Lecture 3

Canonical form and general solution

The homogeneous wave equation in one (spatial) dimension has the form

$$u_{tt} - c^2 u_{xx} = 0 \quad -\infty \le x \le b \le \infty, t > 0$$

where $c \in \mathrm{R}$ is called the *wave speed*, a terminology that will be justified in the discussion below. Using the chain rule for the function $u(x, t)=w(\xi\ (x, t), \eta(x, t))$ and then we obtained the canonical form as

$$-4c^2 w_{\eta\xi} = 0$$

Hence, the general solution is $w(\xi,\eta) = F(\xi) + G(\eta)$,where the $F, G \in C^2(R)$ thus; in the original variables the solution becomes;

$$u(x,t) = F(x+ct) + G(x-ct) \tag{16}$$

In other words, if u is a solution of the one-dimensional wave equation, then there exist two real functions $F, G \in C^2(R)$ such that (16) holds. Conversely, any two functions $F, G \in C^2(R)$ define a solution of the wave equation via formula (16). For a fixed $t_0 > 0$, the graph of the function $G(x-ct_0)$ has the same shape as the graph of the function $G(x)$, except that it is shifted to the right by a distance ct_0. Therefore, the function $G(x-ct)$ represents a wave moving to the right with velocity c, and it is called a *forward wave*. The function $F(x + ct)$ is a wave traveling to the left with the same speed, and it is called a *backward wave*. Indeed c can be called the *wave speed*.

Ex: Show that the Cauchy problem and d'Alembert's formula

$$u_{tt} - c^2 u_{xx} = 0 \quad -\infty \le x \le \infty, t > 0$$

$$u(x,0) = f(x), u_t(x,0) = g(x) \quad -\infty \le x \le \infty$$

Has the general solution $u(x,t) = \dfrac{f(x+ct) + g(x+ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)ds$

Ex: If c=3 and $f(x) = \begin{cases} 1 & |x| \le 2 \\ 0 & |x| > 2 \end{cases}$, $g(x) = \begin{cases} 1 & |x| \le 2 \\ 0 & |x| > 2 \end{cases}$ then;

a) Find $u(0, \frac{1}{6})$

b) Discuss the large time behavior of the solution.

c) Find the maximal value of $u(x, t)$, and the points where this maximum is achieved.

d) Find all the points where $u \in C^2$.

Boundary value problem:

A problem that cosist of finding a solution of partial differential equation,which also satisfies one or more boundary conditions is called Boundary value problem(BVP).

The theory of P.D.Es gives results on the existence of solutions of boundary value problems. But such results are necessarily limited and complicated by the great varity of features types of equations and condition and type of domains. In physical sciences there are many type of heat equation, wave equation and Laplace((or potential)) equation in one or more dimensions with several coordinates for examples:

Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad \to (x, y)$

Cylindrical coordinates: $(x, y, z) \to (r, \phi, z) \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0$, where

$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$

Spherical coordinates:

$(x, y, z) \to (r, \phi, \theta) \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial^2 u}{\partial r^2}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial u}{\partial \theta}) = 0$, where

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\sin\theta$

Cylindrical symmetry: $\frac{1}{r^2}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = 0$, Spherical symmetry : $\frac{\partial}{\partial r}(r^2\frac{\partial^2 u}{\partial r^2}) = 0$,

Type of boundary conditions:

- Dirichlet conditions: where u is specified at each point of a boundary of a region
- Neumann conditions: where the values of the normal derivative of the function $\frac{\partial u}{\partial n}$ are prescribed on the boundary
- Cauchy condition:

Theorem: *Fix T > 0. The above Cauchy problem in the domain* $-\infty \le x \le \infty, 0 \le t \le T$ *is well-posed for* $f \in C^2(R)$, $g \in C^1(R)$.

Proof: The existence and uniqueness follow directly from the d'Alembert formula. Indeed, this formula provides us with a solution, and we have shown that any solution of the Cauchy problem is necessarily equal to the d'Alembert solution.

Note that from our smoothness assumption ($f \in C^2(R)$, $g \in C^1(R)$), it follows that

$u \in C^2(R\times(0,\infty))\bigcap C^1(R\times(0,\infty))$, and therefore, the d'Alembert solution is a classical solution. On the other hand, for $f \in C(R)$ and g that is locally integrable, the d'Alembert solution is a generalized solution.

It remains to prove the stability of the Cauchy problem, i.e. we need to show that small changes in the initial conditions give rise to a small change in the solution.

Let u_i be two solutions of the Cauchy problem with initial conditions f_i, g_i, where $i = 1,2.$ Now, if

$$|f_1(x) - f_2(x)| < \delta \quad |g_1(x) - g_2(x)| < \delta \quad \text{for all} \quad x \in R,$$

then for all $x \in R$ and $0 \le t \le T$ we have

$$|u_1(x) - u_2(x)| \le \frac{|f_1(x+ct) - f_2(x+ct)|}{2} + \frac{|f_1(x-ct) - f_2(x-ct)|}{2}$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} |g_1(s) - g_2(s)| ds < \frac{1}{2}(\delta+\delta) + \frac{1}{2c} 2ct\delta \le (1+T)\delta$$

Therefore, for a given $\varepsilon > 0$, we take $\delta < \dfrac{\varepsilon}{(1+T)}$. Then for all $x \in R$ and $0 \le t \le T$ we have $|u_1(x,t) - u_2(x,t)| < \varepsilon$,

The method of separation of variables

Fourier's method for solving linear PDEs is based on the technique of *separation of variables*. Let us outline the main steps of this technique. First we search for solutions of the homogeneous PDE that are called *product solutions* (or *separatedsolutions*). These solutions have the special form

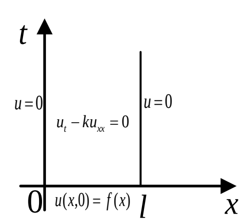
$$u(x,t) = \mathrm{X}(x)\cdot \mathrm{T}(t)$$

and in general they should satisfy certain additional conditions. In many cases,these additional conditions are just homogeneous boundary conditions. It turns out that *X* and *T* should be solutions of linear ODEs that are easily derived from the givenPDE.

Note: Obviously, we are not interested in the zero solution $u(x,t) = 0$. Therefore, we seek functions *X* and *T* that do not vanish identically.

Example: Consider the following heat conduction problem in a finite interval:

$$u_t - ku_{xx} = 0 \quad 0 \le x \le l, t > 0$$
$$u(0,t) = u(l,t) = 0 \quad t \ge 0$$
$$u(x,0) = f(x), \quad 0 \le x \le l$$



$$XT_t = kTX_{xx} \Rightarrow \frac{T_t}{kT} = \frac{X_{xx}}{X} \Rightarrow \frac{T_t}{kT} = \frac{X_{xx}}{X} = -\lambda \quad , \lambda \textit{ is separationconstat}$$

Since u is not the trivial solution $u = 0$, it follows that $X(0) = X(l) = 0$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda X,\ 0 < x < l \qquad with \quad X(0) = X(l) = 0$$ " called an *eigenvalue problem*"

$$\frac{dT}{dt} = -\lambda kT,\ t > 0$$

A nontrivial solution of this system is called an *eigenfunction* of the problem with an *eigenvalue* λ. its general solution of an *eigenvalue problem* (which depends on λ) has the following form:

1- If $\lambda < 0$ then $X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$
2- If $\lambda = 0$ then $X(x) = \alpha + \beta x$
3- If $\lambda > 0$ then $X(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$

where α, β are arbitrary real numbers. The corresponding eigenfunctions are

$$X_n(x) = \sin(\frac{n\pi x}{l}) \quad , \lambda = \left(\frac{n\pi}{l}\right)^2, n = 1,2,3,...$$

The general solution of second ODE has the form $T_n(t) = B_n e^{-k\left(\frac{n\pi}{l}\right)^2 t}, n = 1,2,3,...$
The superposition principle implies that any linear combination

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{l}) e^{-k\left(\frac{n\pi}{l}\right)^2 t} \quad ,$$

of separated solutions is also a solution of the heat equation that satisfies the Dirichlet boundary conditions.

From the initial condition we have $f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{l})$,this is Fourier sine series; Thus ;its coefficient has the form $B_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx.$

Ex: in the above example if $k = 1, l = \pi, f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$,then find $u(x,t)$.

Ex: solve the following problems

1- $u_{tt} - c^2u_{xx} = 0 \quad 0 \le x \le l, t > 0$
$u(x,0) = f(x), u_t(x,0) = g(x) \quad 0 \le x \le l$
$u_x(0,t) = u_x(l,t) = 0, \quad t \ge 0$

2- $r^2u_{rr} + ru_r + u_{\theta\theta} = 0 \quad 0 \le r \le b, 0 \le \theta \le \pi$
$u(r,0) = 0, u(r,\pi) = 0 \quad 0 \le r \le b$
$u(1,\theta) = 0, u(b,\theta) = u_0 \quad 0 \le \theta \le \pi$

3-consider the displacement in a stretched string upon which an external force per unit length acts, parallel to the y axis. Let that the force be proportional to the distance from one end, and let the initial displacement and velocity be zero. Units for x and t can be chosen.

4- let $u(x,y,t)$ denote the transverse displacement at each point (x,y) at time t in a membrane stretched across a rigid square frame in the $xy-$plane. We can select the origin and the point (π,π) as the ends of the diagonal of the frame, and let that the membrane is released at rest with a given initial displacement $f(x,y)$ which is continuous and vanishes on the boundary of the square.

Laplace transformation: Laplace transformation for the function $u(x,t)$ with respect to t is $\{u(x,t)\}=\int_0^\infty u(x,t)e^{-st}dt=U(x,s), \quad t>0$

By using this definition ,we can found the Laplace transforms for $u(x,t)$ derivatives as;

$$\{u_t(x,t)\}=\int_0^\infty u_t(x,t)e^{-st}dt=sU(x,s)-u(x,0,$$

$$\{u_{tt}(x,t)\}=\int_0^\infty u_{tt}(x,t)e^{-st}dt=s^2U(x,s)-su(x,0)-u_t(x,0),$$

$$\vdots$$

Note: other derivatives in the PDE is transforms into ordinary differentional(e.g

$$\{u_x(x,t)\}=\int_0^\infty u_x(x,t)e^{-st}dt=\frac{dU(x,s)}{dx}, \quad \{u_{xx}(x,t)\}=\int_0^\infty u_{xx}(x,t)e^{-st}dt=\frac{d^2U(x,s)}{dx^2},\cdots$$

Example : Solve the following problem

$$u_t-ku_{xx}=0 \quad 0\le x\le l, t>0$$

$$u(0,t)=u_0 \quad t\ge 0$$

$$u(x,0)=0, \quad 0\le x\le l \quad \text{and} \quad u(x,t) \text{ bounded,}$$

Take Laplace transformation for both sides of PDE with respect to t ,we have $\dfrac{d^2U(x,s)}{dx^2}=\dfrac{1}{k}\{sU(x,s)-u(x,0)\}\Rightarrow \dfrac{d^2U}{dx^2}-\dfrac{s}{k}U=0\Rightarrow U(x,s)=\alpha(s)e^{x\sqrt{\frac{s}{k}}}+\beta(s)e^{-x\sqrt{\frac{s}{k}}}$

From $u(x,t)$ must be bounded $\Rightarrow \alpha(s)=0$,thus; $U(x,s)=\beta(s)e^{-x\sqrt{\frac{s}{k}}}$

Furthermore, since $u(0,t)=u_0$,we have $U(0,s)=\int_0^\infty u_0e^{-st}dt=\dfrac{u_0}{s}$. So, $U(0,s)=\beta(s)$

Solution of ODE is $U(x,s)=\dfrac{u_0}{s}e^{-x\sqrt{\frac{s}{k}}}$, by taking inverse Laplace transformation we have, $u(x,t)=u_0\left\{\dfrac{e^{-x\sqrt{\frac{s}{k}}}}{s}\right\}=u_0\, erfc(\dfrac{x}{2\sqrt{kt}})=u_0(1-\int_0^{x/2\sqrt{kt}}e^{-\lambda^2}d\lambda)$ ($erfc=1-erf$)

Ex: Solve the following problem

$$u_t + u_x = x \quad (x,t>0)$$
$$u(x,0)=0, x>0$$
$$u(0,t)=0, t>0$$

Ex: If $u_{tt} = u_{xx} + u_{xxt} \quad (x,t>0)$,and the following data $u_t(x,0)=0$, $u(0,t)=1$, and $u(x,t)\to 0$ as $x\to\infty$ are satisfying, then $u_x(0,t) = -\frac{1}{\sqrt{\pi t}}e^{-t}$.

Fourier transformation: By using this definition of the complex form of the Fourier series, we can obtain Fourier transformation. Therefore, The Fourier transformation for the function $u(x,t)$ with respect to x is given by

$$F\{u(x,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-ipx}dx = U(p,t)$$

And the inverse has the form $$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(p,t)e^{-ipx}dP$$

By using this definition ,we can found the Fourier sine transforms and Fourier cosine transforms as;

$$F\{u(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,t)\sin(px)dx = U_s(p,t)$$

And the inverse has the form $$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} U_s(p,t)\sin(px)dP$$

$$F\{u(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,t)\cos(px)dx = U_c(p,t)$$

And the inverse has the form $$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} U_c(p,t)\cos(px)dP$$

Also ,we can found the Fourier sine and cosine transforms for the derivatives as;

$$F\{u_x(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_x(x,t)\sin(px)dx = -pU_c(p,t)$$
$$F\{u_x(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_x(x,t)\cos(px)dx = -\sqrt{\frac{2}{\pi}}u(0,t) + pU_s(p,t)$$

$$F\{u_{xx}(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_{xx}(x,t)\sin(px)dx = p\sqrt{\frac{2}{\pi}}u(0,t) - p^2U_s(p,t)$$
$$F\{u_{xx}(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_{xx}(x,t)\cos(px)dx = -\sqrt{\frac{2}{\pi}}u_x(0,t) - p^2U_c(p,t)$$

Note: other derivatives in the PDE is transforms into ordinary differential (e.g

$$F\{u_t(x,t)\}=\frac{dU_s(p,t)}{dt},\quad F\{u_{tt}(x,t)\}=\frac{d^2U_s(p,t)}{dt^2},\cdots$$

$$F\{u_t(x,t)\}=\frac{dU_c(p,t)}{dt},\quad F\{u_{tt}(x,t)\}=\frac{d^2U_c(p,t)}{dt^2},\cdots$$

Ex: solve the following problem

$$u_t-ku_{xx}=0\quad 0\le x\le l,t>0$$

$$u(x,0)=0\quad x\ge 0$$

$$u(0,t)=a,\quad t>0$$

$$u\ \&\ u_x\to 0,\quad x\to\infty,$$

Non-linear model of P.D.Es: