Lecture 10: The solution of the Sine-Gordon equation by separation of variables

The form of the solution (38) suggest that the transformation

$$v(x,t) = \tan(\frac{u}{4}) \tag{39}$$

can be used to transform Sine-Gordon equation $u_{xx} - u_{tt} = \sin(u),$ into the form

$$(1+v^2)(v_{xx} - v_{tt} - v) - 2v(v_x^2 - v_t^2 - v^2) = 0, \tag{40}$$

in which the trigonometric identity $\sin(u) = 4v(1-v^2)/(1+v^2),$ has been utilized. we seek solutions by the separation of variables in the form

$$v(x,t) = \tan(\frac{u}{4}) \approx \frac{\phi(x)}{\psi(t)}, \tag{41}$$

for some functions $\phi(x)$ *and* $\psi(t)$ to be determined.

By using equation(41),equation(40) becomes;

$$(\phi^2 + \psi^2)(\frac{\phi_{xx}}{\phi} + \frac{\psi_{tt}}{\psi}) - 2(\phi_x^2 + \psi_t^2) = (\phi^2 + \psi^2) \tag{42}$$

differentiating this equation with respect to x *and* t enable us to separate the variables, so that;

$$\frac{1}{\phi\phi_x}(\frac{\phi_{xx}}{\phi})_x = \frac{1}{\phi\phi_x}(\frac{\psi_{tt}}{\psi})_t = -4k \tag{43}$$

where $-4k$ is a separation constant .Each of these ODEs can be integrated twice to find

$$\left.\begin{aligned} \phi_x^2 &= -k\phi^4 + m^2\phi^2 + n^2 \\ \psi_t^2 &= k\psi^4 + (m^2-1)\psi^2 - n^2 \end{aligned}\right\} \tag{44}$$

where m *and* n are integrating constants. Solve these equations for the following special cases:

Case 1: $k = n = 0, m^2 > 1$

In this case equation (44) take the form

$$\left.\begin{aligned} \phi_x &= \pm m^2 \phi^2 \\ \psi_t &= \pm\sqrt{m^2 - 1}\psi \end{aligned}\right\} \tag{45}$$

which give exponential solutions

$$\left.\begin{aligned} \phi(x) &= a_1 e^{\pm mx} \\ \psi(t) &= a_2 e^{\pm\sqrt{m^2-1}\,t} \end{aligned}\right\} \tag{46}$$

where a_1 *and* a_2 are integrating constants. thus the solution (41) becomes

$$u(x,t) = 4\tan^{-1}(\alpha e^{\pm\frac{x-\mu t}{\sqrt{1-V^2}}}) \tag{47}$$

where $\alpha = \dfrac{a_1}{a_2}$, *and* $\mu = \dfrac{\sqrt{m^2 - 1}}{m}$ are constants.

Evidently, one of these solutions (47)

$$u(x,t) = 4\tan^{-1}(\alpha e^{m(x-\mu t)}) \tag{48}$$

Case 2: If $k = 0, n \neq 0,$ *and* $m^2 > 1$ find

$$u(x,t) = 4\tan^{-1}\left(\frac{\mu\sinh(mx + a_1)}{\cosh(\sqrt{m^2 - 1}\,t + a_2)}\right) \qquad \textbf{((H.w))}$$

Case 3: If $k \neq 0, n = 0,$ *and* $m^2 > 1$ find

$$u(x,t) = -4\tan^{-1}\left(\frac{m}{\sqrt{m^2 - 1}}\,\frac{\sinh(\sqrt{m^2 - 1}\,x + a_1)}{\cosh(mx + a_2)}\right) \textbf{((H.w))}$$

Case 4: If $k = 0, n \neq 0,$ *and* $m^2 < 1$ find

$$u(x,t) \approx -4\tan^{-1}\left(\frac{m}{\sqrt{m^2 - 1}}\,\frac{\sinh(mt + a_1)}{\cosh(mx + a_2)}\right) \textbf{((H.w))}$$

<u>EXs</u>: (1) consider the sine-Gordon equation(SFE) $u_{xx} - c^2 u_{tt} = \sin(u),$ then

a- Show that the transformations $\eta = \frac{1}{2}(x - t/c), \xi = \frac{1}{2}(x + t/c)$ transform the SGE into $u_{\xi\eta} = \sin(u),$

b- Reduce it into the form $u_{xt} = \sin(u),$ by using
$u_x = 2a\sin(u/2), u_t = \frac{2}{a}\sin(u/2)$

then use similarity method to transform the resulting equation into second order ODE. **((H.W))**

(2) Seek a similarity solution of the SGE $u_{tt} - u_{xx} = u^3,$ in the form $u(x,t) = t^m f(xt^n)$ for suitable values of m *and* $n.$ **((H.W))**