Lecture 6:

Self-Similar Scaling Solutions of DEqs.

ODE or PDEs

linear equation — nonlinear equactions

sparation of variables — similarity method

Laplace transformstion — tranlling waive solu

forrier transformtion

⋮

For nonlinear problems, we will show that if the PDE can be scaled in such a way as to exactly peproduce it original form (as suggested by the term 'self-similar'), then it will prossess a class of solutions that share this property. A consequence of the property of self-similarity is the these solutions can be obtained from a reduced version of the original problem. For a PDE with two independent variable, this leads to an ODE problem for the similarity solution. Despite their nature as exact solutions to a give problem under special conditions, they often provide important understanding of the broader behaviour of all solutious of the system.

The process of constructing a similarity solution to a given problem consists of three stages:

(i) Looking for a scaling "symmetry" of the problem to see if similarity solutions are possible.

(ii) Determining the forms of the similarity variables and functions from the scale-invariant Π groups for the problem.

(iii) Solving a reduced problem for the similarity function and then transforming back to give the solution of original problem.

Finding scaling - Invariant Symmetries

For $u = u(x,t)$, will admit a similarity solution by making use of the change of variables

$$u(x,t) = U\,\hat{u}(\hat{x},\hat{t}), \quad x = L\hat{x}, \quad t = T\hat{t} \quad \text{——} (1)$$

where U, L, T are un-determined real positive parameters. We call the problem scale invaraint if relationships exist between the scaling parameters U, L, T in (1) that make the scaled problem take exactly the same as the original problem with at lest one scaling prameter remining undetemined.

problem for $u(x,t)$ = problem for $\tilde{u}(\tilde{x},\tilde{t})$ --- (2)

For equation equations, it expected that the time scale T is the Free prameter and the other scales can be expressed and $L=L(T)$.

Example: $u_t + u u_x = 0 \qquad 0 \le x < \infty$ --- (3)

$u(0,t)=0 \quad , \quad \int_0^\infty u(x,t)\,dx = 4$ --- (4)

Sol By using (1), we have

$$\tilde{u}_{\tilde{t}} + \left(\frac{UT}{L}\right)\tilde{u}\tilde{u}_{\tilde{x}} = 0 \qquad 0 \le \tilde{x} < \infty \text{ --- (5)}$$

$$\tilde{u}(0,\tilde{t}) = 0 \quad , \quad (UL)\int_0^\infty \tilde{u}(\tilde{x},\tilde{t})\,d\tilde{x} = 4 \text{ --- (6)}$$

setting $\frac{UT}{L}=1$ and $UL=1$

element the scaling and make the scaled system identical with (5) and (6) and hence the problem scale invariant.

If $b=0 \Rightarrow \Pi_2 = UT^c \Rightarrow [\Pi_6] = UT^c = T^0$

$\rightarrow [\Pi_2] = T^{-\frac{1}{2}+c} \longrightarrow \boxed{c = \frac{1}{2}}$

If $f = \Pi_2 \Longrightarrow f = t^{\frac{-1}{2}} u$

From above $F(\Pi_1, \Pi_2) = 0$

Applying the implicit function theorem gives

$f = f(\eta) \longrightarrow t^{\frac{1}{2}} u = f(x, t^{\frac{-1}{2}})$

$\longrightarrow u(x,t) = t^{\frac{-1}{2}} f(x, t^{\frac{-1}{2}})$ ——— ③

If we select $c=0 \longrightarrow$??
$u(x,t) = ?$ H.W

Example/ $u_t = u_{xx}$ use the scaling (1) to find η and solve it with

① $\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = \overset{\neq 0}{M}$

② $u(0,t) = 1 \quad u \to 0$ as $x \to \infty$.

$\rightarrow L = T^{\frac{1}{2}} \quad , V = T^{-\frac{1}{2}}$

$\rightarrow u(x,t) = T^{\frac{1}{2}} \tilde{u}(T^{-\frac{1}{2}}x, T^{-1}t)$

for arbitary of T. This transformation is called scaling symmetry of (2).

we define a scale-invariant prameter π as a maximal product powers of the variables of the system

$$\pi = u^a x^b t^c \quad \text{—— (8a)}$$

let $[\pi]$ represent the dependence of π on the scaling

$$[\pi] = U^a L^b T^c \quad \text{—— (8b)}$$

In particular for (7), this yeilds

$$[\pi] = T^{-\frac{a}{2}} T^{\frac{b}{2}} T^c = T^0$$

If $a=0$, $b=1$ $\xrightarrow{\text{from (8)}}$ $\pi = x t^c \Rightarrow [\pi] = L T^c T^0$

$[\pi_1] = T^{\frac{1}{2}+c} \rightarrow \boxed{c = -\frac{1}{2}}$

If $\eta = \pi_1 \Longrightarrow \eta = x t^{-\frac{1}{2}}$

If $b=0 \Rightarrow \Pi_2 = UT^c \Rightarrow [\Pi_6] = UT^c = T^0$

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If we select $c=0 \longrightarrow$??
$u(x,t) = ?$ H.W

Example/ $u_t = u_{xx}$ use the scaling (1) to find η and solve it with

(1) $\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} u(x,0)\, dx = M$

(2) $u(0,t) = 1 \quad u \to 0$ as $x \to \infty$.