Non-linear model of PDEs

The most general first-order non-linear has the form:

$$F(x,y,u,u_x,u_y)=0 \text{ ,} \tag{1}$$

and the most general second- order non-linear has the form:

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=0 \text{ ,} \tag{2}$$

More formally, it is possible to write these equations in the operator form

$$L_x(u(x))=f(x) \tag{3}$$

where L_x is a partial differential operator and $f(x)$ is a given function of two or

more independent variables $x=(x_1,x_2,...)$. Moreover, we indicated that if L_x is not

a linear operator, equ.(3) is called a non-linear PDE and it is called inhomogeneous non-linear PDE if $f(x)\neq 0$,and equ.(3) is called a homogenous nonlinear PDE if $f(x)=0$. Almost there is no general method to solve nonlinear PDEs((finding the analytic (exact) solutions of non-linear PDEs)).

Methods of solutions for nonlinear equations:

Represent only one aspect of the theory of nonlinear PDE like linear equations of existence, uniqueness, and stability of solutions of nonlinear PDEs are fundamental importance. These and other aspects of nonlinear equations have led to the subject into one of the most devise and active areas of modern mathematics. Nonlinear PDEs arise frequently in formulation fundamental laws of nature and in the mathematical analysis of a wide variety of physical problems. For example, the propagation of nonlinear wave((or disturbance wave)), flood waves, Shock wave, waves in traffic flow on high ways can be express by the simplest first-order nonlinear PDE

$$u_t + C(u)u_x, \quad x\in R, t>0 \tag{4}$$

where $C(u)$ is a given function of u. Another example, Burgers' in 1945 introduce the nonlinear PDE as;

$$u_t + uu_x = \nu u_{xx}, \quad x\in R, t>0 \tag{5}$$

where ν is the kinematics viscosity. This equation can be describe the diffusive waves in fluid dynamics ,or sound wave in a viscous medium. Also, Korteweg-de Veries (KdV) nonlinear PDE

$$u_t - \alpha u u_x + \beta u_{xxx} = 0, \quad x \in R, t > 0 \tag{6}$$

where α and β are constants. In 1895 it is introduced to describe the propagation of unidirectional shallow water waves, pressure waves in a liquid -gas bubbles. In the end , we would like to mention that the propagation of a heat pulse in a solid, evolution of water waves, and nonlinear instability problems are description by the Schrödinger's' equation

$$iu_t - u_{xx} + \gamma |u|^2 u = 0, \quad x \in R, t > 0 \tag{7}$$

where $i = \sqrt{-1}$ *and* γ are constants. There are many kinds of nonlinear model of PDEs to describe the physical phenomena's mathematically.

Similarity method: Birkhoof (1950) first recognized that the Boltzmann's method of solving the diffusion equation is based on the algebraic symmetry of the equation, and special solutions of this equation can be obtain by solving a related ODE. such solutions are called similarity solutions become they are geometrically similar.

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*** Similarity transformation**: $u(x,t) = t^p f(\eta), \eta = xt^q$

we consider the classical linear diffusion equation with the constant diffusion coefficient k in the following

$$u_t = k u_{xx}, \quad 0 \le x < \infty, t > 0 \tag{8}$$

subject to the following boundary and initial conditions

$$u(0,t) = 1, \ , u(x,t) \to 0, \, as \; x \to \infty, t > 0, \tag{9}$$

$$u(x,0) = 0, \, 0 \le x < \infty \tag{10}$$

we introduce a one-parameter set of stretching transformations in the $(x,t,u)-$space defined by

$$\tilde{x} = a^{\alpha} x, \quad \tilde{t} = a^{\beta} t, \quad \tilde{u} = a^{\gamma} u, \tag{11}$$

under which equation (8) is invariant, where a is real parameter which belong an open interval I containing $a = 1, \alpha, \beta$ *and* γ as a fixed real constants, usually the set of transformations in the $(x,t,u)-$space is denoted by T_a and we write the set of explicitly as $T_a : R^3 \to R^3$ for each $a \in I$, the set of all such transformation $\{T_a\}$ from a Lie group on R^3 with an identity element T, this can be see clearly as follows;

- $\{T_a\}$ composition(multiplication)

- $\{T_a\}$ Associative law

- $\{T_a\}$ Identity

- $\{T_a\}$ Inverse

Clarification: differentiation the transformations (11) and substituting in heat equation(8), we have

$$\tilde{u}_{\tilde{t}} - k\tilde{u}_{\tilde{x}\tilde{x}} = 0, \quad \text{(provided } \beta = 2\alpha)$$

hence equ.(8) is invariant under the set of transformations

$$\tilde{x} = a^{\alpha}x, \quad \tilde{t} = a^{2\alpha}t, \quad \tilde{u} = a^{\gamma}u$$

for any choice of α *and* γ. The quantities $f(\eta) = t^{p}u, \eta = xt^{-q}$ are invariant under T_a provided $p = -(\gamma/\beta)$ *and* $q = \alpha/\beta$,thus the invariants of transformations are given by

$$f(\eta) = ut^{-\gamma/\beta} = ut^{-\gamma/2\alpha}, \eta = xt^{-\alpha/\beta} = xt^{-1/2} \tag{12}$$

substituting equ.(12) into the original equations(8-10) gives an ordinary differential equation of the form

$$kf''(\eta) + \frac{1}{2}\eta f'(\eta) - \frac{\gamma}{2\alpha}f(\eta) = 0 \tag{13}$$

the transformation data are then given by

$$u(0,t) = t^{\gamma/2\alpha} f(0) = 1 \quad \text{and} \quad f(\infty) = 0 \tag{14}$$

To make the first condition (in equ.(14)) independent of t , we require that $\gamma = 0$ consequently, equs.(13-14) becomes;

$$f''(\eta) + \frac{1}{2k}\eta f'(\eta) = 0 \tag{15}$$

$$f(0) = 1 \quad \text{and} \quad f(\infty) = 0 \tag{16}$$

thus equ.(8) admits the set of transformation(12), which reduces the PDE(8) to the ODE(15). Result (12) is called similarity transformation, and the new independent variable η is called a similarity variable.

Integrating (15) yields the general solution

How to obtain on it? H.W

$$f(\eta) = A + B\int_0^{\eta} e^{-\xi^2/4k} d\xi \qquad (17)$$

where A and B are integrating constants to be determined by using (16). It turn out that $A=1$ and $B=\frac{-1}{\sqrt{\pi k}}$. Thus the solution

$$f(\eta) = 1 - \frac{1}{\sqrt{\pi k}}\int_0^{\eta} e^{-\xi^2/4k} d\xi = 1 - \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-\alpha^2} d\alpha$$
$$= 1 - erf(x/\sqrt{4kt}) = erfc(x/\sqrt{4kt}) \qquad (18)$$

where erf and $erfc$ are the standard error and complmentry error functions respectively. the solution (18) is identical with known solution , which obtained by Laplace transformation method(prove that).

Exs: ((H.W))

1- using similarity transformation $u = t^{-\frac{1}{3}}\omega(xt^{-\frac{1}{3}})$ to solve $u_t = (uu_x)_x,\ x \in R, t > 0$**,** subject to

$u_x(0,t) = 0,\ ,u(\pm\infty,t) = 0,\ t > 0\ ,u(x,0) = f(x),\ x \in R$ and $\int_{-\infty}^{\infty} u(x,t)dx = 1,\ for\ all\ t > 0.$

2- Use Boltzman transformation $\eta = -\frac{1}{2}xt^{-\frac{1}{2}}$ to reduce the nonlinear diffusion equation

$u_t = \frac{\partial}{\partial x}(k(u)\frac{\partial u}{\partial x}),\quad 0 \le x < 1, t > 0$ with boundary and initial conditions
$u(0,t) = u_0, t > 0,\quad u(x,0) = 4, 0 < x < \infty$

to an ODE with independent variable η , then find the solution of ODE.