**Lecture8:**

Example: consider the Sine-Gordon equation

$$u_{tt} - \frac{1}{c^2} u_{xx} = \sin(u), \tag{31}$$

we seek a solution of the form $u(x,t) = \phi(x - \mu t), \quad \xi = x - \mu t$ ,which corresponding to a wave travelling with a velocity $\mu$ and then, substitute in equation(31) to obtain an ODE for $\phi$ in the form

$$(1 - V^2)\phi'' = \sin(\phi) \tag{32}$$

where $V^2 = \frac{\mu^2}{c^2}$.

$$\phi'\phi'' = \frac{\sin(\phi)}{(1 - V^2)}\phi' \tag{33}$$

this equivalently

$$\frac{d}{d\xi}\left[\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{\cos(\phi)}{1 - V^2}\right] = 0 \tag{34}$$

that is ;

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{\cos(\phi)}{1 - V^2} = \beta \tag{35}$$

where $\beta$ is integral constant. Solving for $\frac{d\phi}{d\xi}$ gives a first-order ODE

$$\frac{d\phi}{d\xi} = \sqrt{2A - \frac{2\cos(\phi)}{1 - V^2}}$$

we can separate the variable and then integrated to obtain

$$\int_{\phi_0}^{\phi} \frac{d\psi}{\sqrt{A - \cos\psi}} = \sqrt{\frac{2}{1 - V^2}} \int_{\xi_0}^{\xi} d\eta \tag{36}$$

where $A = \beta\sqrt{1 - V^2}$ . This result depends on the two parameters $V\ ((or\,\mu = cV)),$ the velocity of the solution and $\beta$ an integrating constant.

when $A=1$, a solitary wave solution exists for any velocity $0<|V|<1 \quad ((or\ 0<|\mu|<c))$.

To simplify both sides of equation (6*) by using trigonometric identities, we have

$$\sqrt{\frac{2}{1-V^2}}(\xi-\xi_0)=\sqrt{2}\int_{\phi_0}^{\phi}\frac{d\psi}{2\sin(\frac{\psi}{2})}=\sqrt{2}\log(\frac{\tan(\frac{\phi}{4})}{\tan(\frac{\phi_0}{4})}) \tag{37}$$

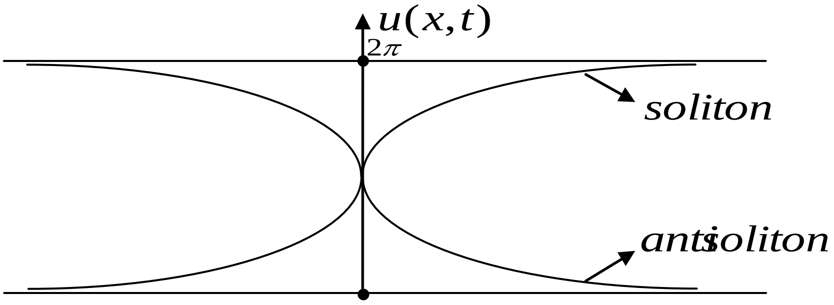
This lends to the solution of $\phi(\xi)$ as;

$$\phi(\xi)=4\tan^{-1}(\alpha e^{\frac{\xi-\xi_0}{\sqrt{1-V^2}}})$$

where $\alpha=\tan(\frac{\phi_0}{4})$, and $\xi_0$ are constants of integration, which can be incorporated in several ways. when choose $\alpha=1$ and $\xi_0=0$ to obtain one simple solitary wave solution for $u(x,t)$ in the form;

$$u(x,t)=4\tan^{-1}(\alpha e^{\frac{x-Vt}{\sqrt{1-V^2}}}) \tag{38}$$

this is called soliton((kink)) solution of sine Gordon equation and represents a continuous profile with $u\rightarrow 0,\quad as\ x\rightarrow -\infty,$ and $u\rightarrow 2\pi,$ as $x\rightarrow +\infty$ shown in the following figure.




the solution properties in the positive $x-$direction with velocity $V$.