Lecture7: **Travelling wave solution**

One of the cornerstones in the study of both linear and nonlinear PDEs is the wave propagation. A wave is a recognizable signal which is transferred from one part of the medium to another part with a recognizable speed of propagation. Energy is often transferred as the wave propagates, but matter may not be. We mention here a few areas where wave propagation is of fundamental importance.

* Fluid mechanics (water waves, aerodynamics)
* Acoustics (sound waves in air and liquids)
* Elasticity (stress waves, earthquakes)
* Electromagnetic theory (optics, electromagnetic waves)
* Biology (epizootic waves)
* Chemistry (combustion and detonation waves)

The simplest form of a mathematical wave is a function of the form

$$u(x,t) = f(\eta), \quad \eta = x - \mu t \tag{19}$$

where $\mu > 0$ is constant(speed of travelling waves). This function is transform the PDE into ODE with one independent variable like the similarity transform.

Example: Find travelling wave solutions to the wave equation

$$u_{tt} = a^2 u_{xx}, \quad x \in R, t > 0\,, \tag{20}$$

with boundary conditions $u(-\infty,t) = const., \quad u(+\infty,t) = const.$

Inserting $u(x,t) = f(x - \mu t),$ into the wave equation, we find

$$u_{tt} - a^2 u_{xx} = \mu^2 f'' - a^2 f'' = (\mu^2 - a^2) f'' = 0 \tag{21}$$

so that either $f(\eta) = A + B\eta$ for some constants A, B or $\mu = \pm a$ and f arbitrary. In the first case we would have

$$u(x,t) = A + B(x \pm at)$$

but the above boundary conditions cannot be satisfied unless $B = 0$. Thus, the only travelling wave solution in this case is constant. For the other case, we see that clearly for any twice differentiable function f such that $\underset{\eta \to \pm\infty}{Lim} = d_{\pm\infty}$ the solution

$$u(x,t) = f(x \pm at)$$

is a travelling wave solution (a pulse if $d_{+\infty} = d_{-\infty}$). In general, it follows that any solution to the wave equation can be obtained as a superposition of two travelling waves: one to the right and one to the left

$$u(x,t) = f(x-a) + g(x+at). \tag{22}$$

Not all equations admit travelling wave solutions, as demonstrated below.

Example: Consider the diffusion equation

$$u_t = Du_{xx}, \quad x \in R, t > 0 \tag{23}$$

Substituting the travelling wave formula $u(x,t) = f(x-\mu t),$ we obtain

$$-\mu f' - Df'' = 0$$

that has the general solution $f(\eta) = a + be^{-\frac{\mu\eta}{D}}$.

It is clear that for f to be constant at both plus and minus infinity it is necessary that $b = 0$. Thus, there are no nonconstant travelling wave solutions to the diffusion equation.

We have already seen that the non-viscid Burger's equation

$$u_t + uu_x = 0, \quad x \in R, t > 0 \tag{24}$$

does not admit a travelling wave solution: any profile will either smooth out or form a shock wave (which can be considered as a generalized travelling wave - it is not continuous!). However, some amount of dissipation represented by a diffusion term allows to avoid shocks.

Example: Find travelling wave solutions of the Burger's equation with viscosity

$$u_t + uu_x = \nu u_{xx}, \quad x \in R, t > 0 \tag{25}$$

for $u \to u_1, as\ x \to -\infty, \quad u \to u_2, as\ x \to +\infty\ and\ u_1 > u_2$. The term uu_x will have a shocking up effect that will cause waves to break and the term νu_{xx} is a diffusion like term.

substituting $u(x,t) = f(\eta), \quad \eta = x - \mu t$ to Burgers' equation, we have

$$-\mu f' + ff' - \nu f'' = 0$$

we re-write this equation as

$$-\mu f' + \frac{1}{2}(f^2)' - \nu f'' = 0$$

hence we can integrate getting

$-\mu f + \frac{1}{2} f^2 - vf' = B$, where B is a constant of integration. Hence

$$\frac{df}{d\eta} = \frac{1}{2v}(f^2 - 2\mu f - 2B) \qquad (26)$$

Ex1: find the solution of this ODE(26)? **((H.W))**

Ex2: Find travelling wave solutions of the KdV equation

$u_t + uu_x + u_{xxx} = 0, \quad x \in R, t > 0$, with $\lim_{x\to\pm\infty} u = 0, \lim_{x\to\pm\infty} u' = 0$

$\lim_{x\to\pm\infty} u'' = 0$ **((H.W))**

Transformations: change of variables may linearize the systems of non-linear PDEs or reduce the non-linear PDE to a non-linear ODE. These change can be classified into three group: transformation of only the dependent variable, transformation of only the independent variable and transformation of both dependent and independent variables(mixed).