2. Solution of linear simulation equation

$$\begin{array}{ll}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = b_1 & (1) \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = b_2 & (2) \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = b_3 & (3) \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mn}x_n = b_m & (m)
\end{array}$$

Where:
m = No. of equations
n = No. of variables (unknowns)

i. If m=n, there is one exact solution.
ii. If m<n, there is infinite number of solutions.
iii. If m>n, there may be an approximate solution.

For n = m only

2.1 Gauss-Seidel Iteration Method

If take n =3 No. of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \Rightarrow \quad x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \Rightarrow \quad x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad \Rightarrow \quad x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Algorithm for the iteration method

1. Put $x_i = \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j \right) \Big/ a_{ii}$.

2. Assume values for x_i , i = 2, 3, ... , n

3. Substitute in the equation to find new set of x_i and continue till we get convergence

$\left|x_i^{k+1} - x_i^{k}\right| < \in$ **where $\in$ rate of convergence (very small)**

Note: To get convergence, the following condition is preferable

$$\left|a_{ii}\right| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{ij}\right|$$

Ex. Solve the set of equations

$$\begin{aligned} x_1 + 4x_2 - x_3 &= 6 \quad \cdots\cdots\cdots \quad (1) \\ 2x_1 - x_2 + x_3 &= 3 \quad \cdots\cdots\cdots \quad (2) \\ -x_1 + 2x_2 + 4x_3 &= 15 \quad \cdots\cdots\cdots \quad (3) \end{aligned}$$

Swap eqs. (1) and (2)

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 3 \quad \cdots\cdots\cdots \quad (1) \\ x_1 + 4x_2 - x_3 &= 6 \quad \cdots\cdots\cdots \quad (2) \\ -x_1 + 2x_2 + 4x_3 &= 15 \quad \cdots\cdots\cdots \quad (3) \end{aligned}$$

n Eqs.	0	1	2	-	-	-
$x_1 = \dfrac{3 + x_2 - x_3}{2}$	-	1.5	0.25	....	1	1
$x_2 = \dfrac{6 - x_1 + x_3}{4}$	0	1.125	2.34	....	2	2
$x_3 = \dfrac{15 + x_1 - 2x_2}{4}$	0	3.625	2.64	....	3	3

Continue until the last two column are identical.

*Convergence at $Max\left|x_i^{k+1} - x_i^{k}\right| \leq \in$

$Max\left|error\right| \leq \in$ Stop iteration

H.W. Write computer program to solve a set of equation by using Gauss-Seidel Iteration.

2.2 Gauss Elimination Method

Forward Elimination and Back substitution

For n = m = 3 .The equations in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Ax = b A: Coefficient matrix

Step (1) Eliminate coefficient of x_1 from equ. (2) & (3)

multiply equ.(1) by $\frac{-a_{21}}{a_{11}}$ and add to equ. (2)

multiply equ.(1) by $\frac{-a_{31}}{a_{11}}$ and add to equ. (3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

Step (2) Eliminate coefficient of x_2 from equ. (3)

multiply equ.(2) by $\frac{-a'_{32}}{a'_{22}}$ and add to equ. (3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a''_{22} & a''_{23} \\ 0 & 0 & a'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b''_2 \\ b''_3 \end{bmatrix}$$

Step (3) Back substitution

$$x_3 = \frac{b''_3}{a''_{33}}$$

$$x_2 = \frac{b'_2 - a'_{22} x_3}{a'_{22}}$$

$$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}$$

Ex. Solve the system by using Gauss elimination method.

$$\begin{aligned} 8x_2 + 2x_3 &= -7 \quad \cdots\cdots\cdots \quad (1) \\ 3x_1 + 5x_2 + 2x_3 &= 8 \quad \cdots\cdots\cdots \quad (2) \\ 6x_1 + 2x_2 + 8x_3 &= 26 \quad \cdots\cdots\cdots \quad (3) \end{aligned}$$

Note.: Pivoting, select the equation with maximum x$_1$ coefficient to eliminate the coefficient in the other equations. (swap eqs. (1), (2) & (3))

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \quad \cdots\cdots\cdots \quad (1) \\ 3x_1 + 5x_2 + 2x_3 &= 8 \quad \cdots\cdots\cdots \quad (2) \\ 8x_2 + 2x_3 &= -7 \quad \cdots\cdots\cdots \quad (3) \end{aligned}$$

Step (1) Eliminate x$_1$ from eq.(2), by multiply $\frac{-3}{6} * (1) + eq.(2)$

$$\begin{bmatrix} 6 & 2 & 8 \\ 3 & 5 & 2 \\ 0 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ -7 \end{bmatrix}$$

Can be rewriting the Augmented matrix $\tilde{A}$

$$\tilde{A} = \left[A \middle| b \right] = \overset{\text{-3/6 *(1)}}{\rightarrow} \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right] \qquad \text{Pivot (6)}$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right] \qquad \text{Swap (2) \& (3) Because } 8 > 4$$

Step (2) Elimination x$_2$ from eq.(3) by multiply $\frac{-4}{8} * (2) + eq.(3)$

$$\text{Pivot (8)} \qquad \overset{\text{-4/8 *(2)}}{\rightarrow} \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}\right]$$

Rewriting in algebraic form

$$\begin{aligned} 6x_1+2x_2+8x_3&=26 \qquad \cdots\cdots\cdots \quad (1)\\ 8x_2+2x_3&=-7 \qquad \cdots\cdots\cdots \quad (2)\\ -3x_3&=-3/2 \qquad \cdots\cdots\cdots \quad (3) \end{aligned}$$

Step (3) Back substitution,

$$x_3=\frac{1}{2}$$

$$x_2=\frac{1}{8}(-7-2x_3)=-1$$

$$x_1=\frac{1}{6}(26-2x_2-8x_3)=4$$

Note: From smaller coefficient of equation can be multiply with factor and after the solution divided it's on the same factor.

$$1000 \quad * \quad \left(0.001x_1+0.02x_2+0.0018x_3=10^{-3}\right) \;\Rightarrow x_1+20x_2+1.8x_3=1$$

2.3 Gauss - Jordan Elimination Method

"Forward & Backward elimination"

Note: Pivoting, selecting the equation with maximum coefficient to eliminate the coefficient in the other equation.

Step (1) Forward elimination the same as in Gauss elimination method.

Step (3) Backward elimination: eliminate all elements above the main diagonal.

Ex. Solve the set of equation using Gauss-Jordan Method

$$\begin{aligned} 2x-y+z&=3 \qquad \cdots\cdots\cdots \quad (1)\\ x+4y-z&=6 \qquad \cdots\cdots\cdots \quad (2)\\ -x+2y+4z&=15 \qquad \cdots\cdots\cdots \quad (3) \end{aligned}$$

Sol.

$$\begin{array}{r} \textbf{-1/2 *eq(1)} \\ \\ \textbf{-1/2 *(1)+eq(3)} \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 4 & -1 & 6 \\ -1 & 2 & 4 & 15 \end{array}\right] \quad \Rightarrow \quad \begin{array}{r} \\ \\ \textbf{-1.5/4.5 *(2)+eq(3)} \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 4.5 & -1.5 & 4.5 \\ 0 & 1.5 & 4.5 & 16.5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 4.5 & -1.5 & 4.5 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

***Back elimination:**

Step (3) eliminate x_3 from eq.(1) & eq.(2)

multiply eq.(3) by $\frac{-1}{5}$ and add to eq. (1)

multiply eq.(3) by $\frac{-1.5}{5}$ and add to eq. (2)

-1/5 *(3)+eq(1)

-1.5/5 *(3)+eq(2)

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 4.5 & -1.5 & 4.5 \\ 0 & 0 & 5 & 15 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

Step (4) eliminate x_2 from eq.(1) by multiply eq. (2) by $\frac{1}{4.5}$ and add to eq. (1)

1/4.5 *eq(2)+eq(1)

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5 & 15 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

Step (5)

$$x = \frac{2}{2} = 1 \quad , \quad y = \frac{9}{4.5} = 2 \quad , \quad z = \frac{15}{5} = 3$$

H.W. Write a computer program to solve N-simultaneous linear equation using Gauss elimination and Gauss-Jordan method.