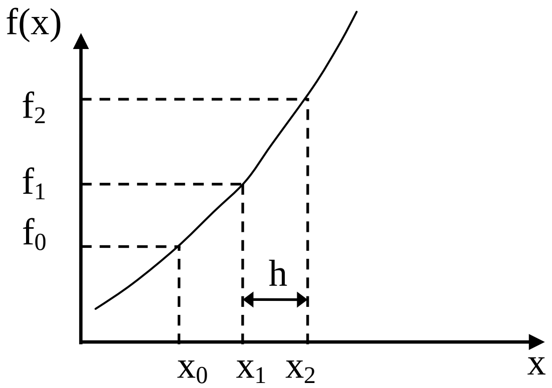
3. Finite Difference, Interpolation and Extrapolation

Finite Differences

Forward difference | **Backward difference** | **Central difference**

1. Forward differences:



$$\Delta f_0 = f_1 - f_0$$
$$\Delta f_1 = f_2 - f_1$$
$$\vdots \qquad \vdots$$
$$\Delta f_n = f_{n+1} - f_n \qquad \text{"1st Forward difference"}$$

$$\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0)$$
$$= f_2 - 2f_1 + f_0$$
$$\vdots$$
$$\Delta^2 f_n = f_{n+2} - 2f_{n+1} + f_n \qquad \text{"2nd Forward difference"}$$

$$\Delta^3 f_n = \Delta\left(\Delta^2 f_n\right) = \Delta f_{n+2} - 2\Delta f_{n+1} + \Delta f_n$$
$$= (f_{n+3} - f_{n+2}) - 2(f_{n+2} - f_{n+1}) + (f_{n+1} - f_n)$$
$$\therefore \Delta^3 f_n = f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n$$

Or $\Delta^3 f_n = \Delta^2 f_{n+1} - \Delta^2 f_n$ "3rd Forward difference"

$$\Delta^4 f_n = \Delta^3 f_{n+1} - \Delta^3 f_n$$

$$\Delta^k f_n = \Delta^{k-1} f_{n+1} - \Delta^{k-1} f_n \qquad (k = 1, 2, 3, \dots)$$

Ex. Construct the forward difference table for the following data;

x	0	0.5	1	1.5	2.0
f(x)	1.2	1.8	3.1	4.6	7.6

n	x_n	f_n	Δf_n	$\Delta^2 f_n$	$\Delta^3 f_n$	$\Delta^4 f_n$
0	0	1.2	0.6	0.7	-0.5	1.8
1	0.5	1.8	1.3	0.2	1.3	-
2	1.0	3.1	1.5	1.5	-	-
3	1.5	4.6	3.0	-	-	-
4	2.0	7.6	-	-	-	-

2. Backward differences: " ∇ "

$$\nabla f_1 = f_1 - f_0$$

$$\nabla f_n = f_n - f_{n-1}$$ " 1st Backward difference"

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1}$$ " 2nd Backward difference"

$$= (f_n - f_{n-1}) - (f_{n-1} - f_{n-2})$$
$$= f_n - 2f_{n-1} + f_{n-2}$$

$$\nabla^3 f_n = \nabla^2 f_n - \nabla^2 f_{n-1}$$ " 3rd Backward difference"

$$\nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}$$

Ex. Construct the backward table for the last data.

n	x_n	f_n	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	$\nabla^4 f_n$
0	0	1.2	-	-	-	-
1	0.5	1.8	0.6	-	-	-
2	1.0	3.1	1.3	0.7	-	-
3	1.5	4.6	1.5	0.2	-0.5	-
4	2.0	7.6	3.0	1.5	1.3	1.8

3. Central differences: "δ"

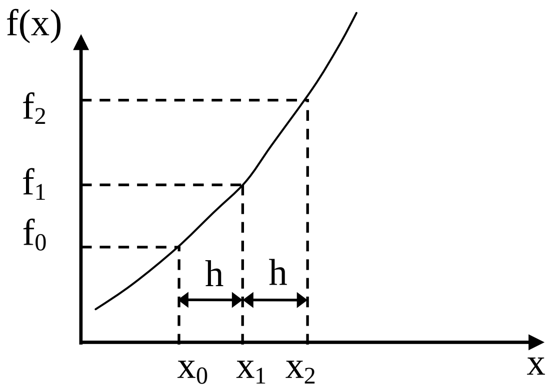
$$\delta f_1 = \frac{f_2 - f_0}{2}$$

Or

$$\delta f_n = \frac{1}{2}(\Delta f_n + \nabla f_n) \quad \text{"1st Central"}$$

$$= \frac{1}{2}\{(f_{n+1} - f_n) - (f_n - f_{n-1})\}$$

$$\therefore \delta f_n = \frac{1}{2}(f_{n+1} + f_{n-1})$$



$$\delta^2 f_n = \delta(\delta f_n) = \frac{1}{2}(\delta f_{n+1} - \delta f_{n-1})$$

$$\delta^3 f_n = \delta^2(\delta f_n) = \frac{1}{2}(\delta^2 f_{n+1} - \delta^2 f_{n-1})$$

$$\delta^k f_n = \frac{1}{2}(\delta^{k-1} f_{n+1} - \delta^{k-1} f_{n-1})$$

H.W. Construct the central difference table for the data in previous example.

Interpolation and Extrapolation

*** Linear interpolation**

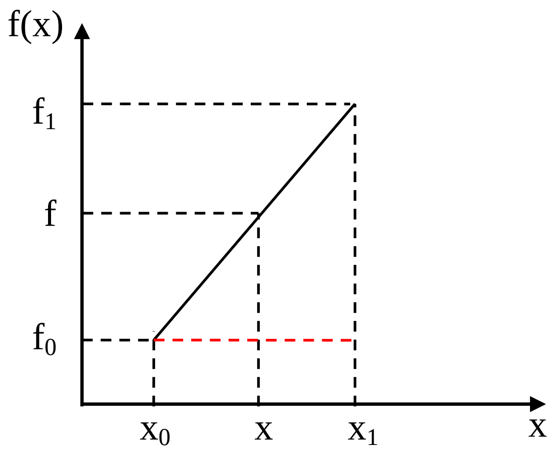
$$\frac{f_1 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$$

$$(f - f_0)h = (x - x_0)\Delta f_0$$

$$f = f_0 + \frac{x - x_0}{h}\Delta f_0$$

$$r = \frac{x - x_0}{h}$$

$$f = f_0 + r\Delta f_0 \quad \textbf{"linear"}$$



$h = x_1 - x_0$

$= x_{n+1} - x_n$ "step"

**** Quadratic interpolation**

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2}\Delta^2 f_0$$

*****Newton Forward interpolation formula**

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f + \cdots$$

*****Newton Backward interpolation formula**

$$f = f_0 + r\nabla f_0 + \frac{r(r-1)}{2!}\nabla^2 f_0 + \frac{r(r+1)(r+2)}{3!}\nabla^3 f_0 + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 f + \cdots$$

Ex. For the same data in last example, find f(- 0.5), f(1.25), f(2.75) using linear and quadratic and 4-term Newton formula.

x	0	0.5	1	1.5	2.0
f(x)	1.2	1.8	3.1	4.6	7.6

Sol.

f(- 0.5) = ?   extrapolation

linear : $f = f_0 + r\Delta f_0$ $r = \dfrac{x - x_0}{h}$

$x_0 = 0$, h = 0.5 "step" $h = x_{n+1} - x_n$ $r = \dfrac{-0.5}{0.5} = -1$

$$f(-0.5) = 1.2 + (-1)(0.6) = 0.6$$

Quadratic : $f = f_0 + r\Delta f_0 + \dfrac{r(r-1)}{2}\Delta^2 f_0$

$$f(-0.5) = 1.2 + (-1)(0.6) + \frac{(-1)(-2)}{2} * 0.7 = 1.3$$

4-term Newton Forward formula

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0$$

$$f(-0.5) = 1.2 + (-1)(0.6) + \frac{(-1)(-2)}{2}(0.7) + \frac{(-1)(-2)(-3)}{6}(-0.5) = 1.8$$

f(1.25) = ?   Let $x_0 = 1.0$ **Note:** Assume the x_0 nearest to the desired value

linear :

$$r = \frac{x - x_0}{h} = \frac{1.25-1}{0.5} = 0.5$$

$$f(1.25) = f_0 + r\Delta f_0 = 3.1 + 0.5*1.5 = 3.85$$

Quadratic :

$$f(1.25) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2}\Delta^2 f_0 = 3.1 + 0.5*1.5 + \frac{0.5*(-0.5)}{2}*1.5 = 3.66$$

4-term Newton formula   Let $x_0 = 0.5$ ***Because** x_0=1 not found $\Delta^3 f_0$

$$r = \frac{x - x_0}{h} = \frac{1.25-0.5}{0.5} = 1.5$$

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0$$

$$f(1.25) = 1.8 + 1.5*1.3 + \frac{1.5*0.5}{2}*0.2 + \frac{1.5*0.5*(-0.5)}{6}*1.3 = 3.74$$

f(2.7) = ?   Let $x_0 = 2.0$ **Note:** Using Backward Table because containing all terms

$$r = \frac{x - x_0}{h} = \frac{2.75-2}{0.5} = 1.5$$

linear : $f(2.75) = f_0 + r\nabla f_0 = 7.6 + 1.5*3.0 = 12.1$

Quadratic :

$$f(2.75) = f_0 + r\nabla f_0 + \frac{r(r+1)}{2}\nabla^2 f_0 = 7.6 + 1.5*3.0 + \frac{1.5*2.5}{2}*1.5 = 14.91$$

4-term Newton formula

$$f = f_0 + r\nabla f_0 + \frac{r(r+1)}{2!}\nabla^2 f_0 + \frac{r(r+1)(r+2)}{3!}\nabla^3 f_0$$

$$f(2.75) = 7.6 + 1.5*3.0 + \frac{1.5*2.5}{2}*1.5 + \frac{1.5*2.5*3.5}{6}*1.3 = 17.75$$

Note: Newton interpolation formula can be used only when the point are equally spaced (h = constant).

H.W. Write a computer program to interpolate f(x) using Newton forward and backward formula using n-term.

Lagrange Interpolation formula

This method has less accuracy them Newton formula, but it has advantage of it can be used for equally or non-equally spaced data.

$$f(x) = \sum_{k=0}^{n} N_k(x) f_k$$

$$N_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)} = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

where, n: number of terms - 1

Ex. if take n = 2

$$f(x) = \sum_{k=0}^{2} N_k(x) f_k = N_0(x) f_0 + N_1(x) f_1 + N_2(x) f_2$$

$$\underset{k=0}{N_0} = \prod_{\substack{j=0 \\ j \neq k}}^{2} \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)}$$

$$\underset{k=1}{N_1} = \prod_{\substack{j=0 \\ j \neq k}}^{2} \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\underset{k=2}{N_2} = \prod_{\substack{j=0 \\ j \neq k}}^{2} \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)}$$

$$\therefore f(x) = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)} . f_0 + \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)} . f_1 + \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)} . f_2$$

Ex. For the last example, find f(1.25) using 3-term Lagrange interpolation formula.

x = 1.25

		x_0	x_1	x_2	
x	0	0.5	1	1.5	2.0
f(x)	1.2	1.8	3.1	4.6	7.6
		f_0	f_1	f_2	

n = No. of terms - 1 = 3 - 1 = 2

$$f(x) = \sum_{k=0}^{2} N_k(x) f_k = N_0(x) f_0 + N_1(x) f_1 + N_2(x) f_2$$

$$\underset{k=0}{N_0} = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)} = \frac{(1.25-1)}{(0.5-1)} \cdot \frac{(1.25-1.5)}{(0.5-1.5)} = -0.125$$

$$\underset{k=1}{N_1} = \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)} = \frac{(1.25-0.5)}{(1-0.5)} \cdot \frac{(1.25-1.5)}{(1-1.5)} = 0.75$$

$$\underset{k=2}{N_2} = \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)} = \frac{(1.25-0.5)}{(1.5-0.5)} \cdot \frac{(1.25-1)}{(1.5-1)} = 0.375$$

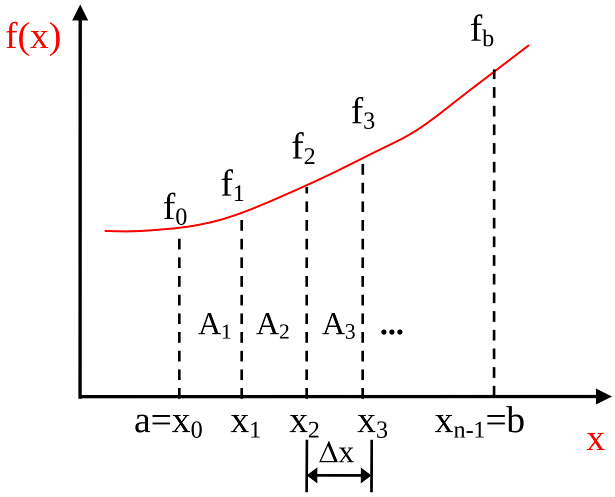
$$\therefore f(1.25) = -0.125*1.8 + 0.75*3.1 + 0.375*4.6 = 3.825$$

N.B. Using Lagrange interpolation up to term. (Max **n**=No. of point - 1)

H.W. Write a general computer program to read points (x_n, f_n) then construct the difference table up $\Delta^{n-1}f$, then estimate y(x) for any x using 4-term Newton and Lagrange interpolation.

4. Numerical Integration and Differentiation

4.1 Integration:



1. Trapezoidal Rule

$$\int_a^b f(x)dx$$

n: No. of divisions (strips)

$$\Delta x = \frac{b-a}{n}$$

Total area = $\int_a^b f(x)dx = \sum_{i=1}^{n} A_i$

$$A_1 = \frac{f_0 + f_1}{2}.\Delta x \quad , \quad A_2 = \frac{f_1 + f_3}{2}.\Delta x \quad , \cdots \quad A_n =$$

$x_1 = x_0 + \Delta x$ $= a + \Delta x$ $x_2 = x_1 + \Delta x$ …