7. Numerical Solution of Partial Differential Equation

The PDEs serve as models for elliptic, parabolic, and hyperbolic. For example, the Laplace equation is a representative example of an elliptic type of PDE, and so forth.

A PDE is called **quasilinear** if it is linear in the highest derivatives. Hence a second-order quasilinear PDE in two independent variables *x*, *y* is of the form;

$$au_{xx} + 2bu_{xy} + cu_{yy} = F\left(x, y, u, u_x, u_y\right)$$

Note:

$$u_x = \frac{\partial u}{\partial x} \quad , u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_y = \frac{\partial u}{\partial y} \quad , u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \, \partial y}$$

u is the unknown function. This equation is said to be of;

Elliptic type	if	$ac - b^2 > 0$	(example: Laplace equation)
Parabolic type	if	$ac - b^2 = 0$	(example: heat equation
Hyperbolic type	if	$ac - b^2 < 0$	(example: wave equation)

7.1 Solution of 2-D Laplace Equation;

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

To obtain methods of numerical solution. we replace the partial derivatives by corresponding **difference quotients**, as follows. By the **Taylor** formula,

$$u(x+h, y) = u(x, y) + hu_x(x, y) + \frac{h^2}{2} u_{xx}(x, y) + \frac{h^3}{6} u_{xxx}(x, y) + \cdots \quad (1)$$

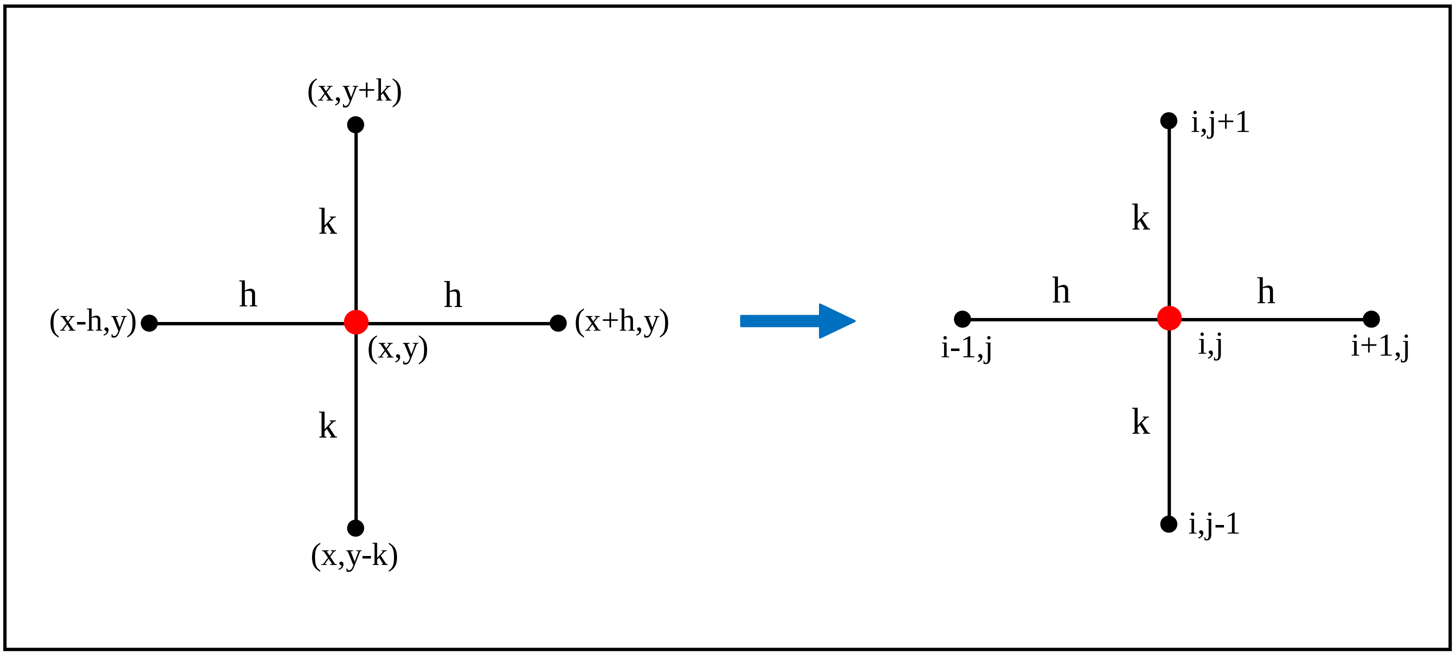
$$u(x-h, y) = u(x, y) - hu_x(x, y) + \frac{h^2}{2} u_{xx}(x, y) - \frac{h^3}{6} u_{xxx}(x, y) + \cdots \quad (2)$$

Subtract Eq.(2) from Eq.(1), neglect terms in h^2, h^3, ...,

$$u_x(x,y) \approx \frac{1}{2h}\left[u(x+h,y) - u(x-h,y)\right] \quad \Rightarrow \quad u_x\Big|_{j,i} = \frac{1}{2h}\left(u_{i+1,j} - u_{i-1,j}\right)$$

Similarly

$$u_y(x,y) \approx \frac{1}{2k}\left[u(x,y+k) - u(x,y-k)\right] \quad \Rightarrow \quad u_y\Big|_{j,i} = \frac{1}{2k}\left(u_{i,j+1} - u_{i,j+1}\right)$$



To obtain second derivative. Adding Eq.(1) and Eq.(2);

$$u_{xx}(x,y) \approx \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]$$

$$u_{xx}\Big|_{i,j} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] \qquad (4)$$

Similarly

$$u_{yy}(x,y) \approx \frac{1}{k^2}\left[u(x+k,y) - 2u(x,y) + u(x-k,y)\right]$$

$$u_{xx}\Big|_{i,j} = \frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] \qquad (5)$$

Substitute Eqs. (4) &(5) into Laplace Equation, get;

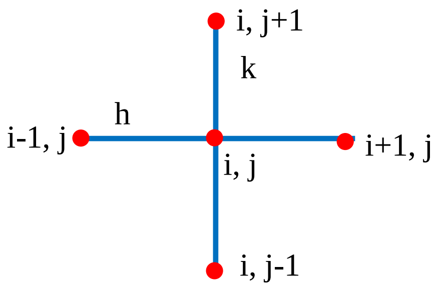
$$\frac{1}{h^2}\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right]+\frac{1}{k^2}\left[u_{i,j+1}-2u_{i,j}+u_{i,j-1}\right]=0$$

*** If h = k**

$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=0$$

$$\therefore u_{i,j}=\frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}}{4}$$

Note:



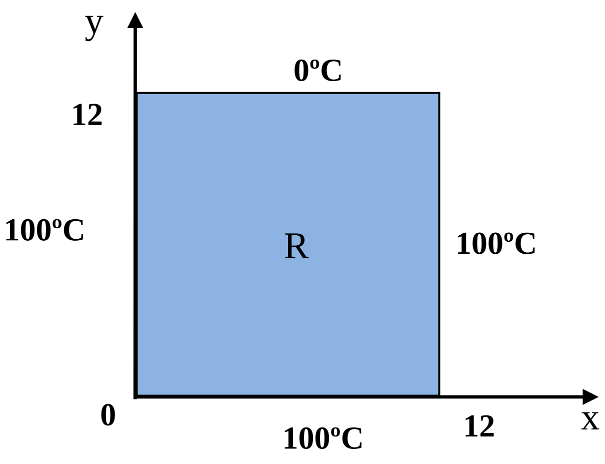
$$\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}$$

(Called Stencil, pattern, molecule, or star)

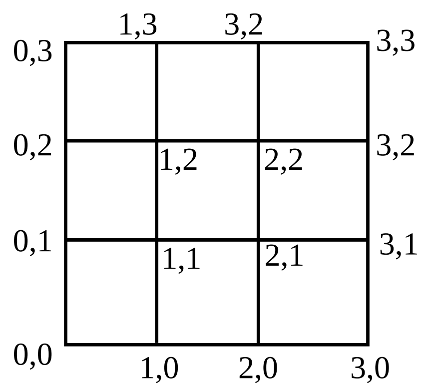
Can be rewritten the Laplace Equation in stencil as;

$$\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}u=0$$

Ex. The four sides of a square plate of side 12 cm made of homogeneous material are kept at constant temperature 0 ºC and 100 ºC as shown figure. Using a grid mesh 4 cm. Find the (steady state) temperature at the mesh point.



Given Problem



Grid and mesh points

Sol. The equations for 2-D in steady state as;

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Consider the mesh points;

h = k = 4

$$4u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}$$

*** Boundary Condition**

$$u_{1,0} = u_{2,0} = 100\ ^0C$$
$$u_{1,3} = u_{2,3} = 0\ ^oC$$

$$u_{0,1} = u_{0,1} = 100\ ^0C$$
$$u_{3,1} = u_{3,2} = 100\ ^oC$$

At point (1,1)

$$4u_{1,1} = u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} \quad \Rightarrow \quad 4u_{1,1} = u_{2,1} + 100 + u_{1,2} + 100$$

$$4u_{1,1} - u_{2,1} - u_{1,2} = 200 \qquad (1)$$

At point (2,1)

$$4u_{2,1} = u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} \quad \Rightarrow \quad 4u_{2,1} = 100 + u_{1,1} + u_{2,2} + 100$$

$$4u_{2,1} - u_{1,1} - u_{2,2} = 200 \qquad (2)$$

At point (1,2)

$$4u_{1,2} = u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} \quad \Rightarrow \quad 4u_{1,2} = u_{2,2} + 100 + 0 + u_{1,1}$$

$$4u_{1,2} - u_{2,2} - u_{1,1} = 100 \qquad (3)$$

At point (2,2)

$$4u_{2,2} = u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} \quad \Rightarrow \quad 4u_{2,2} = 100 + u_{1,2} + 0 + u_{2,1}$$

$$4u_{2,2} - u_{1,2} - u_{2,1} = 100 \qquad (4)$$

Solving Eqs. (1), (2), (3), and (4) by Iteration method or Gauss-Jordan method to finding the interior point $u_{1,1}, u_{2,1},\ u_{1,2}, \textit{and}\ \ u_{22}$

Rearrangement above equations

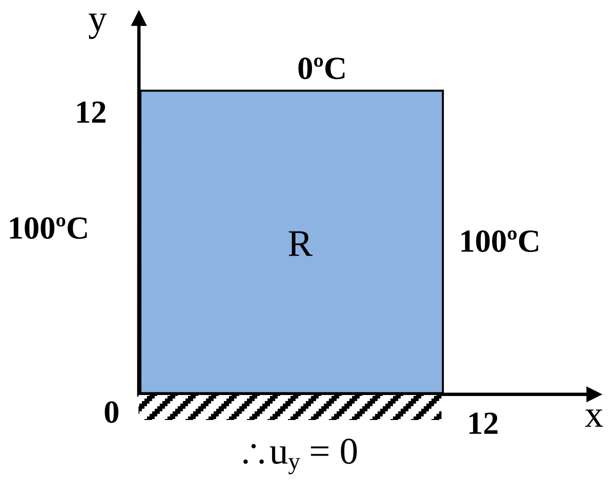
(1) If using Gauss-Jordan elimination method (Practical for small system of equation) as;

$$\begin{aligned} -4u_{11} + u_{21} + u_{12} \qquad\qquad &= -200 \\ u_{11} - 4u_{21} \qquad\quad + u_{22} &= -200 \\ u_{11} \qquad - 4u_{12} + u_{22} &= -100 \\ u_{21} + u_{12} - 4u_{22} &= -100 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} -4 & 1 & 1 & 0 & -200 \\ 1 & -4 & 0 & 1 & -200 \\ 1 & 0 & -4 & 1 & -100 \\ 0 & 1 & 1 & -4 & -100 \end{array}\right]$$

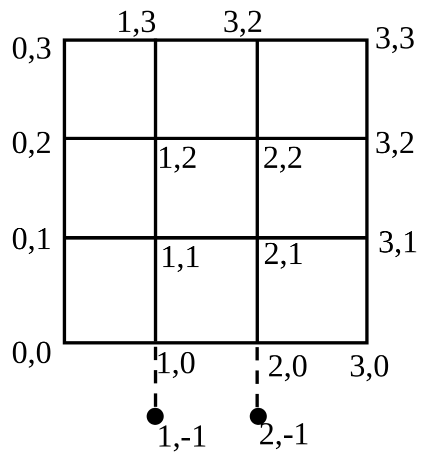
(2) If using Iteration method (Practical for large system of equation) as;

$$\begin{aligned} u_{11} &= 0.25u_{21} + 0.25u_{12} + 50 \\ u_{21} &= 0.25u_{11} + 0.25u_{22} + 50 \\ u_{12} &= 0.25u_{11} + 0.25u_{22} + 25 \\ u_{22} &= 0.25u_{21} + 0.25u_{12} + 25 \end{aligned}$$

Ex. Find the temperature distribution for the following plate.



Given Problem



Grid and mesh points

To find the temperature at isolation side

$$u_y\Big|_{1,0} = \frac{u_{1,1} - u_{1,0}}{2k} = 0 \qquad \Rightarrow \qquad u_{1,1} = u_{1,0} \qquad \ldots(a)$$

$$u_y\Big|_{2,0} = \frac{u_{2,1} - u_{2,0}}{2k} = 0 \qquad \Rightarrow \qquad u_{2,1} = u_{2,0} \qquad \ldots(b)$$

At point (1,0)

$$4u_{1,0}=u_{2,0}+u_{0,0}+u_{1,1}+u_{1,-1} \quad \Rightarrow \quad 4u_{1,0}=u_{2,0}+100+u_{1,1}+u_{1,1}$$

$$4u_{1,0}-u_{2,0}-2u_{1,1}=100 \qquad (1)$$

At point (2,0)

$$4u_{2,0}=u_{3,0}+u_{1,0}+u_{2,1}+u_{2,-1} \quad \Rightarrow \quad 4u_{2,0}=100+u_{1,0}+u_{2,1}+u_{2,-1}$$

$$4u_{2,0}-u_{1,0}-2u_{2,1}=100 \qquad (2)$$

The other points same the above example

H.W. Complete the other points and solution using Gauss-Seidel Method.

H.W. Write a computer programming to solve 2-D Laplace equation in case of Dirichlet or Neumann Conditions.

7.2 Solution of Parabolic Equation; (1-D Diffuse equaton)

In this section we explain the numerical solution of the prototype of parabolic equation, the one dimensional heat equation;

$$\frac{1}{\alpha}u_t=u_{xx} \qquad \textbf{(1)}$$

x :*space* $0\le x\le L$, t :*time* $t\ge 0$ α : thermal diffusivity $\left(\frac{m^2}{s}\right)$

The initial condition $u(x,0)=f\ (x\ )$ (given) and boundary condition at x=0 and x = L for all $t\ge 0$. Using $\Delta x=h,\quad \Delta t=k$

$$u_t\big|_{i,j}=\frac{u_{i,j+1}-u_{i,j}}{k} \qquad \textbf{Forward difference} \qquad \textbf{(a)}$$

$$u_{xx}\big|_{i,j}=\frac{1}{h^2}\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right] \qquad \textbf{(b)}$$

Subs. above Eqs. (a) & (b) in Eq.(1);

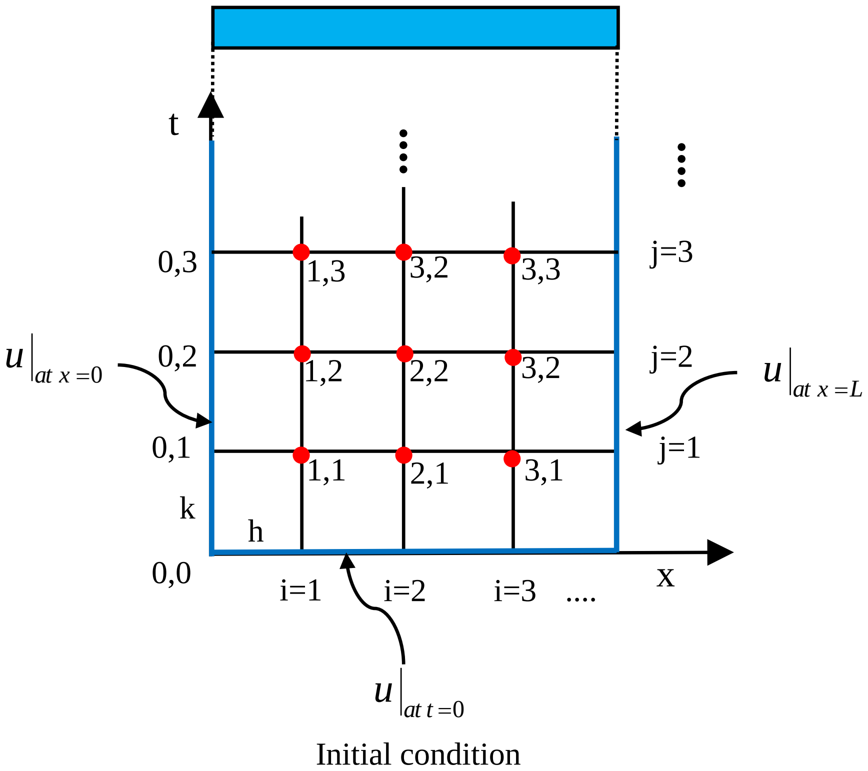
$$\frac{1}{\alpha k}\left[u_{i,j+1}-u_{i,j}\right]=\frac{1}{h^2}\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right]$$

$$u_{i,j+1}=\frac{\alpha k}{h^2}\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right]+u_{i,j}$$

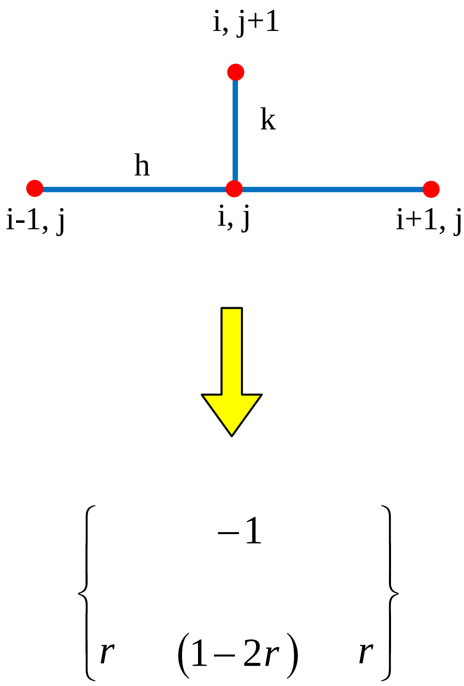
$$u_{i,j+1}=\frac{\alpha k}{h^2}\left[u_{i+1,j}+u_{i-1,j}\right]+\left(1-\frac{2\alpha k}{h^2}\right)u_{i,j}$$

$$u_{i,j+1} = (1-2r)u_{i,j} + r\left[u_{i+1,j} + u_{i-1,j}\right]$$

where $r = \dfrac{\alpha k}{h^2}$



Figure; Grid and mesh points



Figure; stencil for Four Points

N. B. For convergence and stability of explicit method is the condition;

$$r \le \frac{1}{2} \quad \Rightarrow \quad \frac{\alpha k}{h^2} \le \frac{1}{2}$$

$$\therefore k \le \frac{h^2}{2\alpha} \quad time\ \ step$$

Ex. A bar is initially at 100 ºC, it's ends are attached to a constant temperature body at 0 ºC. Find the temperature distribution assuming $\alpha = 0.1$, Take $\Delta x = h = 0.25$.

Sol. If h= 0.25 and $\alpha = 0.25$

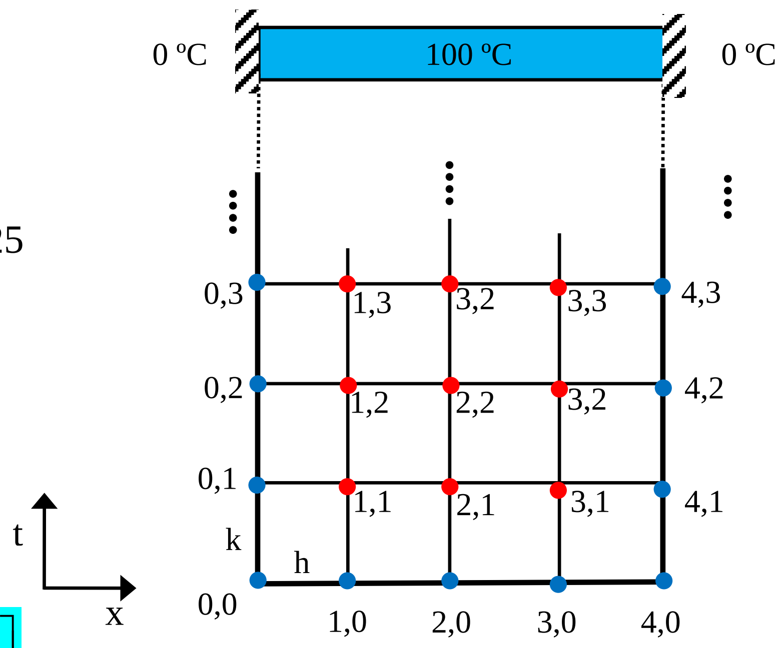
The condition for stability is;

$$\frac{\alpha k}{h^2} \le \frac{1}{2} \quad \Rightarrow \therefore k \le \frac{h^2}{2\alpha}$$

$$k \le \frac{0.25^2}{2*0.1} \quad \rightarrow \quad k \le 0.3125$$

$$Take \quad k = 0.25$$

$$r = \frac{\alpha k}{h^2} = \frac{0.1*0.25}{0.25^2} = 0.4$$



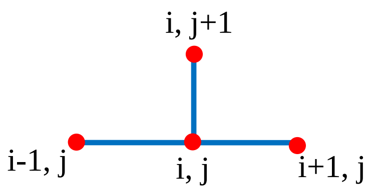
$$u_{i,j+1} = (1-2r)u_{i,j} + r\left[u_{i+1,j} + u_{i-1,j}\right]$$

Figure; Grid and mesh points

*** Boundary conditions**

At x = 0 $\quad u_{0,1} = u_{0,2} = u_{0,3} = \ldots = u_{0,j} = 0$

At x = L $\quad u_{4,1} = u_{4,2} = u_{4,3} = \ldots = u_{4,j} = 0$



*** Initial conditions** **At t = 0**

$$u_{1,0} = u_{2,0} = u_{3,0} = \ldots = u_{i,0} = 100$$

Compute the other points using the above formula

At t = 0.25, j =1

$$u_{1,1} = 0.2u_{1,0} + 0.4\left[u_{2,0} + u_{0,0}\right] = 0.2*100 + 0.4(100+0) = 60$$

$$u_{2,1} = 0.2u_{2,0} + 0.4\left[u_{3,0} + u_{1,0}\right] = 0.2*100 + 0.4(100+100) = 100$$

$$u_{3,1} = 0.2u_{3,0} + 0.4\left[u_{4,0} + u_{2,0}\right] = 0.2*100 + 0.4(0+100) = 60$$

At t = 0.5, j =2

$$u_{2,1} = 0.2u_{1,1} + 0.4\left[u_{2,1} + u_{0,1}\right] = 0.2*60 + 0.4\left(100+0\right) = 52$$

$$u_{2,2} = 0.2u_{2,1} + 0.4\left[u_{3,1} + u_{1,1}\right] = 0.2*100 + 0.4\left(60+60\right) = 68$$

$$u_{3,2} = 0.2u_{3,1} + 0.4\left[u_{4,1} + u_{2,1}\right] = 0.2*60 + 0.4(0+100) = 52$$

At t = 0.75, j = 3

H.W. Complete the other points

continue the procedure for until the steady state condition that is not change the value of u at increase the time t.

H.W. Write a computer programming to solve 1-D heat equation.

7.3 Solution of Hyperbolic Equation; (1-D Wave equaton)

In this section we explain the numerical solution of the prototype of hyperbolic equation, the one dimensional wave equation;

$$\frac{1}{c^2}u_{tt} = u_{xx} \qquad (1)$$

x :*space* $0 \le x \le L$, t :*time* $t \ge 0$ c : speed$\left(\frac{m}{s}\right)$

Initial displacement $u(x,0) = f(x)$ (Given)

Initial velocity $u_t(x,0) = v(x)$ (Given)

boundary condition $u(0,t) = u(L,t) = 0$

Using $\Delta x = h, \quad \Delta t = k$

$$u_{xx}\big|_{i,j} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] \qquad (a)$$

$$u_{tt}\big|_{i,j} = \frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] \qquad (b)$$

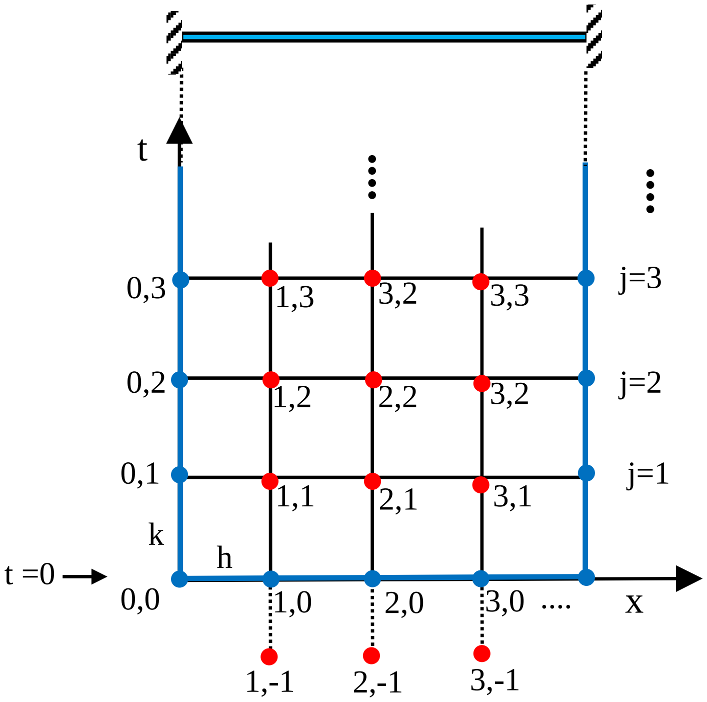
Subs. above Eqs. (a) & (b) in Eq.(1);

$$\frac{1}{c^2k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

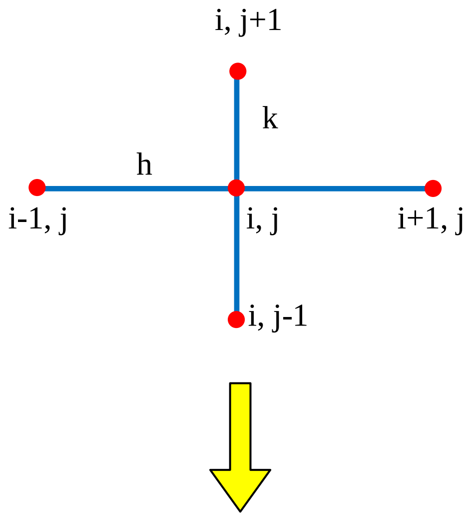
$$u_{i,j+1} = \frac{c^2k^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] + 2u_{i,j} + u_{i,j-1}$$

$$u_{i,j+1} = 2(1-r^2)u_{i,j} + r^2\left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1}$$

where $r = \frac{ck}{h}$



Figure; Grid and mesh points



$$\begin{Bmatrix} & -1 & \\ r^2 & 2\left(1-r^2\right) & r^2 \\ & -1 & \end{Bmatrix}$$

Figure; stencil for five Points

* The condition for convergence and stability of the solution;

$$\left(1-r^2\right) \geq 0 \quad \Rightarrow \quad r \leq 1 \quad \Rightarrow \frac{ck}{h} \leq 1 \quad \therefore k \leq \frac{h}{c}$$

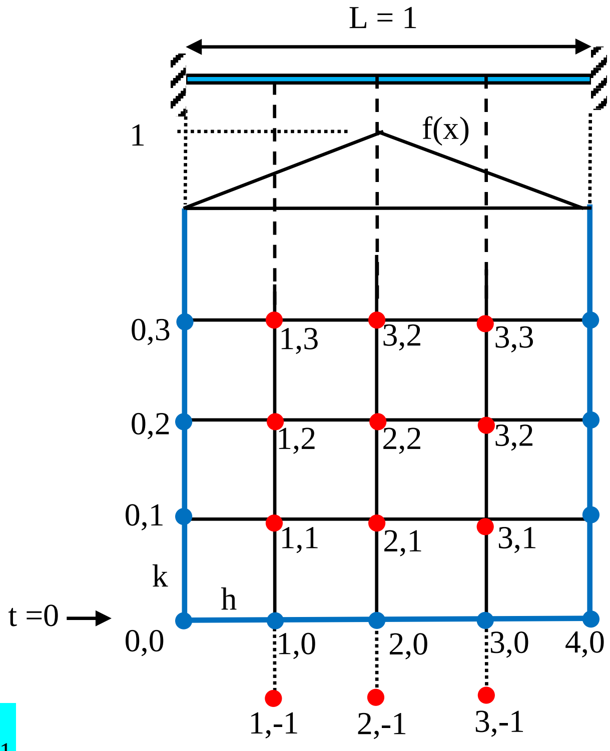
At initial velocity v(x) given (At t = 0 , j = 0)

$$v\left(x_i\right) = u_t\big|_{i,0} = \frac{u_{i,0} - u_{i,-1}}{k}$$ **Using Backward difference**

$$\therefore u_{i,-1} = u_{i,0} - kv\left(x_i\right)$$

Ex. For the string shown, find the motion using Δx = 0.25, speed c = 1 m/s.

$$v(x) = \begin{cases} 2 & 0.25 \le x \le 0.75 \\ 0 & otherwise \end{cases}$$



Figure; Grid and mesh points

Sol. h = Δx = 0.25

$$r \le 1 \Rightarrow \therefore k \le \frac{h}{c}$$

$$k \le 0.25 \quad \Rightarrow take \quad k = 0.2$$

$$r = \frac{ck}{h} = \frac{1*0.2}{0.25} = 0.8$$

$$r^2 = 0.64$$

$$u_{i,j+1} = 2(1-r^2)u_{i,j} + r^2\left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1}$$

$$u_{i,j+1} = 0.72u_{i,j} + 0.64\left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1}$$

* Boundary conditions;

$$u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = ... = 0$$
$$u_{4,0} = u_{4,1} = u_{4,2} = u_{4,3} = ... = 0$$

At t = 0 , j = 0

Initial conditions (I.C.)

$$u_{i,0} = f(ih)$$
$$u_{1,0} = f(0.25) = 0.5$$
$$u_{2,0} = f(0.5) = 1$$
$$u_{3,0} = f(0.75) = 0.5$$

Note: Some time the function f(x) given as; f(x)= sin πx

$$u_{i,-1} = u_{i,0} - kv\,(ih)$$
$$u_{1,-1} = u_{1,0} - kv\,(0.25) = 0.5 - 0.2*2 = 0.1$$
$$u_{2,-1} = u_{2,0} - kv\,(0.5) = 1 - 0.2*2 = 0.6$$
$$u_{3,-1} = u_{3,0} - kv\,(0.75) = 0.5 - 0.2*2 = 0.1$$

$$u_{1,1} = 0.72u_{1,0} + 0.64\left(u_{2,0} + u_{0,0}\right) - u_{1,-1} = 0.72*0.5 + 0.64(1+0) - 0.1 = 0.9$$
$$u_{2,1} = 0.72u_{2,0} + 0.64\left(u_{3,0} + u_{1,0}\right) - u_{2,-1} = 0.72*1 + 0.64\left(0.5+0.5\right) - 0.6 = 0.76$$
$$u_{3,1} = 0.72u_{3,0} + 0.64\left(u_{4,0} + u_{2,0}\right) - u_{3,-1} = 0.72*0.5 + 0.64\left(0+1\right) - 0.1 = 0.9$$

At t = 0.2, j =2

$$u_{1,2} = 0.72u_{1,1} + 0.64\left(u_{2,1} + u_{0,1}\right) - u_{1,0} = 0.72*0.9 + 0.64(0.76+0) - 0.5 = 0.6344$$
$$u_{2,2} = 0.72u_{2,1} + 0.64\left(u_{3,1} + u_{1,1}\right) - u_{2,0} = 0.72*0.76 + 0.64\left(0.9+0.9\right) - 1 = 0.6992$$
$$u_{3,2} = 0.72u_{3,1} + 0.64\left(u_{4,1} + u_{2,1}\right) - u_{3,0} = 0.72*0.9 + 0.64\left(0+0.76\right) - 0.5 = 0.6344$$

At t = 0.4, j = 3

Note: Can be calculate half points and using the symmetry to obtained the others points

H.W. Complete the other points

H.W. Write a computer programming to solution the hyperbolic equation.