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Introduction to Electrical Networks

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Chapter One

Coupling Circuits

Part 2

- ***Energy in a Coupled Circuit***
- ***Coupling Coefficient***
- ***Linear Transformers***
- ***T & π Equivalent Circuit for Linear Transformers***
- ***Ideal Transformers***
- ***Compute Z_{in} of Ideal Transformers***
- ***Complex Power of Ideal Transformers***
- ***Ideal Autotransformers***

1.3 Energy in a Coupled Circuit

We saw that the energy stored in an inductor is given by

$$w = \frac{1}{2}Li^2 \qquad \dots (13)$$

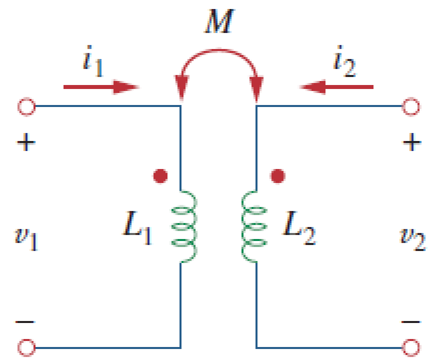


Fig 1.10 The circuit for deriving energy stored in a coupled circuit.

Consider the circuit in Fig. 1.10. We assume that currents i_1 and i_2 are zero initially, so that the energy stored in the coils is zero. If we let i_1 increase from zero to I_1 while maintaining $i_2 = 0$, the power in coil 1 is

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt} \qquad \dots (14)$$

and the energy stored in the circuit is

$$w_1 = \int p_1 \, dt = L_1 \int_0^{I_1} i_1 \, di_1 = \frac{1}{2} L_1 I_1^2 \qquad \dots (15)$$

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2 the mutual voltage induced in coil 1 is $M_{12} \, di_2/dt$,while the mutual voltage induced in coil 2 is zero, since i_1 does not change. The power in the coils is now

$$p_2(t) = i_1 M_{12} \frac{di_2}{dt} + i_2 v_2 = I_1 M_{12} \frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt} \qquad \dots (16)$$

and the energy stored in the circuit is

$$w_2 = \int p_2 dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2 = M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2 \qquad \dots (17)$$

The total energy stored in the coils when both and have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2 \qquad \dots (18)$$

If we reverse the order by which the currents reach their final values, the total energy will be

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2 \qquad \dots (19)$$

Since the total energy stored should be the same regardless of how we reach the final conditions, comparing Eqs. (18) and (19) leads us to conclude that

$$M_{12} = M_{21} = M \qquad \& \qquad w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

This equation was derived based on the assumption that the coil currents both entered the dotted terminals. If one current enters one dotted terminal while the other current leaves the other dotted terminal, in this case the total energy will be

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

1.3.1 Coupling Coefficient

The energy stored in the circuit cannot be negative because the circuit is passive. This means that

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0 \qquad \ldots (20)$$

To complete the square, we both add and subtract the term $i_1 i_2\sqrt{L_1 L_2}$ on the right-hand side of Eq. (20) and obtain

$$\frac{1}{2}(i_1\sqrt{L_1} - i_2\sqrt{L_2})^2 + i_1 i_2(\sqrt{L_1 L_2} - M) \geq 0 \qquad \ldots (21)$$

The squared term is never negative; at its least it is zero. Therefore, the second term on the right-hand side of Eq. (21) must be greater than zero; that is,

$$\sqrt{L_1 L_2} - M \geq 0 \;\Rightarrow\; M \leq \sqrt{L_1 L_2} \;\Rightarrow\; M = k\sqrt{L_1 L_2} \qquad \ldots (22)$$

Where k is the coupling coefficient and $0 \leq k \leq 1$ equivalently $0 \leq M \leq \sqrt{L_1 L_2}$. The coupling coefficient is the fraction of the total flux emanating from one coil that links the other coil. For example, in Fig. 1.2, and in Fig. 1.3,

$$k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}} \qquad \& \qquad k = \frac{\phi_{21}}{\phi_2} = \frac{\phi_{21}}{\phi_{21} + \phi_{22}}$$

Depending on the value of the coupling coefficient, we can classify the status of the mutual coupling into three cases:-

1- If the entire flux produced by one coil links another coil, then $\boldsymbol{k = 1}$ and we have 100 percent coupling, or the coils are said to be ***perfectly coupled***.
2- For $\boldsymbol{k > 0.5}$ **,** they are said to be ***tightly coupled.***
3- For $\boldsymbol{k < 0.5}$ **,** they are said to be ***loosely coupled.***

We expect k to depend on ***the closeness of the two coils***, ***their core***, ***their orientation***, ***and their windings***. Figure 1.11 shows loosely coupled windings and tightly coupled windings. The air-core transformers used in radio frequency circuits are loosely coupled, whereas iron-core transformers used in power systems are tightly coupled. The linear transformers discussed in Section 1.4 are mostly air-core; the ideal transformers discussed in Sections 1.5 &1.6 are principally iron-core.

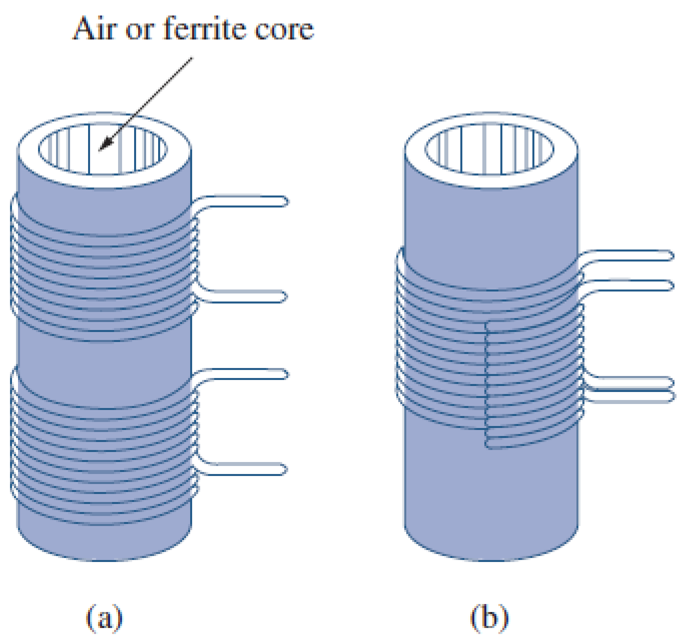


Fig 1.11 Windings: (a) loosely coupled, (b) tightly coupled; cutaway view demonstrates both windings.

Example 1.3/ Consider the circuit in Fig. 1.12. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time $t = 1\ s$ if $v = 60\cos(4t + 30^{\circ})$

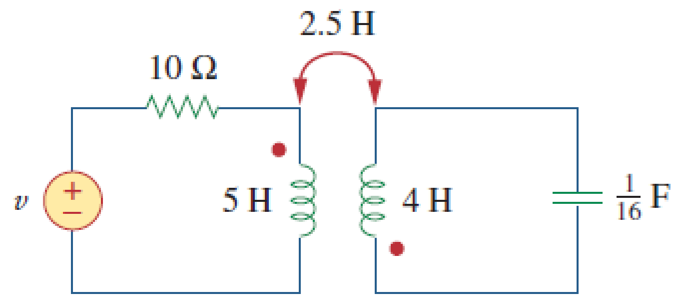


Fig 1.12 For Example 1.3.

1.4 Linear Transformers

Here we introduce the transformer as a new circuit element. A transformer is a magnetic device that takes advantage of the phenomenon of mutual inductance. A transformer is generally a four-terminal device comprising two (or more) magnetically coupled coils. As shown in Fig. 1.13, the coil that is directly connected to the voltage source is called the ***primary winding***. The coil connected to the load is called the ***secondary winding***.

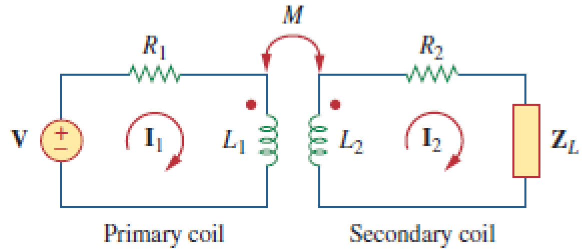
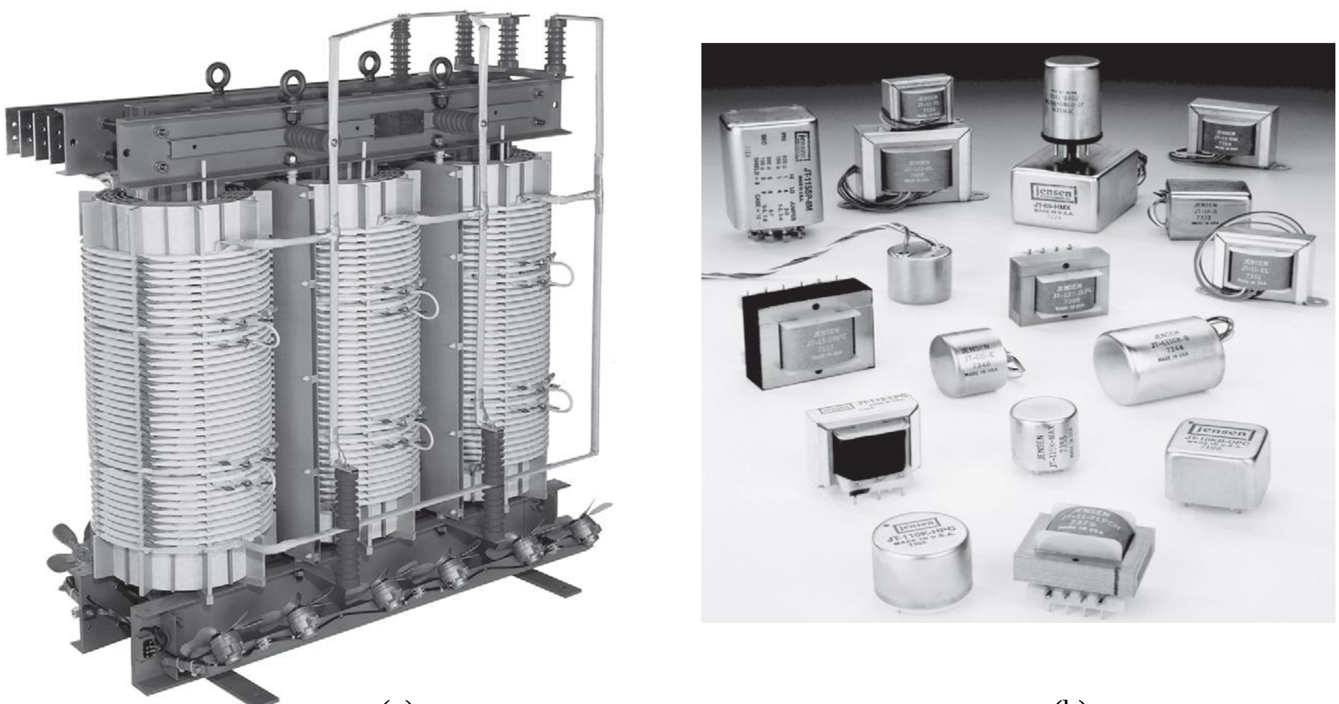


Fig 1.13 A linear transformer.

The resistances R_1 and R_2 are included to account for the losses (power dissipation) in the coils. The transformer is said to be linear if the coils are wound on a magnetically linear material—a material for which the magnetic permeability is constant. Such materials include air, plastic, Bakelite, and wood. In fact, most materials are magnetically linear. Linear transformers are sometimes called air-core transformers, although not all of them are necessarily air-core. They are used in radio and TV sets. Figure 1.4 portrays different types of transformers.



(a) (b)

Fig 1.14 Different types of transformers: (a) copper wound dry power transformer, (b) audio transformers.

We would like to obtain the input impedance Z_{in} as seen from the source, because Z_{in} governs the behavior of the primary circuit. Applying KVL to the two meshes in Fig. 1.13 gives

$$\mathbf{V} = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 \qquad \ldots (23)$$

$$0 = -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2 \qquad \ldots (24)$$

From Eq. (24), $\boldsymbol{I_2} = \frac{\boldsymbol{j\omega M}}{\boldsymbol{R_2 + j\omega L_2 + Z_L}}\boldsymbol{I_1}$, if we substitute it into Eq. (23). We get the input impedance as

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}}{\mathbf{I}_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L} \qquad \ldots (25)$$

Notice that the input impedance comprises two terms. The first term, ($\boldsymbol{R_1 + j\omega L_1}$), is the primary impedance. The second term is due to the coupling between the primary and secondary windings. It is as though this impedance is reflected to the primary. Thus, it is known as the reflected impedance $\boldsymbol{Z_R}$ and

$$\mathbf{Z}_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L}$$

Example 1.4/ In the circuit of Fig. 1.15, calculate the input impedance and current $\boldsymbol{I_1}$. Take $Z_1 = 60 - j100\ \Omega$, $Z_2 = 30 + j40\ \Omega$ and $Z_L = 80 + j60\ \Omega$

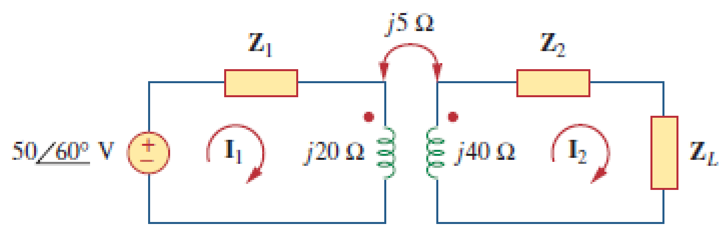


Fig 1.15 For Example 1.4.

1.4.1 *T & π Equivalent Circuit for Linear Transformer*

The little bit of experience gained in the previous sections in analyzing magnetically coupled circuits is enough to convince anyone that analyzing these circuits is not as easy as circuits in previous chapters. For this reason, it is sometimes convenient to replace a magnetically coupled circuit by an equivalent circuit with no magnetic coupling. We want to replace the linear transformer in Fig. 1.16 by an equivalent **T** or **π** circuit, a circuit that would have no mutual inductance.

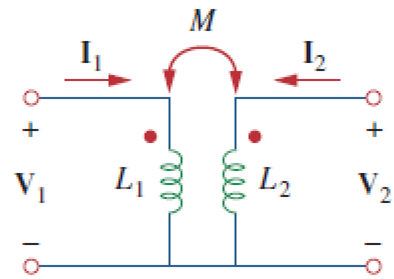


Fig 1.16 Determining the equivalent circuit of a linear transformer.

The voltage-current relationships for the primary and secondary coils give the matrix equation

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \qquad \ldots (26)$$

By matrix inversion, this can be written as

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \qquad \ldots (27)$$

Our goal is to match Eqs. (26) and (27) with the corresponding equations for the **T** and **π** networks.

A- T Equivalent Circuit

For the **T** (or **Y**) network of Fig. 1.17, mesh analysis provides the terminal equations as

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \qquad \ldots (28)$$

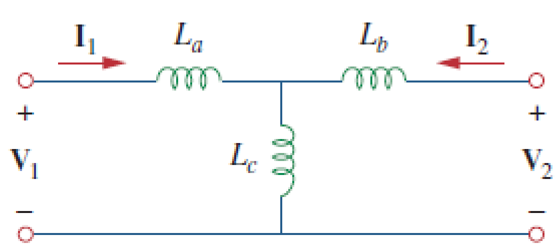


*Fig 1.17 An equivalent **T** circuit.*

If the circuits in Figs. 116 and 1.17 are equivalents, Eqs. (26) and (28) must be identical. Equating terms in the impedance matrices of Eqs. (26) and (28) leads to

$$L_a = L_1 - M, \qquad L_b = L_2 - M, \qquad L_c = M$$

B- π Equivalent Circuit

For the **π** (or **Δ**) network in Fig. 1.18, nodal analysis gives the terminal equations as

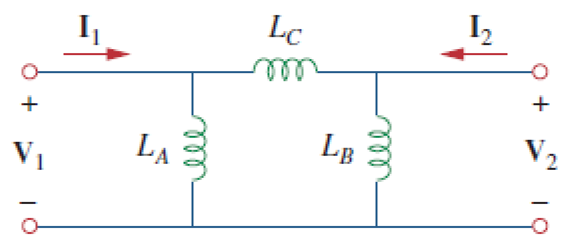


*Fig 1.18 An equivalent **π** circuit.*

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \quad \dots (29)$$

Equating terms in admittance matrices of Eqs. (27) and (29), we obtain

$$L_A = \frac{L_1 L_2 - M^2}{L_2 - M}, \qquad L_B = \frac{L_1 L_2 - M^2}{L_1 - M}$$
$$L_C = \frac{L_1 L_2 - M^2}{M}$$

Note that in Figs. 1.17 and 1.18, the inductors are not magnetically coupled. Also note that changing the locations of the dots in Fig. 1.16 can cause M to become $-M$.

Example 1.5/ Solve for I_1, I_2 and V_0 in Fig. 1.19 using the T-equivalent circuit for the linear transformer.

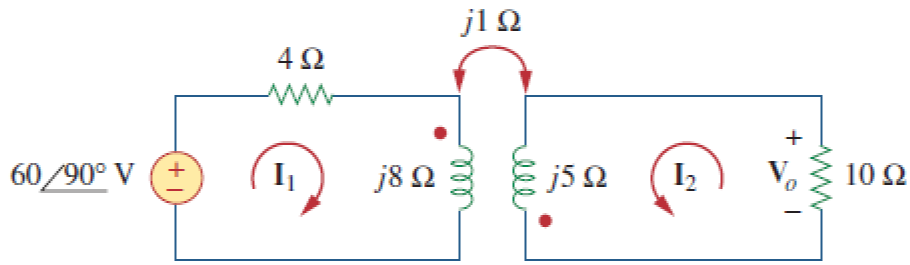


Fig 1.19 For Example 1.5.

1.5 Ideal Transformers

An ideal transformer is one with perfect coupling ($k = 1$). It consists of two (or more) coils with a large number of turns wound on a common core of high permeability. Because of this high permeability of the core, the flux links all the turns of both coils, thereby resulting in a perfect coupling, so a transformer is said to be ideal if it has the following properties:

1- Coils have very large reactances $(L_1, L_2, M \to \infty)$
2- Coupling coefficient is equal to unity $(K = 0)$
3- Primary and secondary coils are lossless $(R_1 = R_2 = 0)$

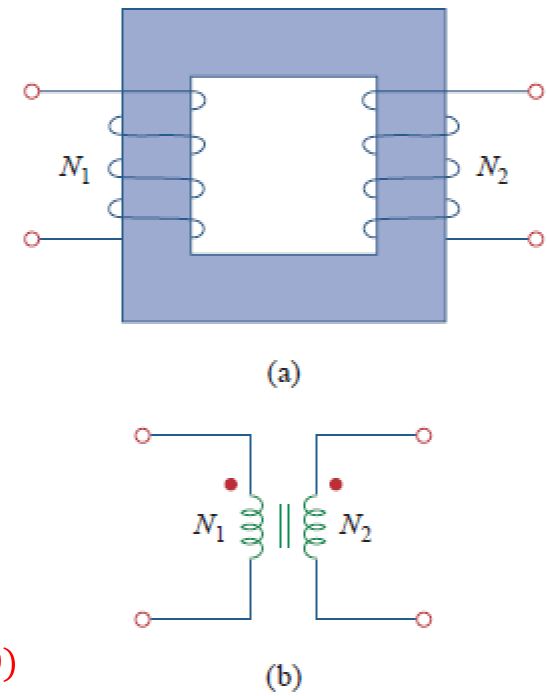


Fig 1.20 (a) Ideal transformer, (b) circuit symbol for ideal transformers.

Figure 1.20(a) shows a typical ideal transformer; the circuit symbol is in Fig. 1.20(b). The vertical lines between the coils indicate an iron core as distinct from the air core used in linear transformers. The primary winding has N_1 turns; the secondary winding has N_2 turns.

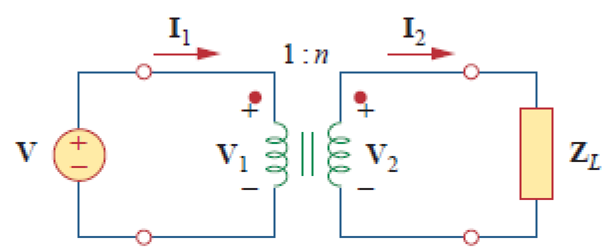


Fig 1.21 Relating primary and secondary quantities in an ideal transformer.

When a sinusoidal voltage is applied to the primary winding as shown in Fig. 1.21, the same magnetic flux ϕ goes through both windings. According to Faraday's law, the voltage across the primary winding and secondary winding are

$$v_1 = N_1\frac{d\phi}{dt}, \quad v_2 = N_2\frac{d\phi}{dt} \Rightarrow \frac{v_2}{v_1} = \frac{N_2}{N_1} = n \xrightarrow{\text{In Phasor}} \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} = n \quad \dots (30)$$

where n is the turns ratio or transformation ratio.

For the reason of power conservation, the energy supplied to the primary must equal the energy absorbed by the secondary, since there are no losses in an ideal transformer. This implies that

$$v_1 i_1 = v_2 i_2 \xrightarrow{\text{In Phasor}} \frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{V}_2}{\mathbf{V}_1} = n \quad \dots (31)$$

So, depending on the value of ***n***, we can know the type of transformer as the following:-

1- $n = 1$ *or* $V_2 = V_1$, we generally call the transformer ***an isolation transformer.***
2- $n > 1$ *or* $V_2 > V_1$, we have ***a step-up transformer.***
3- $n < 1$ *or* $V_2 < V_1$, we have ***a step-down transformer.***

Power companies often generate at some convenient voltage and use a step-up transformer to increase the voltage so that the power can be transmitted at very high voltage and low current over transmission lines, resulting in significant cost savings. Near residential consumer premises, step-down transformers are used to bring the voltage down to 220 V.

It is important that we know how to get the proper polarity of the voltages and the direction of the currents for the transformer in Fig. 1.21. If the polarity of V_1 or V_2 or the direction of I_1 or I_2 is changed, ***n*** in Eqs. (30) and (31) may need to be replaced by $-n$ .The two simple rules to follow are:

1- If V_1 and V_2 are both positive or both negative at the dotted terminals, the transformer ratio will be $+n$. Otherwise, use $-n$.
2- 2. If I_1 and I_2both enter into or both leave the dotted terminals, the transformer ratio will be $+n$. Otherwise, use $-n$.

The rules are demonstrated with the four circuits in Fig. 1.22.

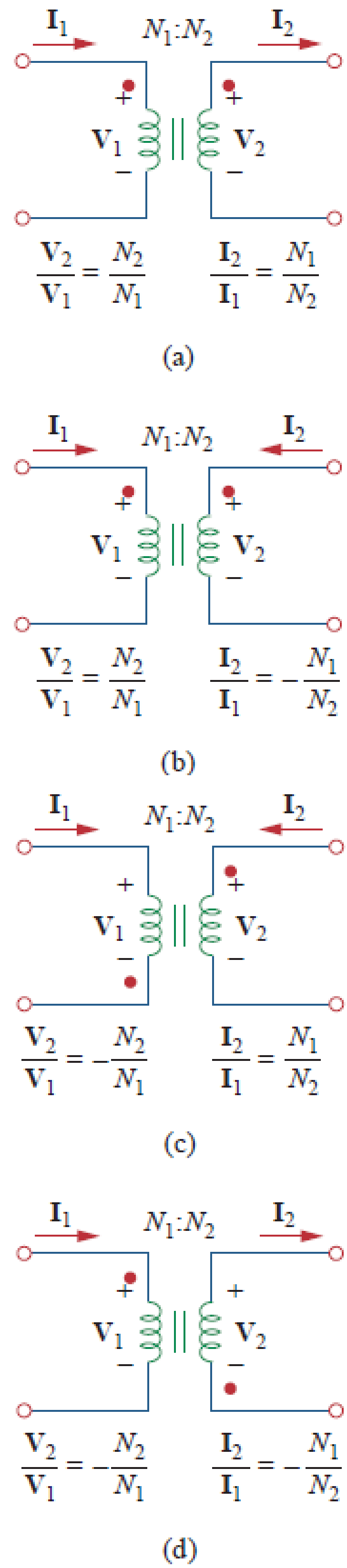


Fig 1.22 Typical circuits illustrating proper voltage polarities and current directions in an ideal transformer.

1.5.1 Compute Z_{in} of Ideal Transformers

The input impedance as seen by the source in Fig. 1.21 is found from

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{n^2}\frac{\mathbf{V}_2}{\mathbf{I}_2} \quad \dots (32)$$

It is evident from Fig. 1.21 that $V_2/I_2 = Z_L$ so that

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_L}{n^2} \quad \dots (33)$$

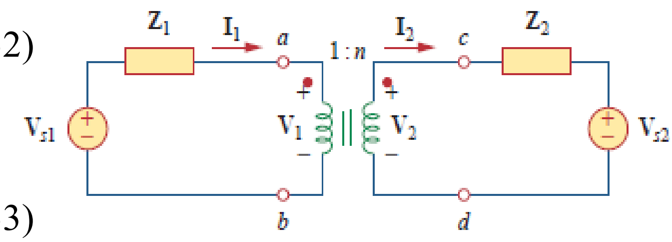


Fig 1.23 Ideal transformer circuit whose equivalent circuits are to be found.

The input impedance is also called the ***reflected impedance***, since it appears as if the load impedance is reflected to the primary side. This ability of the transformer to transform a given impedance into another impedance provides us a means of ***impedance matching*** to ensure ***maximum power transfer***.

In analyzing a circuit containing an ideal transformer, it is common practice to eliminate the transformer by reflecting impedances and sources from one side of the transformer to the other. In the circuit of Fig. 1.23, suppose we want to reflect the ***secondary side*** of the circuit to the ***primary side***. We find the Thevenin equivalent of the circuit to the right of the terminals a-b. We obtain V_{Th} as the open-circuit voltage at terminals a-b, as shown in Fig. 1.24(a).

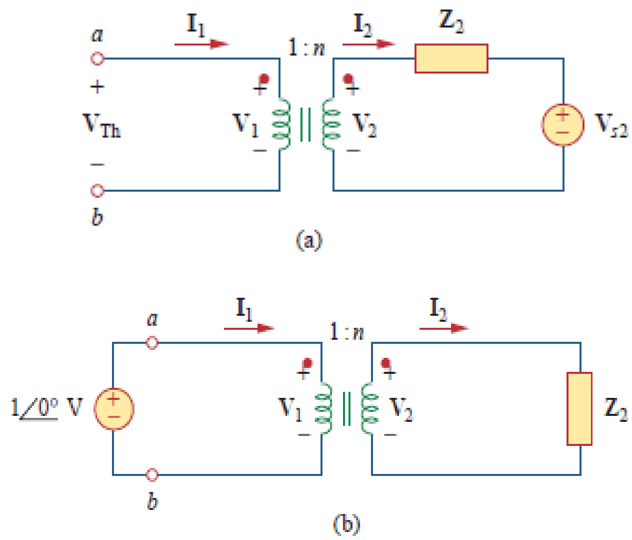


Fig 1.24 (a) Obtaining V_{Th} for the circuit in Fig. 1.23, (b) obtaining Z_{Th} for the circuit in Fig. 1.23.

Since terminals a-b are open $I_1 = 0 = I_2$, so that $V_2 = V_{s2}$. Hence,

$$\mathbf{V}_{\text{Th}} = \mathbf{V}_1 = \frac{\mathbf{V}_2}{n} = \frac{\mathbf{V}_{s2}}{n} \quad \dots (34)$$

To get we remove the voltage source in the secondary winding and insert a unit source at terminals a-b, as in Fig. 1.24(b).

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{V}_2/n}{n\mathbf{I}_2} = \frac{\mathbf{Z}_2}{n^2}, \qquad \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 \quad \dots (35)$$

Once we have V_{Th} and Z_{Th} we add the Thevenin equivalent to the part of the circuit in Fig. 1.23 to the left of terminals a-b. Figure 1.25 shows the result.

In the same strategy, we can also reflect the primary side of the circuit in Fig. 1.23 to the secondary side. Figure 1.26 shows its equivalent circuit.

Also note that if the locations of the dots in Fig. 1.23 are changed, we might have to replace $\boldsymbol{n}$ by $-\boldsymbol{n}$ in order to obey the dot rule, illustrated in Fig. 1.22.

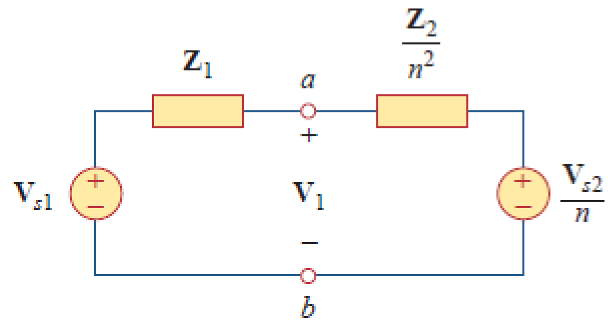


Fig 1.25 Equivalent circuit for Fig. 1.23 obtained by reflecting the secondary circuit to the primary side.

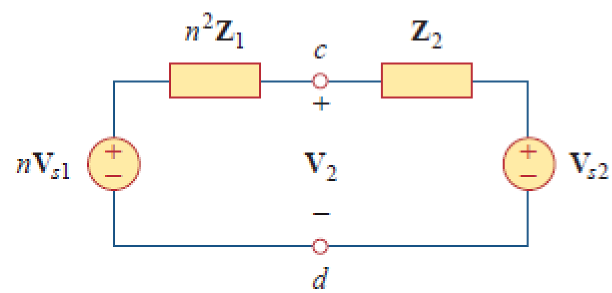


Fig 1.26 Equivalent circuit for Fig. 1.23 obtained by reflecting the primary circuit to the secondary side.

1.5.2 ***Complex Power of Ideal Transformers***

The complex power in the primary winding is

$$\mathbf{S_1} = \mathbf{V_1}\ \mathbf{I_1^*} \qquad \boldsymbol{KVA} \qquad \dots (36)$$

As $\mathbf{V_1} = \mathbf{V_2/n}$, $\mathbf{I_1} = \mathbf{nI_2}$, so Eq. (36) will be

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{I}_1^* = \frac{\mathbf{V}_2}{n}(n\mathbf{I}_2)^* = \mathbf{V}_2\mathbf{I}_2^* = \mathbf{S}_2 \qquad \dots (37)$$

Eq. (37) is showing that the complex power supplied to the primary is delivered to the secondary without loss. The transformer absorbs no power. Of course, we should expect this, since the ideal transformer is lossless.

Example 1.6/ Calculate the power supplied to the 10 Ω resistor in the ideal transformer circuit of Fig. 1.27.

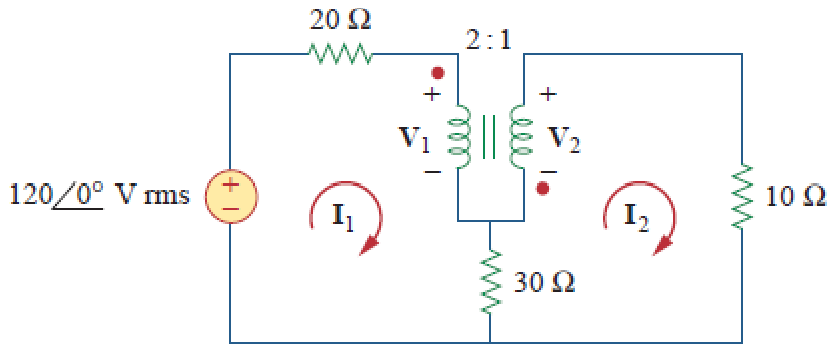


Fig 1.27 For Example 1.6.

1.6 Ideal Autotransformers

Unlike the conventional two-winding transformer we have considered so far, an autotransformer has a single continuous winding with a connection point called a tap between the primary and secondary sides. The tap is often adjustable so as to provide the desired turns ratio for stepping up or stepping down the voltage. This way, a variable voltage is provided to the load connected to the autotransformer.

Fig 1.28 A typical autotransformer.

Figure 1.28 shows a typical autotransformer. As shown in Fig. 1.29, the autotransformer can operate in the step-down or step up mode. The autotransformer is a type of power transformer. Its major advantage over the two-winding transformer is its ability to transfer larger apparent power. Another advantage is that an autotransformer is smaller and lighter than an equivalent two-winding transformer. However, since both the primary and secondary windings are one winding, electrical isolation (no direct electrical connection) is lost.

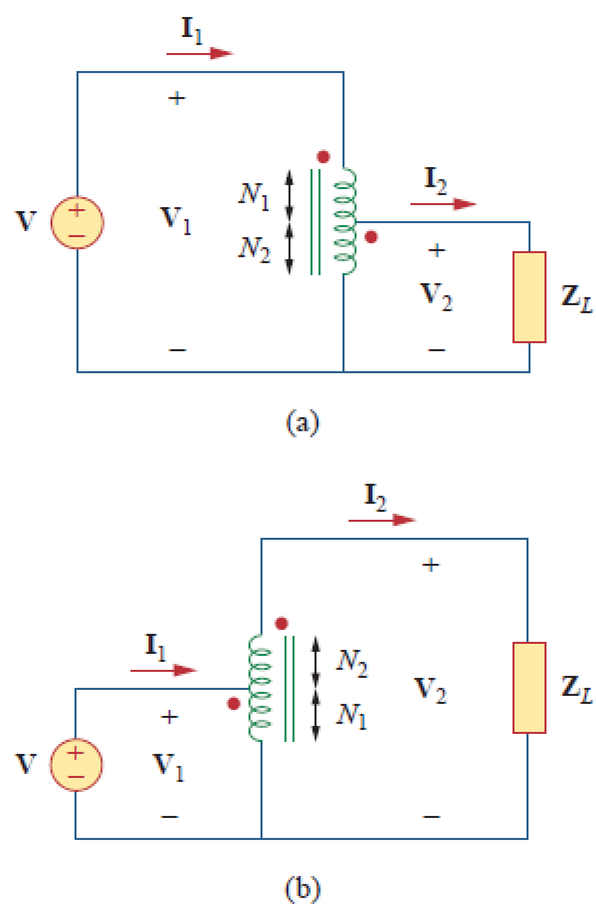


Fig 1.29 (a) Step-down autotransformer, (b) step-up autotransformer.

Some of the formulas we derived for ideal transformers apply to ideal autotransformers as well. ***For the step-down autotransformer circuit***:-

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1 + N_2}{N_2} = 1 + \frac{N_1}{N_2} \qquad \dots (38)$$

As an ideal autotransformer, there are no losses, so the complex power remains the same in the primary and secondary windings:

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{I}_1^* = \mathbf{S}_2 = \mathbf{V}_2\mathbf{I}_2^* \Rightarrow V_1I_1 = V_2I_2 \Rightarrow \frac{V_2}{V_1} = \frac{I_1}{I_2} \Rightarrow \frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_2}{N_1 + N_2} \qquad \dots (39)$$

For the step-up autotransformer circuit:-

$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_1 + N_2} \quad \text{or} \quad \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1}{N_1 + N_2} \qquad \ldots (40)$$

The complex power given by Eq. (39) also applies to the step-up autotransformer so that Eq. (40) again applies. Hence, the current relationship is

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_1 + N_2}{N_1} = 1 + \frac{N_2}{N_1} \qquad \ldots (41)$$

A major difference between conventional transformers and autotransformers is that the primary and secondary sides of the autotransformer are not only coupled magnetically but also coupled conductively. The autotransformer can be used in place of a conventional transformer when electrical isolation is not required.

Example 1.7/ Refer to the autotransformer circuit in Fig. 1.30. Calculate: (a) $\mathbf{I_1}$and $\mathbf{I_2}$ $\mathrm{I_o}$ if $\mathbf{Z_L} = \mathbf{8 + j6\ \Omega}$,and (b) the complex power supplied to the load.

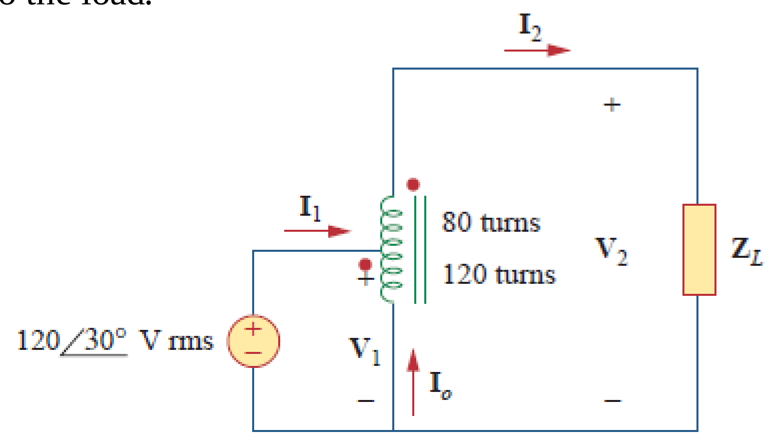


Fig 1.30 For Example 1.7.