Basrah University
College of Engineering
Electrical Engineering Department

Electric Circuits Analysis

Asst. Lect: Hamzah Abdulkareem

Chapter Two
Part 1

Transient Circuits

- ***First-Order Circuits***
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- ***The Source Free - RL Circuit***
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Chapter Two

Transient Circuits

2.1 First-Order Circuits

Now that we have considered the three passive elements (resistors, capacitors, and inductors individually, we are prepared to consider circuits that contain various combinations of two or three of the passive elements. In this part, we shall examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor. These are called $\boldsymbol{RC}$ and $\boldsymbol{RL}$ circuits, respectively. Before begin in analysis the mentioned circuits, some definitions should be discussed :-

1- **A first-order circuit** is characterized by a first-order differential equation.
2- **The natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
3- **A source-free circuit** occurs when its dc source is suddenly disconnected.
4- **A capacitor** consists of two conducting plates separated by an insulator (or dielectric). Fig 2.1 shows different types of capacitors.
5- **Capacitance** is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).
6- The relationship between capacitor current & voltage is:- $i = C\frac{dv}{dt}$.
7- **An inductor** consists of a coil of conducting wire as shown in Fig 2.2.
8- **Inductance** is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).
9- The relationship between indicator current & voltage is:- $v = L\frac{di}{dt}$.

Fig 2.1 Several different types of commercially available capacitors.

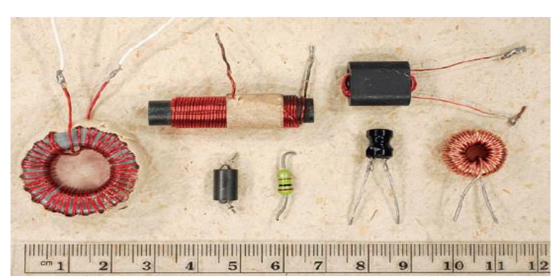

Fig 2.2 Several different types of commercially available inductors.

2.2 The Source Free - RC Circuit

A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors. Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 2.3. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be $v(t)$ the voltage across the capacitor. Since the capacitor is initially charged, we can assume that at time $t = 0$ the initial voltage is

$$v(0) = V_o \qquad \ldots (2.1)$$

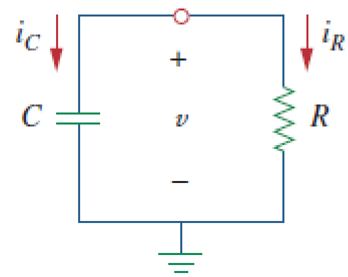

Fig 2.3 A source-free RC circuit.

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_o^2 \qquad \ldots (2.2)$$

Applying KCL at the top node of the circuit in Fig. 2.3 yields

$$i_R + i_C = 0 \qquad \ldots (2.3)$$

By definition $i_C = C\frac{dv}{dt}$, and $i_R = \frac{v}{R}$ Thus,

$$C\frac{dv}{dt} + \frac{V}{R} = 0 \qquad \ldots (2.4), \qquad \text{or} \qquad \frac{dv}{dt} + \frac{v}{RC} = 0 \qquad \ldots (2.5)$$

This is a first-order differential equation, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad \ldots (2.6)$$

Ingrate it →

$$\ln v = -\frac{t}{RC} + \ln A$$

where is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC}$$

Taking powers of e produces →

$$v(t) = \mathrm{A}e^{-\frac{t}{RC}} \qquad \ldots (2.7)$$

But from the initial conditions $v(0) = A = V_o$, so the circuit response is

$$v(t) = \mathrm{V_o}e^{-\frac{t}{RC}} \qquad \ldots (2.8)$$

Eq. (2.8) shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the natural response of the circuit.

The natural response is illustrated graphically in Fig. 2.4. Note that at we have the correct initial condition $t = 0$ as in Eq. (2.1). As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the time constant, denoted by $\tau = RC$, so Eq. (2.8) will be :-

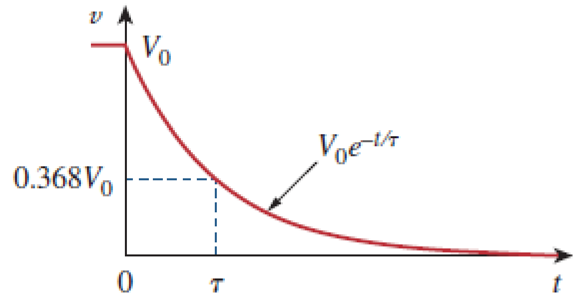

Fig 2.4 The voltage response of the RC circuit.

$$v(t) = V_o e^{-\frac{t}{\tau}} \qquad \ldots (2.9)$$

This implies that at $t = \tau$, so $v(t)$ will be equal to

$$v(t) = 0.368V_o$$

So the time constant τ of a circuit is the time required for the response to decay to a factor of $1/e$ or 36.8 percent of its initial value.

With the voltage in Eq. (2.9), we can find the current i_R as :-

$$i_R = \frac{v(t)}{R} = \frac{V_o}{R} e^{-\frac{t}{\tau}} \qquad \ldots (2.10)$$

The power dissipated in the resistor is

$$p(t) = vi_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p(\lambda) d\lambda = \int_0^t \frac{V_0^2}{R} e^{-2\lambda/\tau} d\lambda$$

$$= -\frac{\tau V_0^2}{2R} e^{-2\lambda/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}),$$

As t ⟶ ∞

$$w_R = \frac{1}{2} C V_o^2$$

which is $w(0)$ the same as the energy initially stored in the capacitor.

Example 2.1 :- In Fig. 2.5, let $v_C(0) = 15$ Find v_C, v_x and i_x for $t \geq 0$

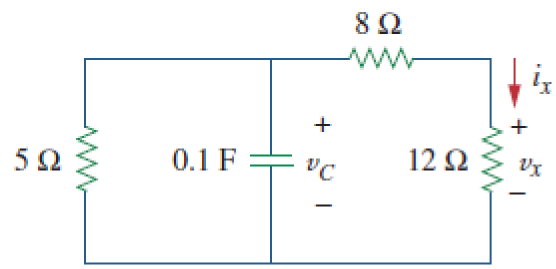

Fig 2.5 For Example 2.1.

Example 2.2:- The switch in the circuit in Fig. 2.6 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$ Calculate the initial energy stored in the capacitor.

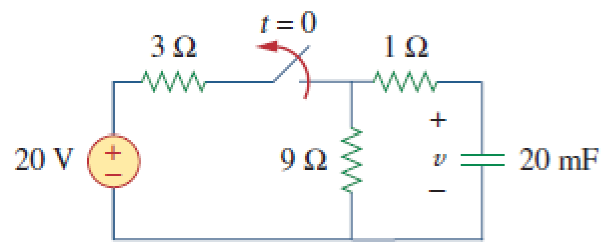

Fig 2.6 For Example 2.2.

2.3 The Source Free - RL Circuit

Consider the series connection of a resistor and an inductor, as shown in Fig. 2.7. Our goal is to determine the circuit response, which we will assume to be the current through the inductor.

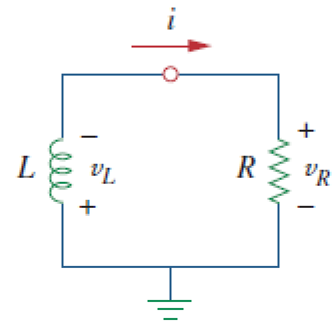

Fig 2.7 A source-free RL circuit.

At $t = 0$ we assume that the inductor has an initial current I_o or

$$i(0) = I_0 \qquad \ldots (2.11)$$

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2 \qquad \ldots (2.12)$$

Applying KVL around the loop in Fig.2.7 ,

$$v_L + v_R = 0 \qquad \ldots (2.13)$$

But $v_L = L\, di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

or

$$\frac{di}{dt} + \frac{R}{L} i = 0 \qquad \ldots (2.14)$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \qquad \ldots (2.15)$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \qquad \ldots (2.16)$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig. 2.8. It is evident from Eq. (2.16) that the time constant for the RL circuit is

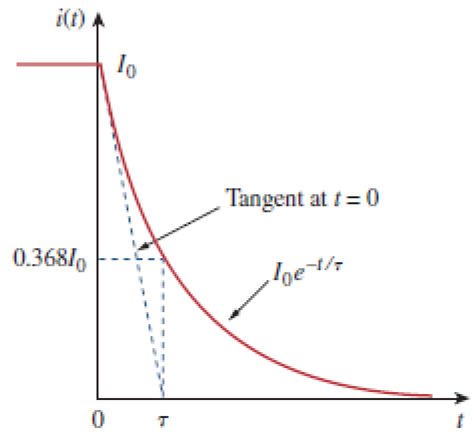

Fig 2.8 The current response of the RL circuit.

$$\tau = \frac{L}{R} \qquad \dots (2.17)$$

Thus, Eq. (2.16) may be written as

$$i(t) = I_o e^{-\frac{t}{\tau}} \qquad \dots (2.18)$$

With the current in Eq. (2.18), we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \qquad \dots (2.19)$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \qquad \dots (2.20)$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p(\lambda) d\lambda = \int_0^t I_0^2 e^{-2\lambda/\tau} d\lambda = -\frac{\tau}{2} I_0^2 R e^{-2\lambda/\tau}\Big|_0^t,$$

or

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

As t ⟶ ∞

$$w_R = \frac{1}{2} L I_o^2$$

which is the same as the initial energy stored in the inductor as in Eq. (2.12).

Example 2.3 :- In Fig. 2.9, let $i(0) = 10$ Find $i(t), i_x$ and $i_x(t)$.

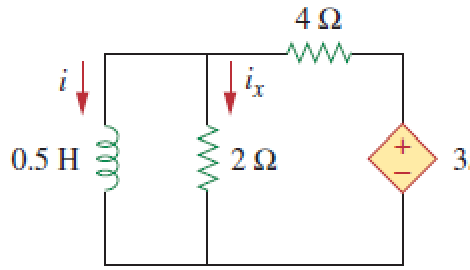

Fig 2.9 For Example 2.3.

Example 2.4:- The switch in the circuit in Fig. 2.6 has been closed for a long time, and it is opened at $t = 0$. Find $i(t)$ for $t \geq 0$ Calculate the initial energy stored in the capacitor.

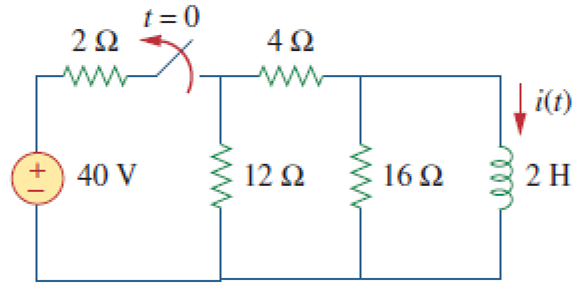

Fig 2.10 For Example 2.4.

2.4 The Unit-Step Function

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t. The mathematical representation of unit-step function can be written in three ways according to the shift status, these status are :-

1- General form (There is not any shifting) as shown in Fig. (2.11):-

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

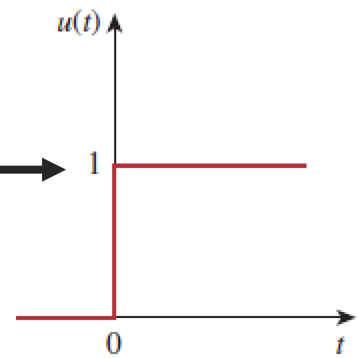

Fig 2.11 The unit step function without shifting

2- Unit step function delayed by t_0 shown in Fig. (2.12):-

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

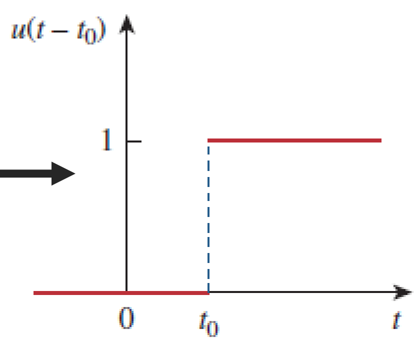

Fig 2.12 The unit step delayed by t_0.

3- Unit step function advanced by t_0 shown in Fig. (2.13):-

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

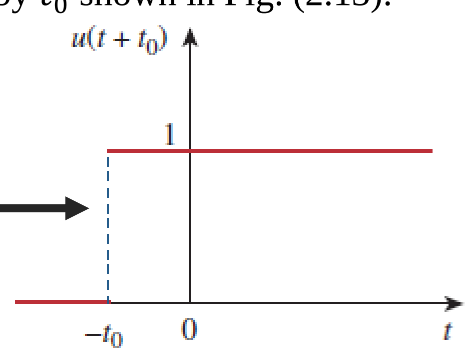

Fig 2.13 The unit step advanced by t_0.

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \quad \ldots (2.21)$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \quad \ldots (2.22)$$

If we let $t_0 = 0$ then $v(t)$ is simply the step voltage $V_0 u(t)$ .A voltage source of is shown in Fig. 2.14(a); its equivalent circuit is shown in Fig. 2.14(b). Similarly, a current source $I_0 u(t)$ of is shown in Fig. 2.15(a), while its equivalent circuit is in Fig. 2.15(b).

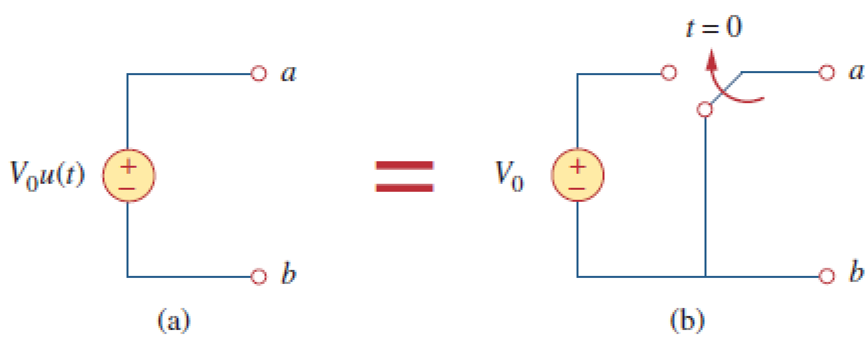

Fig 2.14 (a) Voltage source of $V_0 u(t)$, (b) its equivalent circuit.

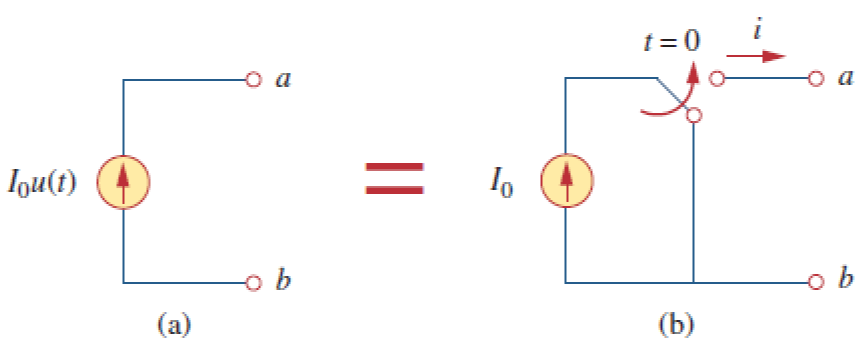

Fig 2.15 (a) Current source of $I_0 u(t)$, (b) its equivalent circuit.

Example 2.5:- Express the voltage pulse in Fig. 2.16 in terms of the unit step.

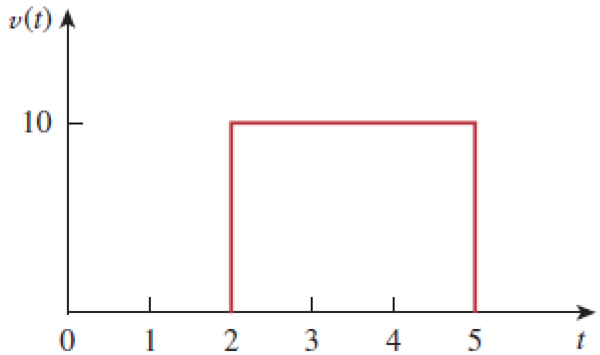

Fig 2.16 For Example 2.5.

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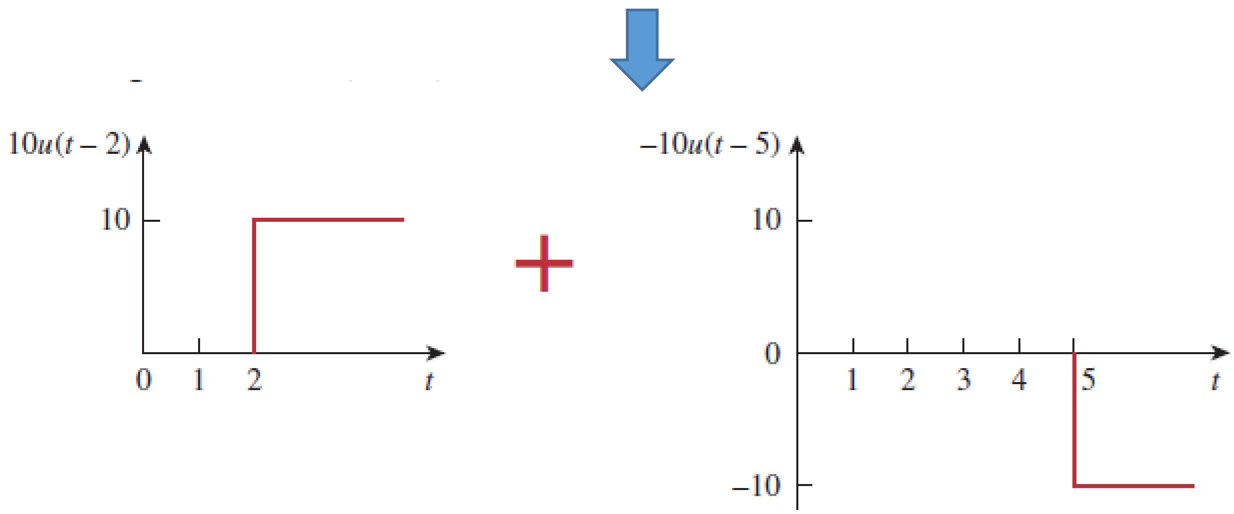

2.5 Step Response of an RC circuit

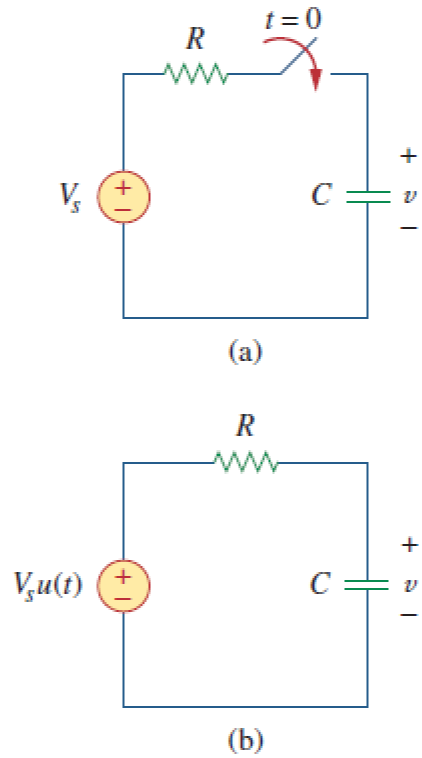

Fig 2.17 An RC circuit with voltage step input.

When the dc source of an RC circuit is suddenly applied, the voltageor current source can be modeled as a step function, and the response is known as a step response.

So, the step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

Consider the RC circuit in Fig. 2.17(a) which can be replaced by the circuit in Fig. 2.17(b), where V_s is a constant dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume V_0 an initial voltage on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad \ldots (2.23)$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \qquad \ldots (2.24)$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (2.24) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad \ldots (2.25)$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \qquad \ldots (2.26)$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \qquad \ldots (2.27)$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \qquad \ldots (2.28)$$

Thus,

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases} \qquad \ldots (2.29)$$

This is known as the complete response (or total response) of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. Assuming that $V_s > V_0$, a $v(t)$ plot of is shown in Fig. 2.18.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (2.29) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \qquad \ldots (2.30)$$

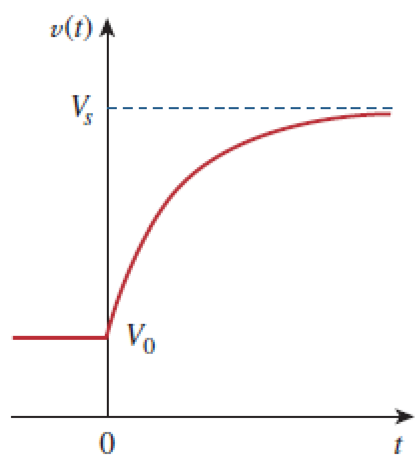

Fig 2.18 Response of an RC circuit with initially charged capacitor.

Eq. (2.30) is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (2.30) using $i(t) = C\frac{dv}{dt}$ We get

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t) \qquad \dots (2.31)$$

Figure 2.19 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.

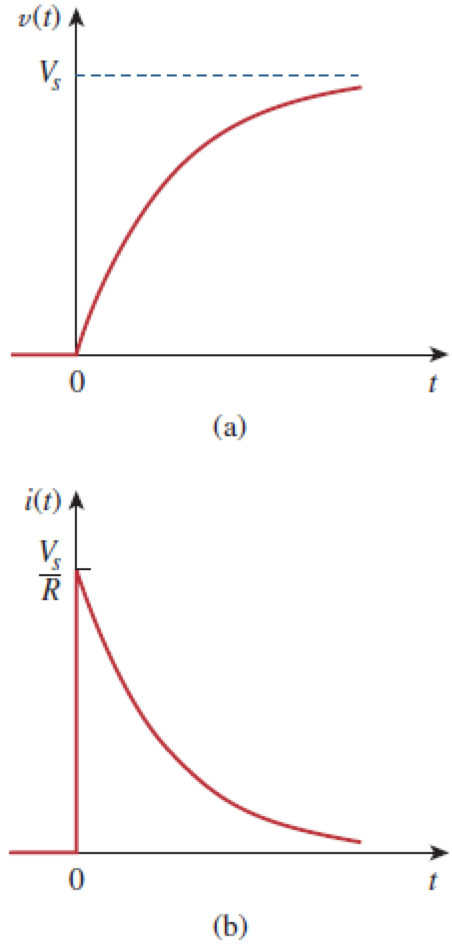

Fig 2.19 Step response of an RC circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

From Eq. (2.28), it is evident that has two components. Classically there are two ways of decomposing this into two components. The first is to break it into a "natural response and a forced response'' and the second is to break it into a "transient response and a steady-state response.'' .Table 2.1 shows these responses.

Table 2.1 Time steps response components of an RC circuit

Natural response and Forced response	**Transient response and Steady-state response**
Complete response = natural response (stored energy) + forced response (independent source) or $v = v_n + v_f$ where $v_n = V_o e^{-t/\tau}$ and $v_f = V_s(1 - e^{-t/\tau})$ Where v_n & v_f are natural response and forced response respectively.	Complete response = transient response + steady-state response or $v = v_t + v_{ss}$ where $v_t = (V_o - V_s)e^{-t/\tau}$ and $v_{ss} = V_s$ Where v_t & v_{ss} are transient response and steady state response respectively.

Thus, the transient response is the circuit's temporary response that will die out with time while the steady-state response is the behavior of the circuit a long time after an external excitation is applied. Whichever way we look at it, the complete response in Eq. (2.28) may be written as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \qquad \ldots (2.32)$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady state value.

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$ there is a time delay in the response so that Eq. (2.32) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \qquad \ldots (2.33)$$

where $v(t_0)$ is the initial voltage at $t = t_0$ and $v(\infty)$ is the final or steady state value.

Example 2.6:- The switch in Fig. 2.20 has been in position A for a long time. At $t = t_0$ the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $= 1\ s\ and\ 4\ s$.

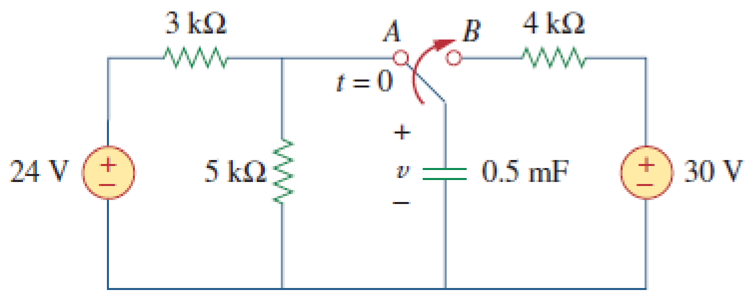

Fig 2.20 For Example 2.6.

Example 2.7:- In Fig. 2.21, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

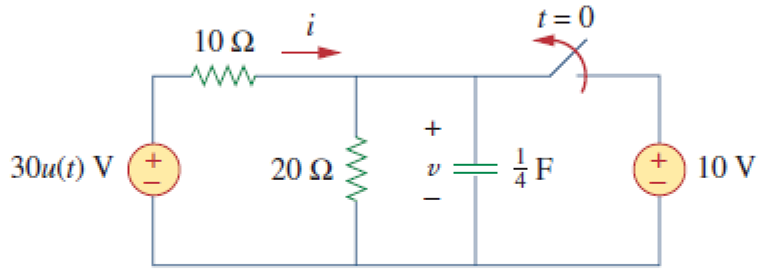

Fig 2.21 For Example 2.7.

2.6 Step Response of an RL circuit

Consider the RL circuit in Fig. 2.22(a), which may be replaced by the circuit in Fig. 2.22(b). Again, our goal is to find the inductor current i as the circuit response. Rather than apply Kirchhoff's laws, we will use the technique of two components as in the last section . Let the response be the sum of the transient response and the steady-state response,

$$i = i_t + i_{ss} \qquad \ldots(2.34)$$

As $i_t = Ae^{-t/\tau}$, & $i_{ss} = \frac{V_s}{R}$, so the above equation will be

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \qquad \ldots(2.35)$$

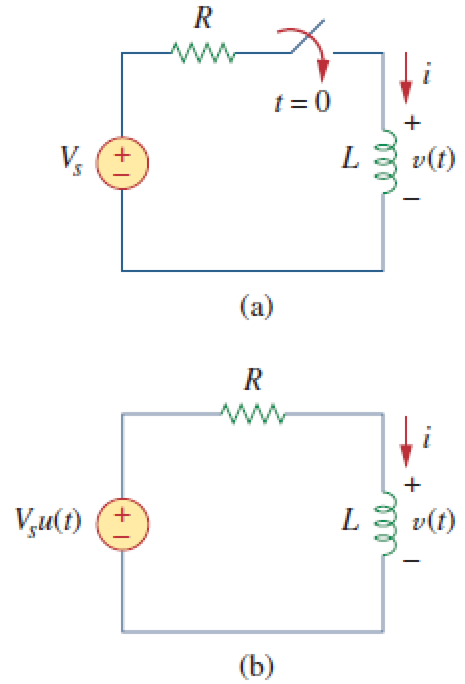

Fig 2.22 An RL circuit with voltage step input.

We now determine the constant A from the initial value of i. Let be I_0 the initial current through the inductor, which may come from a source other than V_s .Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \qquad \ldots(2.36)$$

Thus, at = 0 , Eq. (2.35) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

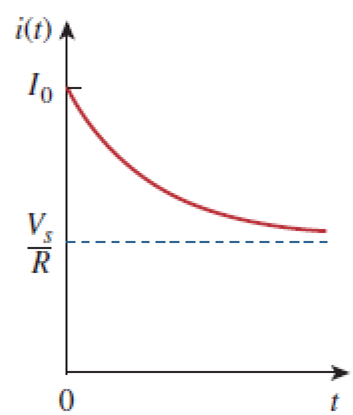

Fig 2.23 Total response of the RL circuit with initial inductor currentI_0.

Substituting for A in Eq. (2.35), we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$$

Or can be written

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

This is the complete response of the RL circuit. It is illustrated in Fig. 2.23.

Again, if the switching takes place at time $t = t_0$ instead of at $t = 0$, the last equation becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau} \qquad \ldots(2.37)$$

If $I_0 = 0$ then

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad \ldots(2.38)$$

Or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \qquad \ldots(2.39)$$

This is the step response of the RL circuit with no initial inductor current. The voltage across the inductor is obtained from Eq. (2.39) using $v = L\dfrac{di}{dt}$ .We get

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$

Or

$$v(t) = V_s e^{-t/\tau}u(t) \qquad \ldots(2.40)$$

Figure 2.24 shows the step responses in Eqs. (2.39) and (2.40).

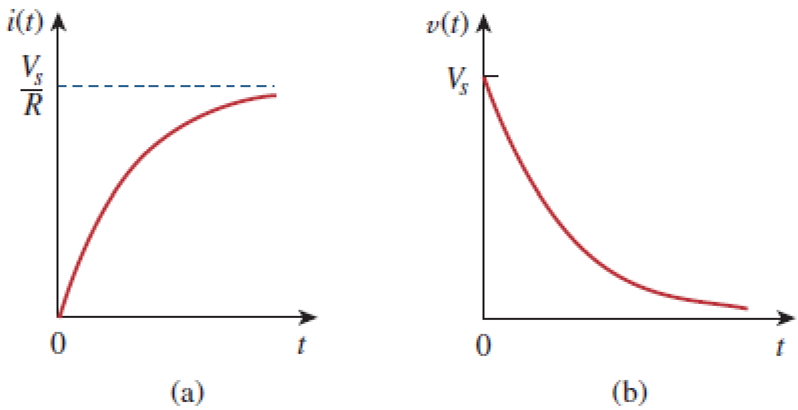

Fig 2.24 Step responses of an RL circuit with no initial inductor current:
(a) current response, (b) voltage response.

Example 2.8:- Find $i(t)$ in the circuit of Fig. 2.25 for $t > 0$ .Assume that the switch has been closed for a long time.

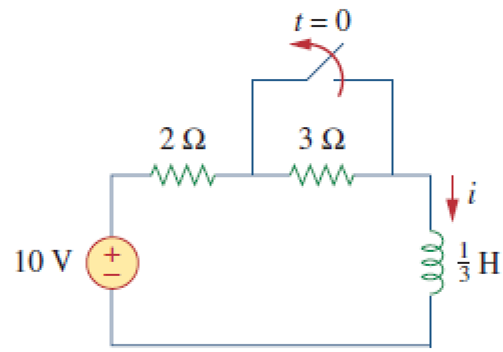

Fig 2.25 For Example 2.8.

Example 2.9:- At $t = 0$, switch 1 in Fig. 2.26 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2\ s$ and $t = 5\ s$.

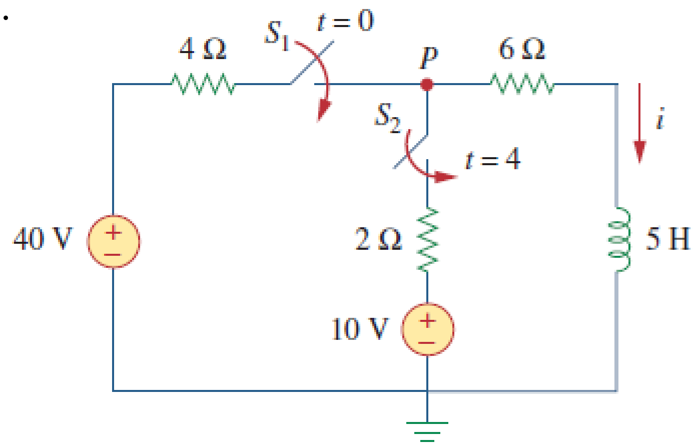

Fig 2.26 For Example 2.9.