6. Numerical Solution of ODE

6.1 First order DE.

We know that an ODE of the first order is of the form F(x, y, y')=0 and can often be written in the explicit form $y' = f(x, y)$. An **initial value problem** for this equation is of the form;

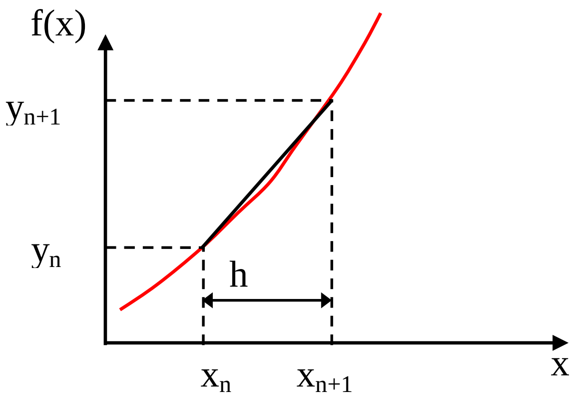
$$y' = f(x, y), \qquad y(x_0) = y_0$$

We shall discuss methods of computing numerical values of the solution. these methods are step-by-step methods, that is, we start from the given $y_0=y(x_0)$ and process stepwise, computing approximate values of the solution of y(x) at the mesh point.

$$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \ \cdots$$

Ex. $y' - 2y = x \quad \Rightarrow \quad y' = f(x, y) = x + 2y$

(1) Euler Method



Using Taylor series

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \cdots$$

For a small value of h, the highest powers h^2, h^3, ... are very small

$$y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n)$$

In the first step we compute

$$y_1 = y_0 + hf(x_0, y_0)$$

In the second step we compute

$$y_2 = y_1 + hf(x_1, y_1)$$

In general;

$$y_{n+1} = y_n + hf\,(x_n, y_n) \qquad \textbf{n = 0, 1 , 2, ...}$$

This is called the **Euler method** or **Euler-Cauchy method**.

Ex. Solve $y' - 2(x + y) = 0$ for x from 0 to 1. given that y(0) = 1, use h = 0.1.

$y' = f\,(x, y) = 2x + 2y$

Using Euler method $y_{n+1} = y_n + hf\,(x_n, y_n)$

n	x_n	y_n	hf_n $0.1(2x_n+2y_n)$	y_{n+1}
0	0	1	0.2	1.2
1	0.1	1.2	0.26	1.46
2	0.2	1.46	0.332	1.792
3	0.3			
4	0.4			
5	0.5			
6	0.6			
7	0.7			
8	0.8			
9	0.9			
10	1.0			

(2) Modified Euler Method (Heun's Mehtod)

Euler's method is generally much too inaccurate. For a large and small *h* the computation becomes prohibitive.

The improved Euler method we compute two values;

Step (1): **Predictor**

$$K_n = hf\,(x_n, y_n)$$
$$L_n = hf\,(x_n + h, y_n + K_n)$$

Step (2): **Corrector**

$$y_{n+1} = y_n + \frac{1}{2}(K_n + L_n)$$

Hence the **improved Euler method** is a **predictor–corrector method**: In each step we predict a value and then we correct it.

Ex. Solve the least example using modified Euler method.

$$y' = f(x, y) = 2x + 2y \qquad , y(0) = 1, \quad h = 0.1$$

$$K_n = hf(x_n, y_n) \qquad = 0.1(2x_n + 2y_n)$$

$$L_n = hf(x_n + h, y_n + K_n) = 0.1[2(x_n + 0.1) + 2(y_n + K_n)]$$

$$y_{n+1} = y_n + \frac{1}{2}(K_n + L_n)$$

$$K_0 = 0.1(2*0 + 2*1) = 0.2$$

$$L_0 = 0.1[2(0 + 0.1) + 2(1 + 0.2)] = 0.26$$

$$y_1 = 1 + \frac{1}{2}(0.2 + 0.26) = 1.23$$

⋮

n	x_n	y_n	K_n	L_n	y_{n+1}
0	0	1	0.2	0.26	1.23
1	0.1	1.23	0.266	0.3392	1.533
2	0.2	1.533			
3	0.3				
4	0.4				
5	0.5				
6	0.6				
7	0.7				
8	0.8				
9	0.9				
10	1.0				

(3) Runge - Kutta Method

A method of great practical importance and much greater accuracy than that of the *improved Euler method* is the *classical Runge–Kutta method of fourth order*, which we call briefly the **Runge–Kutta method**.

The following fourth order Runge-Kutta algorithm, with $x_n = x_0 + nh$.

Step (1): **Calculate**

$$A_n = hf\,(x_n, y_n)$$
$$B_n = hf\left(x_n + \frac{h}{2}, y_n + \frac{A_n}{2}\right)$$
$$C_n = hf\left(x_n + \frac{h}{2}, y_n + \frac{B_n}{2}\right)$$
$$D_n = hf\,(x_n + h, y_n + C_n)$$

Step (2): **Calculate**

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

Ex. Use the fourth order Runge-Kutta algorithm with a step size h = 0.2 to sole initial value problem

$$\frac{dy}{dx} + 2y = \sin 3x \qquad with \;\; y(0) = 1$$

in the interval $0 \le x \le 1$.

Sol.

$$\therefore y' = f\,(x, y) = \sin 3x - 2y$$

$$A_n = 0.2 * (\sin 3x_n - 2y_n)$$
$$B_n = 0.2\left[\sin 3(x_n + 0.1) - 2\left(y_n + \frac{A_n}{2}\right)\right]$$
$$C_n = 0.2\left[\sin 3(x_n + 0.1) - 2\left(y_n + \frac{B_n}{2}\right)\right]$$
$$D_n = 0.2[\sin 3(x_n + 0.2) - 2(y_n + C_n)]$$

$$A_0 = 0.2*\left(\sin 3*0 - 2*1\right) = -0.4$$

$$B_0 = 0.2\left[\sin 3\left(0+0.1\right) - 2\left(1+\frac{-0.4}{2}\right)\right] = -0.2609$$

$$C_0 = 0.2\left[\sin 3\left(0+0.1\right) - 2\left(1+\frac{-0.2609}{2}\right)\right] = -0.28872$$

$$D_0 = 0.2\left[\sin 3(0+0.2) - 2(1-0.28872)\right] = -0.17158$$

$$\therefore y_1 = 1 + \frac{1}{6}\left[\left(-0.4\right) + 2\left(-0.2609\right) + 2\left(-0.28872\right) + \left(-0.17158\right)\right] = 0.72153$$

$$\vdots$$

n	x_n	y_n	A_n	B_n	C_n	D_n	y_{n+1}
0	0	1	-0.4	-0.261	-0.288	-0.171	0.721
1	0.2	0.721	-0.175	-0.097	-0.112	-0.057	0.613
2	0.4	0.613	-0.058	-0.034	-0.038	-0.0348	0.573
3	0.6	0.573					
4	0.8						
5	1.0						

H.W. Write a computer program to solve and first order differential equation using Euler, modified Euler and Runge-Kutta methods.

6.2 Second Order Ordinary Differential Equation:

We know that an ODE of the second order is of the form F(x, y, y',y")=0 and can often be written in the explicit form;

$$y'' = f\left(x, y, y'\right) \qquad with \;\; y\left(x_0\right) = y_0 \;\; and \quad y'\left(x_0\right) = y'_0$$

(1) Euler Method

Using Taylor series

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \cdots$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n$$

The **Euler method** we compute two steps;

Step (1) $$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} f(x_n, y_n, y'_n)$$

Step (2) $$y'_{n+1} = y'_n + hf(x_n, y_n, y'_n)$$

Ex. Solve $y'' - 4y' - 2xy = 0$ given that $y(0) = 1,\ y'(0) = 1$ for x from 0 to 1 using h = 0.1.

Sol.

$$y'' = f(x, y, y') = 2xy + 4y'$$

$$y_{n+1} = y_n + 0.1y'_n + 0.005(2x_n y_n + 4y'_n)$$

$$y'_{n+1} = y'_n + 0.1(2x_n y_n + 4y'_n)$$

Calculate

$$y_1 = 1 + 0.1*1 + 0.005(2*0*1 + 4*1) = 1.12$$

$$y'_1 = 1 + 0.1(2*0*1 + 4*1) = 1.4$$

⋮ ⋮

n	x_n	y_n	y'_n	$f(x_n, y_n, y'_n)$	y_{n+1}	y'_{n+1}
0	0	1	1	4	1.12	1.4
1	0.1	1.12	1.4	5.824	1.289	1.982
2	0.2	1.289	1.982			
3	0.3					
4	0.4					
5	0.5					
6	0.6					
7	0.7					

8	0.8					
9	0.9					
10	1.0					

(2) Runge-Kutta-Nyström Method (RKN Method)

RKN methods are direct extensions of RK methods (Runge–Kutta methods) to second-order ODEs $y'' = f(x, y, y')$, as given by Nyström (1925). The best known of these uses the following formulas;

Step (1): **Calculate**

$$A_n = hf(x_n, y_n, y'_n)$$

$$B_n = hf\left(x_n + \frac{h}{2}, y_n + \frac{K}{2}, y'_n + \frac{A_n}{2}\right) \quad \textbf{where} \quad K = h\left(y'_n + \frac{A_n}{4}\right)$$

$$C_n = hf\left(x_n + \frac{h}{2}, y_n + \frac{K}{2}, y'_n + \frac{B_n}{2}\right)$$

$$D_n = hf(x_n + h, y_n + L, y'_n + C_n) \quad \textbf{where} \quad L = h(y'_n + C_n)$$

Step (2): **Calculate**

$$y_{n+1} = y_n + h\left(y'_n + \frac{1}{6}(A_n + B_n + C_n)\right)$$

Step (3): **Calculate**

$$y'_{n+1} = y'_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

Ex. Resolve the last example using RKN method, take h = 0.2.

$$y'' = f(x, y, y') = 2xy + 4y' \qquad , y(0) = y'(0) = 1$$

$$A_n = 0.2(2x_n y_n + 4y'_n)$$

$$B_n = 0.2\left(2(x_n + 0.1)\left(y_n + \frac{K}{2}\right) + 4\left(y'_n + \frac{A_n}{2}\right)\right) \quad \textbf{where} \quad K = 0.2\left(y'_n + \frac{A_n}{4}\right)$$

$$C_n = 0.2\left(2(x_n+0.1)\left(y_n+\frac{K}{2}\right)+4\left(y'_n+\frac{B_n}{2}\right)\right)$$

$$D_n = 0.2\left(2(x_n+0.2)(y_n+L)+4(y'_n+C_n)\right)$$ **where** $$L = 0.2\left(y'_n+\frac{C_n}{2}\right)$$

$$A_0 = 0.2[2*0*1+4*1] = 0.8$$

$$K = 0.2\left(1+\frac{0.8}{4}\right) = 0.24$$

$$B_n = 0.2\left[2(0+0.1)\left(1+\frac{0.24}{2}\right)+4\left(1+\frac{0.8}{2}\right)\right] = 1.165$$

$$C_n = 0.2\left[2(0+0.1)\left(1+\frac{0.24}{2}\right)+4\left(1+\frac{1.165}{2}\right)\right] = 1.311$$

$$L = 0.2\left(1+\frac{1.311}{2}\right) = 0.311$$

$$D_n = 0.2\left[2(0+0.2)(1+0.331)+4(1+1.311)\right] = 1.971$$

$$y_1 = 1+0.2\left(1+\frac{1}{6}(0.8+1.165+1.311)\right) = 1.309$$

$$y'_1 = 1+\frac{1}{6}(0.8+2*1.165+2*1.311+1.971) = 2.287$$

n	x_n	y_n	y'_n	A_n	B_n	C_n	D_n	y_{n+1}	y'_{n+1}
0	0	1	1	0.8	1.165	1.311	1.971	1.309	2.287
1	0.2	1.309	2.287						
2	0.4								
3	0.6								
4	0.8								
5	1.0								

H.W. Write a computer program to solve any second order differential equation using Euler and Runge-Kutta Nyström methods.