Lecture 1(review)

Introduction: Overview

Historically, partial differential equations originated from the study of surfaces in geometry and for solving a wide variety of problems in mechanics. During the second half of the nineteenth century, a large number of mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations. The primary reason for this research was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering. The next phase of the development of linear partial differential equations is characterized by the efforts to develop the general theory and various methods of solutions of these linear equations. In fact, partial differential equations have been found to be essential to develop the theory of surfaces on the one hand and to the solution of physical problems on the other. These two areas of mathematics can be seen as linked by the bridge of the calculus of variations.

With the discovery of the basic concepts and properties of distributions, the modern theory of linear partial differential equations is now well established. The subject plays a central role in modem mathematics, especially in physics, geometry, and analysis. Although origin of nonlinear partial differential equations is very old, they have undergone remarkable new developments during the last half of the twentieth century. One of the main impulses for developing nonlinear partial differential equations has been the study of nonlinear wave propagation problems. These problems arise in different areas of applied mathematics, physics, and engineering, including fluid dynamics, nonlinear optics, solid mechanics, plasma physics, quantum field theory, and condensed-matter physics. Nonlinear wave equations in particular have provided several examples of new solutions that are remarkably different from those obtained for linear wave problems. The best known examples of these are the corresponding shock waves, water waves, solitons and solitary waves. One of the remarkable properties of solitons is a localized wave form that is retained after interaction with other solitons, confirming solitons' 'particle-like' behavior. Indeed, the theory of nonlinear waves and soliions has experienced a revolution over the past three decades. During this revolution, many remarkable and unexpected phenomena have also been observed in physical, chemical, and biological systems. Other major achievements of twentieth-century applied mathematics include the discovery of soliton interactions, the Inverse Scattering Transform AST) method for finding the explicit exact solution for several canonical partial differential equations, and asymptotic perturbation analysis for the investigation of nonlinear evolution equations.

Definitions:

- Any differential equation containing partial derivatives is called a partial differential equation(P.D.E)
- The order of the P.D.E equal to the order of highest partial derivatives occurring in it.
- The dependent variable ((unknown function)) must be a function of at least two independent variables.
- A solution of the P.D.E in a region $\Re$ of (x, y) plane, we mean a function $u(x, y,)$ for which u, u_x, u_y are defined at each point (x, y) in $\Re$, and which equation reduced to an identity at each point such of function u is said to satisfy the equation in $\Re$.
- Derivation of **P.D.Es**:
  - From elimination of arbitrary constants.
  - From elimination of arbitrary functions.
  - From elimination of arbitrary constants and arbitrary functions.
  - From the connection between the physical and engineering problems.

Linear P.D.Es of first order:

Linear P.D.E of the first order can be written as;

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \qquad (1)$$

Equ.(1) is called Lagrange equation and it is generalized to n independent variables as;

$$X_1 P_1 + X_2 P_2 + \cdots + X_n P_n = R) \qquad (2)$$

Where $X_1, X_2, \cdots, X_n$ and R are functions of $x_1, x_2, \cdots, x_n$ and a dependent variables z ; $P_i = \frac{\partial z}{\partial x_i}, i = 1,2,..., n$

Theorem: The general solution of Equ.(1) is $F(u_1, u_2) = 0$,where F is an arbitrary function and $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$ form a solutions of equation

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} \qquad \textbf{(3)}$$

Proof: ((see Senddon page 50))

Theorem: **I**if $u_i(x_1, x_2, ..., x_n, z) = c_i$, $i = 1,2,...,n$ are independent solutions of the equations $\frac{dX_1}{P_1} = \frac{dX_2}{P_2} = \cdots = \frac{dX_n}{P_n}$, then the relation $\phi(u_1, u_2, ..., u_n) = 0$ in which the function ϕ is arbitrary , is a general solution of linear P.D.E

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R . \qquad (4)$$

Proof:((see Senddon)
Ex: page 55 and 57 H.W ((Senddon)).

Non-Linear P.D.Es of the first order:

Let $$F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0 \qquad (5)$$

is a non-linear P.D.E of the first order ((in which F is not necessarily linear in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$)) be derived from

$$g(x, y, z, a, b) = 0 \qquad (6)$$

By eliminate the arbitrary constants a and b .then g is called a complete solution of non-linear P.D.E(5)
The general method for obtaining complete solution of (5) is called Charpit's method. Before considering a general solution by this method, we give special procedure for handling four types of equations,

Type I: if the P.D.E in the form $f(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$, then $z = ax + h(a)y + c$, where $b = h(a)$

Type II: if the P.D.E in the form $z = ax + by + f(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$, then $z = ax + by + f(a,b)$

Type III: if the P.D.E in the form $f(z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$ then $z = z(x + ay)$,such that $\frac{\partial z}{\partial x} = \frac{dz}{du}$ and $\frac{\partial z}{\partial y} = a\frac{dz}{du}$,where $u = x + ay$

Type IV: if the P.D.E in the form $f_1(x, \frac{\partial z}{\partial x}) = f_2(y, \frac{\partial z}{\partial y})$, then $f_1(x, \frac{\partial z}{\partial x}) = a$ and $f_2(y, \frac{\partial z}{\partial y}) = a$ with $F_1(x,a) = \frac{\partial z}{\partial x}$ and $F_2(y,a) = \frac{\partial z}{\partial y}$ and $z = F_1(x,a)dx + F_2(y,a)dy$

Charpit's formulation is;

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-(pf_p + qf_q)} = \frac{dF}{0}$$

We can solve these equations to obtained p and q associated with original P. D.E.