac{V_s}{R+Ls} = \frac{1}{L}\frac{V_s}{R/L+s} \quad \dots (14)$$

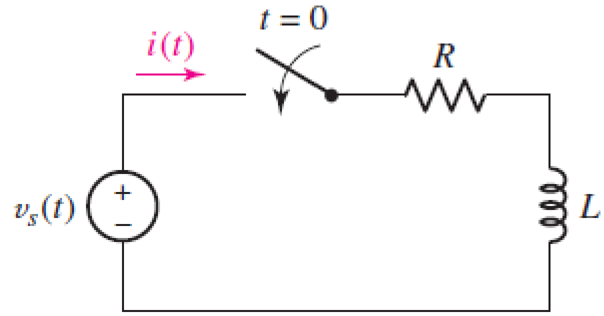

Fig 2.11 An example that illustrates the determination of the complete response through a knowledge of the critical frequencies of the impedance faced by the source.

Next we consider the natural response. From previous experience, we know that the form will be a decaying exponential with the time constant L/R, but let's pretend that we are finding it for the first time. The form of the natural (source-free) response is, by definition, independent of the forcing function; the forcing function contributes only to the magnitude of the natural response. To find the proper form, we turn off all independent sources; here, $v_s(t)$ is replaced by a short circuit. Next, we try to obtain the natural response as a limiting case of the forced response. Returning to the frequency-domain expression of Eq. 14, we obediently set $V_s\ =\ 0$. On the surface, it appears that $I(s)$ must also be zero, but this is not necessarily true if we are operating at a complex frequency that is a simple pole of $I(s)$. That is, the denominator and the numerator may both be zero so that $I(s)$ need not be zero.

Let us inspect this new idea from a slightly different vantage point. We fix our attention on the ratio of the desired forced response to the forcing function. We designate this ratio H(s) and define it to be the circuit transfer function. Then,

$$\frac{I_f(s)}{V_s} = H(s) = \frac{1}{L(R/L+s)} \qquad \dots (15)$$

In this example, the transfer function is the input admittance faced by V_s . We seek the natural (source-free) response by setting $V_s = 0$. However, $I_f(s) = V_s H(s)$, and if $V_s = 0$, a nonzero value for the current can be obtained only by operating at a pole of $H(s)$. The poles of the transfer function therefore assume a special significance.

In this particular example, we see that the pole of the transfer function occurs at $s = -R/L + j0$, as shown in Fig. 2.12. If we choose to operate at this particular complex frequency, the only finite current that could result must be a constant in the s-domain (i.e., frequency-independent). We thus obtain the natural response.

$$I\left(s = -\frac{R}{L} + j0\right) = A \qquad \dots (16)$$

where A is an unknown constant. We next desire to transform this natural response to the time domain. Our knee-jerk reaction might be to attempt to apply inverse Laplace transform techniques in this situation. However, we have already specified a value of $\boldsymbol{s}$, so that such an approach is not valid. Instead, we look to the real part of our general function e^{st} , such that

$$i_n(t) = Re\{Ae^{st}\} = Re\{Ae^{-\frac{Rt}{L}}\} \qquad \dots (17)$$

In this case we find

$$i_n(t) = Ae^{-\frac{Rt}{L}} \qquad \dots (18)$$

so that the total response is then

$$i(t) = Ae^{-\frac{Rt}{L}} + i_f(t) \qquad \dots (19)$$

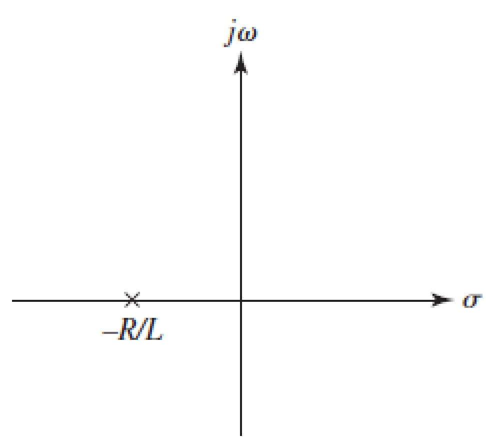

Fig 2.12 Pole-zero constellation of the transfer function H(s)

Example 2.7: For the source-free circuit of Fig. 2.13, determine expressions for i_1 and i_2 $for\ t > 0$, given the initial conditions $i_1(0) = i_2(0) = 11\ A$.

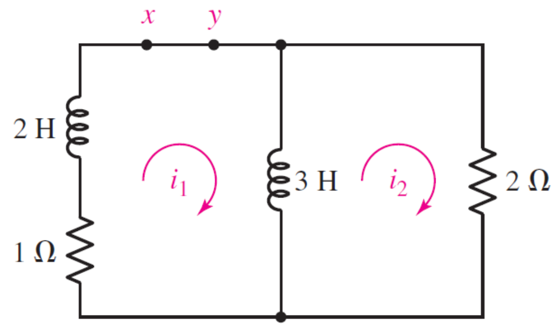

Fig 2.13 For Example 2.7