## Lecture 3

We shall show in the next the hyperbolic (resp., parabolic,elliptic) in a domain *D*, is one can find a coordinate system in which the equation has a simpler form that we call the ***canonical form*** of the equation. Moreover, in such a case the principal part of the canonical form is equal to the principal part of the fundamental equation of mathematical physics of the same type. This is one of the reasons for studying these fundamental equations.

**Definition:** The transformation $(\xi, \eta)=(\xi(x, y), \eta(x, y))$ is called a ***change of coordinates*** (or a nonsingular transformation) if the Jacobian $J := \xi x\eta y - \xi y\eta x$ of the transformation does not vanish at any point $(x, y)$.

**Lemma :** *The type of a linear second-order PDE in two variables is invariant under a change of coordinates. In other words, the type of the equation is an intrinsic property of the equation and is independent of the particular coordinate system used.*

*Proof:*

$$L[u] = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \qquad (10)$$

and let $(\xi, \eta)=(\xi(x, y), \eta(x, y))$ be a nonsingular transformation.

Write $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$. We claim that $w$ is a solution of a second-order equation of the same type. Using the chain rule one finds that

$$u_x = w_\xi \xi_x + w_\eta \eta_x,$$
$$u_y = w_\xi \xi_y + w_\eta \eta_y,$$
$$u_{xx} = w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx},$$
$$u_{xy} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy},$$
$$u_{yy} = w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy},$$

Substituting these formulas in (10), we see that $w$ is satisfies the linear equation

$$\ell[w] = Aw_{\xi\xi} + Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G \qquad (11)$$

Where

$$A(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2,$$
$$B(\xi,\eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$$
$$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2,$$

Note: we do not need to compute the coefficients of the lower-order derivatives (D,E,F)

An elementary calculation shows that these coefficients satisfy the following matrix equation:

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix},$$

Denote by $J$ the Jacobian of the transformation. Taking the determinant of the two sides of the above matrix equation, we find

$$-\delta(\ell) = AC - B^2 = J^2(ac - b^2) = -J^2\delta(L)$$

Therefore, the type of the equation is invariant under nonsingular transformations.

**Definition 3.4** The *canonical form of hyperbolic, parabolic and elliptic equations are*

$$\ell[w] = w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$$
$$\ell[w] = w_{\xi\xi} + \ell_1[w] = G(\xi,\eta)$$
$$\ell[w] = w_{\xi\xi} + w_{\eta\eta} + \ell_1[w] = G(\xi,\eta)$$

where $\ell_1$ is a first-order linear differential operator, and $G$ is a function.

**<u>Theorem</u>**: *Suppose that (10) is hyperbolic in a domain D. There exists a coordinate system* $(\xi, \eta)$ *in which the equation has the canonical form*

$$w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$$

*where* $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, $\ell_1$ is a first-order linear differential operator, and $G$ is a function *which depends on (10)*.

**Proof:**

Without loss of generality, we may assume that $a(x, y) \neq 0$ for all $(x, y) \in D$. We need to find two functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that

$$A(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$$
$$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$$

The equation that was obtained for the function $\eta$ is actually the same equation as for $\xi$; therefore, we need to solve only one equation. It is a first-order equation that is not quasilinear; but as a quadratic form in $\xi$ it is possible to write it as a product of two linear terms

$$\frac{1}{a}[a\xi_x + (b - \sqrt{b^2 - ac})\xi_y][a\xi_x + (b + \sqrt{b^2 - ac})\xi_y] = 0$$

Therefore, we need to solve the following linear equations:

$$a\xi_x + (b - \sqrt{b^2 - ac})\xi_y = 0 \qquad (12\#)$$

$$a\xi_x + (b + \sqrt{b^2 - ac})\xi_y = 0 \qquad (13\#\#)$$

In order to obtain a nonsingular transformation ($\xi(x, y), \eta(x, y)$) we choose $\xi$ to be a solution of (12#) and $\eta$ to be a solution of (13##).The characteristic equations for (#) are

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b + \sqrt{b^2 - ac}, \quad \frac{d\xi}{dt} = 0$$

Therefore, $\xi$ is constant on each characteristic. The characteristics are solutions of the equation

$$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a} \qquad (14\$)$$

The function $\eta$ is constant on the characteristic determined by

$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a} \qquad (15\$\$)$$

**Definition :** The solutions of ($) and ($$) are called the two families of the *characteristics* (or *characteristic projections*) of the equation $L[u] = g$.

**Example**: Consider the *Tricomi equation:*

$$u_{xx} + xu_{yy} = 0 \quad x < 0$$

Find a mapping $q=q(x, y)$, $r=r(x, y)$ that transforms the equation into its canonical form, and present the equation in this coordinate system.

$$\frac{dy}{dx} = \pm\sqrt{x} \Rightarrow q(x,y) = \frac{3}{2}y + (-x)^{\frac{3}{2}} \text{ and } r(x,y) = \frac{3}{2}y - (-x)^{\frac{3}{2}}$$

Define $v(q, r) = u(x, y)$. By the chain rule

$$u_x = -\frac{3}{2}(-x)^{\frac{1}{2}}v_q + \frac{3}{2}(-x)^{\frac{1}{2}}v_r \qquad u_y = \frac{3}{2}(v_q + v_r)$$

$$u_{xx} = -\frac{9}{4}xv_{qq} - \frac{9}{4}xv_{rr} + \frac{9}{2}xv_{qr} + \frac{3}{4}(-x)^{\frac{-1}{2}}(v_q - v_r)$$

$$u_{yy} = -\frac{9}{4}(v_{qq} + 2v_{qr} + v_{rr})$$

Substituting these expressions into the Tricomi equation we obtain

$$u_{xx} + xu_{yy} = -9(q-r)^{\frac{2}{3}}(v_{qr} + \frac{v_q - v_r}{6(q-r)}) = 0$$

**Ex:** Consider the equation

$$u_{xx} - 2\sin(x)u_{xy} - \cos(2x)u_{yy} - \cos(x)u_y = 0$$

Find a coordinate system $s = s(x, y)$, $t = t(x, y)$ that transforms the equation into its canonical form. Show that in this coordinate system the equation has the form $v_{st} = 0$, and find the general solution.

**<u>Theorem</u>**: *Suppose that (10) is parabolic in a domain D. There exists a coordinate system* ($\xi$, $\eta$) *in which the equation has the canonical form* $w_{\xi\xi} + \ell_1[w] = G(\xi,\eta)$ *where* $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, $\ell_1$ is a first-order linear differential operator, and $G$ is a function *which depends on (10)*.

**Proof: Use the Same procedure that use in previous theorem.**

**Ex :** Prove that the equation $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ is parabolic and find its canonical form; find the general solution on the half-plane $x > 0$.

**<u>Theorem</u>**: *Suppose that (10) is elliptic in a domain D. There exists a coordinate system* ($\xi$, $\eta$) *in which the equation has the canonical form* $w_{\xi\xi} + w_{\eta\eta} + \ell_1[w] = G(\xi,\eta)$ *where* $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, $\ell_1$ is a first-order linear differential operator, and $G$ is a function *which depends on (10)*.

**Proof: Use the Same procedure that use in previous theorem.**

**Ex:** Consider the *Tricomi equation:* $u_{xx} + xu_{yy} = 0 \quad x > 0$

Find a mapping $q=q(x, y)$, $r=r(x, y)$ that transforms the equation into its canonical form, and present the equation in this coordinate system.