Lecture 2(Review)

High order linear P.D.Es with constant and/or variable coefficients:

Consider the P.D.E

$$f(D,\overline{D})z = F(x,y) \tag{7}$$

Where $f(D,\overline{D})$ denotes a differential operator of the type

$$f(D,\overline{D}) = \sum_r \sum_s C_{rs} D^r \overline{D}^s$$

In which C_{rs} are constants , $D = \dfrac{\partial}{\partial x}$ and $\overline{D} = \dfrac{\partial}{\partial y}$. The general solution of equ.(7) may be written as; z=[C.F]+[P.I], complementary function(C.F) can be obtained from $f(D,\overline{D})z = 0$, and particular integral (P.I) can be obtained from $P.I = \dfrac{1}{f(D,\overline{D})} F(x,y)$.

Theorem : if u is the complementary function and z_1 is the particular integral of a linear P.D.E ,then $u + z_1$ is a general solution of equation.

Proof: H.w

Theorem :if $u_1, u_2, ..., u_n$ are the solutions of linear P.D.E $f(D,\overline{D})z = 0$ then $\sum_{r=1}^{n} C_r u_r$,where $C's$, is also a solution.

Proof: using equation ( ) for each solution, then obtain the aims of proof due to each one is satisfies equ.( ).

We can classify linear differential operators $f(D,\overline{D})$ into two types (reducible and irreducible)

Definition: $f(D,\overline{D})$ is irreducible if it can be written as a product of linear factors of the form $aD + b\overline{D} + c$,while is irreducible, where a,b,c are constants.

Theorem: if the operator $f(D,\overline{D})$ is reducible, the order in which the linear factor occur is unimportant. The theorem will be proved if can show that $(a_r D + b_r \overline{D} + c_r)(a_s D + b_s \overline{D} + c_s) = (a_s D + b_s \overline{D} + c_s)(a_r D + b_r \overline{D} + c_r)$ for any variable operator can be written in the form $f(D,\overline{D}) = \prod_{r=1}^{n} (a_r D + b_r \overline{D} + c_r)$.

Theorem:

a) if $a_r D + b_r \overline{D} + c_r$ is a factor of $f(D,\overline{D})$ and $\phi(\xi)$ is an arbitrary function of the single

value ξ , then if $a_r \neq 0$, $u_r = e^{\frac{-c_r}{a_r}x} \phi(b_r x - a_r y)$ is a solution of $f(D,\overline{D})z = 0$

b) if $a_r D + b_r \overline{D} + c_r$ is a factor of $f(D, \overline{D})$ and $\phi(\xi)$ is an arbitrary function of the single value ξ , then if $b_r \neq 0$, $u_r = e^{\frac{-c_r}{b_r}x}\phi(b_r x - a_r y)$ is a solution of $f(D, \overline{D})z = 0$

Theorem: if $(a_r D + b_r \overline{D} + c_r)^n, (a_r \neq 0)$ is a factor of $f(D, \overline{D})$ and if the functions $\phi_{r_1}, \cdots \phi_{r_n}$ are arbitrary functions, then $e^{\frac{-c_r}{a_r}x}\sum_{s=1}^{n} x^{s-1}\phi_{rs}(b_r x - a_r y)$ is a solution of $f(D, \overline{D})z = 0$

EX: pages 70,73,105 H.W
Ex: pages 125 H.W

Review:

The French mathematician Jacques Hadamard (1865–1963) coined the notion of *well-posedness*. According to his definition, a problem is called well-posed if it satisfies all of the following criteria

1. **Existence** The problem has a solution.
2. **Uniqueness** There is no more than one solution.
3. **Stability** A small change in the equation or in the side conditions gives rise to a small change in the solution.

If one or more of the conditions above does not hold, we say that the problem is*ill-posed*. One can fairly say that the fundamental problems of mathematical physics are all well-posed. However, in certain engineering applications we might tackle problems that are ill-posed. In practice, such problems are unsolvable. Therefore, when we face an ill-posed problem, the first step should be to modify it appropriately in order to render it well-posed.

Classification

We pointed out in the previous section that PDEs are often classified into different types. In fact, there exist several such classifications. Some of them will be described here. Other important classifications will be described later.

- **The order of an equation**

The first classification is according to the *order* of the equation. for example, the equation $u_{tt} - u_{xx} = f(x,t)$ is called a second-order equation, while $u_t + u_{xxxx} = 0$ is called a fourth-order equation.

- **Linear equations**

Another classification is into two groups: linear versus nonlinear equations. for example, the equation $x^7 u_x + e^{xy} u_{xy} + \sin(x^2 + y^2) = x^3$ is a linear equation, while $u_x^2 + u_y^2 = 1$ is a nonlinear equation. The nonlinear equations are often further classified into subclasses according to the type of the nonlinearity. Generally speaking, the nonlinearity is more pronounced when it appears in a higher derivative. For example, the following two equations are both nonlinear:

$$u_{xx} + u_{yy} = u^3, \tag{8}$$

$$u_{xx} + u_{yy} = |\nabla u|^2 u \tag{9}$$

Here $|\nabla u|$ denotes the norm of the gradient of u. While (9) is nonlinear, it is still linear as a function of the highest-order derivative. Such nonlinearity is called ***quasilinear.*** On the other hand in (8) the nonlinearity is only in the unknown function. Such equations are often called ***semilinear.***

Scalar equations versus systems of equations

A single PDE with just one unknown function is called a ***scalar equation***. In contrast, a set of m equations with l unknown functions is called a ***system* of *m* equations**.

Differential operators and the superposition principle

A function has to be k times differentiable in order to be a solution of an equation of order k. For this purpose we define the set $C^k(D)$ to be the set of all functions that are k times continuously differentiable in D. In particular, we denote the set of continuous functions in D by $C^0(D)$, or $C(D)$. A function in the set C^k that satisfies a PDE of order k, will be called a *classical* (or *strong*) solution of the PDE. It should be stressed that we sometimes also have to deal with solutions that are not classical. Such solutions are called *weak* solutions.

Mappings between different function sets are called *operators*. L [u] will denote the operation of an operator L on a function u. In particular, we shall deal in this course with operators defined by partial derivatives of functions. Such operators, which are in fact mappings between different C^k classes, are called ***differential operators***.