5. Partial Differential Equations (PDEs)

PDE: These equations contains more than one independent variable.

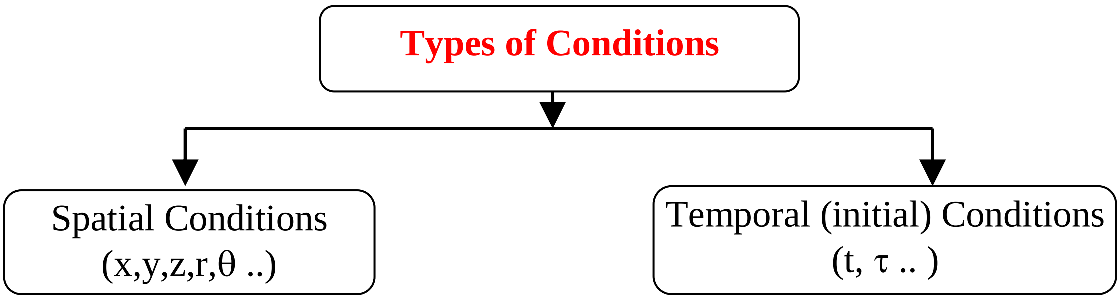
Ex.

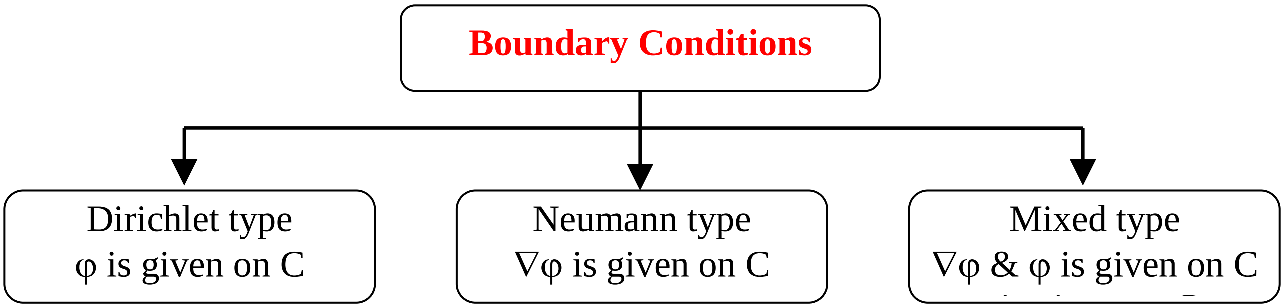
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$

T: dependent variable

x, y, z : Independent var.

$$* \quad \frac{\partial \phi}{\partial x}=\phi_x \quad , \quad \frac{\partial^2 \phi}{\partial x^2}=\phi_{xx} \qquad Or \qquad \frac{\partial \phi}{\partial x}=D_x\phi \quad , \quad \frac{\partial}{\partial x}=D_x$$

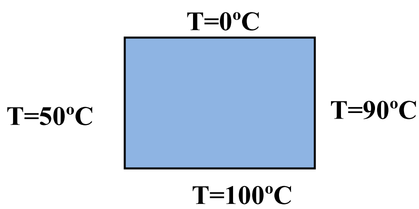




Ex.

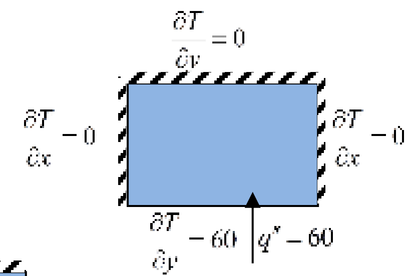
1. Dirichlet type

T is given in C



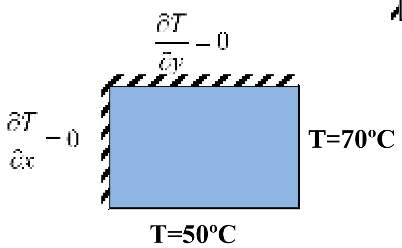
2. Neumann type

∇T is given in C



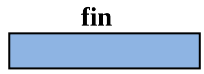
3. Mixed type

T & ∇T is given in C



N.B. Classification of PDE is similar to the ODE.

Type of PDEs:



1. Heat diffusion equation

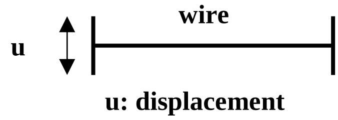
$$\frac{1}{\alpha}u_t = u_{xx}$$ heat diffuse equ. in one dimension

Where; $u_t = \dfrac{\partial u}{\partial t}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ and α : diffusivity

$$\frac{1}{\alpha}u_t = u_{xx} + u_{yy}$$ heat diffuse equ. in two dimension

$$\frac{1}{\alpha}u_t = u_{xx} + u_{yy} + u_{zz}$$ heat diffuse equ. in three dimension

2. Wave equation



$$\frac{1}{c^2}u_{tt} = u_{xx}$$ wave equ. in 1D

Where; c : speed

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy}$$ wave equ. in 2D

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy} + u_{zz}$$ wave equ. in 3D

3. Laplace equation

$u_{xx} + u_{yy} = 0$ Laplace equ. in 2D

$u_{xx} + u_{yy} + u_{zz} = 0$ Laplace equ. in 3D

Or $\nabla^2 u = 0$, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

4. Poisson's equation

$u_{xx} + u_{yy} = f(x, y)$ Poisson's equ. in 2D

$u_{xx} + u_{yy} + u_{zz} = f(x, y)$ Or $\nabla^2 u = f(x, y)$ in 3D

Conic Section Classification

Using for 2nd order, 2 variable, PDE

$$A(x,y)\phi_{xx} + B(x,y)\phi_{xy} + C(x,y)\phi_{yy} = F\left(x, y, \phi_x, \phi_y\right)$$

x, y are any independent variables (Spatial or Temporal)

$$Z = B^2 - 4AC \begin{cases} < 0 & \textit{elliptic eq.} \\ = 0 & \textit{parabolic eq.} \\ > 0 & \textit{hyperbolic eq.} \end{cases}$$

Z:discrimination factor

Elliptic : Usually steady state problem required spatial B.C. only

Ex. Laplace equ.

$$\phi_{xx} + \phi_{yy} = 0 \qquad \therefore A = 1, \quad B = 0, \quad C = 1$$

$$Z = B^2 - 4AC = 0 - 4*1*1 = -4 < 0 \qquad elliptic \quad eq.$$

Poisson's equ. $\phi_{xx} + \phi_{yy} = f(x, y)$ elliptic eq.

Parabolic & hyperbolic equation represent Unsteady state conditions, therefore require both spatial and initial B.C.

Ex. Diffusion equ.

$$\phi_{xx} = \frac{1}{\alpha}\phi_t \qquad \therefore A = 1, \quad B = 0, \quad C = 0$$

$$Z = B^2 - 4AC = 0 - 4*1*0 = 0 \qquad parabolic \quad eq.$$

Ex. wave equ. in 1D

$$\phi_{xx} - \frac{1}{c^2}\phi_{tt} = 0 \qquad \therefore A = 1, \quad B = 0, \quad C = \frac{-1}{c^2}$$

$$Z = B^2 - 4AC = 0 - 4*1*\left(\frac{-1}{c^2}\right) = \frac{4}{c^2} > 0 \qquad hyperbolic \quad eq.$$

Ex. wave equ. 2D

$$\frac{1}{c^2}\phi_{tt} = \phi_{xx} + \phi_{yy}$$

elliptic equ. in (x, y) & hyperbolic eq. in (x, t) , (y , t) coordinates.

SOLUTION OF PDEs;

Separation of Variables or Product Method

In order to use the method of separation of variables we must be working with a linear homogenous partial differential equations with linear homogeneous boundary conditions.

if take wave equation in 1D.

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

Assume $u(x,t) = X(x).T(t)$

$$u_{xx} = X''(x).T(t)$$
$$u_{tt} = X(x).T''(t)$$

Subs. in the diff. eq.

$$X''(x).T(t) = \frac{1}{c^2} X(x).T''(t)$$ divided by $X(x).T(t)$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

The left side is independent of t, but the right side must also be independent of x. Therefore, being independent of both x and t, each side of equation must be a constant.

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \text{constant}\begin{cases} \lambda^2 & \textit{positive con.} \\ 0 & \textit{zero con.} \\ -\lambda^2 & \textit{negative con.} \end{cases}$$

Taking negative constant

$$\frac{X''(x)}{X(x)} = -\lambda^2 \quad \Rightarrow X''(x) + \lambda^2 X(x) = 0 \ \rightarrow \ (D^2 + \lambda^2)X = 0$$

$$(D - i\lambda)(D + i\lambda)X = 0 \quad \Rightarrow \ \therefore X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$$

$$X(x) = c_1(\cos\lambda x + i\sin\lambda x) + c_2(\cos\lambda x - i\sin\lambda x)$$

$$= \underbrace{(c_1 + c_2)}^{=A}\cos\lambda x + i\underbrace{(c_1 - c_2)}^{=B}\sin\lambda x$$

$$\therefore X(x) = A\cos\lambda x + B\sin\lambda x$$

$$\frac{1}{c^2}\frac{T''(t)}{T(t)}=-\lambda^2 \qquad \Rightarrow T''(t)+\lambda^2c^2T(t)=0 \;\rightarrow\; (D^2+\lambda^2c^2)T=0$$

$$(D-i\lambda c)(D+i\lambda c)T=0 \qquad \Rightarrow \;\therefore T(t)=c_1e^{i\lambda ct}+c_2e^{-i\lambda ct}$$

$$\therefore T(t)=E\cos\lambda ct+F\sin\lambda ct$$

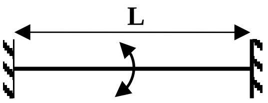
$$or \quad \therefore T(t)=E\cos kt+F\sin kt \qquad where\;\; k=\lambda c$$

In this case the solution

$$u(x,t)=X(x).T(t)=\left(A\cos\lambda x+B\sin\lambda x\right)\left(E\cos\lambda ct+F\sin\lambda ct\right)$$

**** To apply the boundary conditions, there are three cases:**

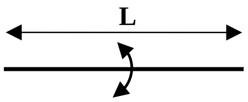
(a)



$u(0,t)=0 \quad , \quad u(x,0)=f(x)$

$u(L,t)=0 \quad , \quad u_t(x,0)=v(x)$

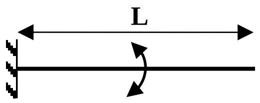
(b)



$u_x(0,t)=0 \quad , \quad u(x,0)=f(x)$

$u_x(L,t)=0 \quad , \quad u_t(x,0)=v(x)$

(c)



$u(0,t)=0 \quad , \quad u(x,0)=f(x)$

$u_x(L,t)=0 \quad , \quad u_t(x,0)=v(x)$

If apply the case (a)

B.C. $\quad u(0,t)=u(L,t)=0$

$$u(x,t)=X(x).T(t)=\left(A\cos\lambda x+B\sin\lambda x\right)\left(E\cos\lambda ct+F\sin\lambda ct\right)$$

(1) apply the $u(0,t)=0$

$$u(0,t)=0=\left(A*1+B*0\right)\left(E\cos\lambda ct+F\sin\lambda ct\right)$$

$$u(0,t)=0=A\left(E\cos\lambda ct+F\sin\lambda ct\right)$$

$\because T(t)\neq 0$ since this will give trivial solution $\qquad \rightarrow \quad \therefore A=0$

$$u(x,t)=B\sin\lambda x\left(E\cos\lambda ct+F\sin\lambda ct\right)$$

$$\therefore u(x,t)=\sin\lambda x\left(E^*\cos\lambda ct+F^*\sin\lambda ct\right)$$ where $E^*=B.E$, $\quad F^*=B.F$

(2) apply the $u(L,t)=0$

$$\therefore u(l,t)=0=\sin\lambda l\left(E^{*}\cos\lambda ct+F^{*}\sin\lambda ct\right)$$

$\because T(t)\neq 0$ since this will give trivial solution $\rightarrow \quad \therefore \sin\lambda L=0$

$$\because \sin\lambda L=0 \;\rightarrow\; \lambda L=n\pi \;\rightarrow \lambda=\frac{n\pi}{L} \;\Rightarrow \therefore \lambda_n=\frac{n\pi}{L} \qquad n=1,\,2,\,3,\cdots$$

$$\therefore u_n(x,t)=\sin\lambda_n x\left(E_n^{\ *}\cos\lambda_n ct+F_n^{\ *}\sin\lambda_n ct\right)$$
$$=\sin\frac{n\pi x}{L}\left(E_n^{\ *}\cos\frac{n\pi ct}{L}+F_n^{\ *}\sin\frac{n\pi ct}{L}\right)$$

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\sin\lambda_n x\left(E_n^{\ *}\cos\lambda_n ct+F_n^{\ *}\sin\lambda_n ct\right)$$

Apply the initial conditions $u(x,0)=f(x)$ *and* $u_t(x,0)=v(x)$

(3) Apply the $u(x,0)=f(x)$

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}\left(E_n^{\ *}\times 1+F_n^{\ *}\times 0\right)\sin\frac{n\pi x}{L}$$

$$\therefore f(x)=\sum_{n=1}^{\infty}E_n^{*}\sin\frac{n\pi x}{L} \qquad\qquad w_n=\frac{n\pi}{p}$$

Apply the four series half-range sine series.

$$E_n^{*}=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

(4) Apply the $u_t(x,0)=v(x)$

$$u_t(x,t)=\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(-E_n^{\ *}\sin\frac{n\pi ct}{L}+F_n^{\ *}\cos\frac{n\pi ct}{L}\right)\frac{n\pi c}{L}$$

$$u_t(x,0)=v(x)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(-E_n^{\ *}\times 0+F_n^{\ *}\times 1\right)\frac{n\pi c}{L}$$

$$v(x)=\sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}F_n^{*}\right)\sin\frac{n\pi x}{L}$$

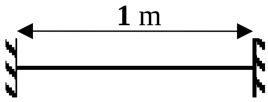
Apply the half-range sine series

$$\frac{n\pi c}{L}F_n^* = \frac{2}{L}\int_0^L v(x)\sin\frac{n\pi x}{L}dx \quad \text{--->} \quad F_n^* = \frac{2}{n\pi c}\int_0^L v(x)\sin\frac{n\pi x}{L}dx$$

H.W. case (b) and (c) find the solution

H.W. Problems P.309 "Wylie"

Ex.



$c = 2 \quad m/s$

$$f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 2x - 0.5 & \frac{1}{4} < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x < 1 \end{cases} \qquad v(x) = f(x) = \begin{cases} 1 & \frac{1}{4} < x < \frac{1}{2} \\ 0 & otherwise \end{cases}$$

$$\therefore \lambda_n = \frac{n\pi}{L} = n\pi$$

$$E_n^* = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = 2\int_{0.25}^{0.5}(2x-0.5)\sin n\pi x dx$$

$$= 4\left[\frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} + \frac{0.25\cos n\pi x}{n\pi}\right]_{0.25}^{0.5}$$

$$=$$

$$F_n^* = \frac{2}{n\pi c}\int_0^L v(x)\sin\frac{n\pi x}{L}dx = \frac{1}{n\pi}\int_{0.25}^{0.5} 1\times\sin n\pi x dx = \frac{1}{n\pi}\left[\frac{-\cos n\pi x}{n\pi}\right]_{0.25}^{0.5}$$

$$= \frac{-1}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right]$$

Separation of Variables for 2D Laplace equation

$$u_{xx}+u_{yy}=0$$

Assume $u(x,t)=X(x).Y(y)$

$$u_{xx}=X''.Y$$
$$u_{yy}=X.Y''$$

Subs. in the diff. eq.

$X''.Y+X.Y''=0$ divided by $X(x).Y(y)$

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\text{constant}\begin{cases}\lambda^2 & \textit{positive con.}\\ 0 & \textit{zero con.}\\ -\lambda^2 & \textit{negative con.}\end{cases}$$

Note: Select the constant dependent on B.C

1- If the symmetric about x-coordinate should be take negative constant $=-\lambda^2$

2- If the symmetric about y-coordinate should be take positive constant $=\lambda^2$

***If taking negative constant**

$$\frac{X''(x)}{X(x)}=-\lambda^2 \quad \Rightarrow X''(x)+\lambda^2X(x)=0 \;\rightarrow\; (D^2+\lambda^2)X=0$$

$$(D-i\lambda)(D+i\lambda)X=0 \quad \Rightarrow \therefore X(x)=c_1e^{i\lambda x}+c_2e^{-i\lambda x}$$

$$X(x)=c_1(\cos\lambda x+i\sin\lambda x)+c_2(\cos\lambda x-i\sin\lambda x)$$

$$=\overbrace{(c_1+c_2)}^{=A}\cos\lambda x+\overbrace{i(c_1-c_2)}^{=B}\sin\lambda x$$

$$\therefore X(x)=A\cos\lambda x+B\sin\lambda x$$

$$-\frac{Y''(y)}{Y(y)}=-\lambda^2 \quad \Rightarrow Y''(y)-\lambda^2Y(y)=0 \;\rightarrow\; (D^2-\lambda^2)Y=0$$

$$(D-\lambda)(D+\lambda)Y=0 \quad \Rightarrow \therefore Y(y)=c_3e^{\lambda y}+c_4e^{-\lambda y}$$

$$Y(y)=c_3(\cosh\lambda y+\sinh\lambda y)+c_4(\cosh\lambda y-\sinh\lambda y)$$

$$=\overbrace{(c_3+c_4)}^{=E}\cosh\lambda y+\overbrace{i(c_3-c_4)}^{=F}\sinh\lambda y$$

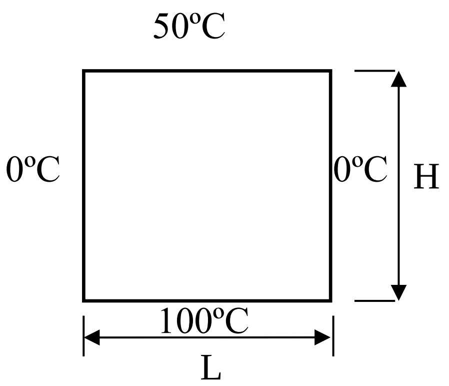
$$\therefore Y(y)=E\cosh\lambda y+F\sinh\lambda y$$

In this case the solution

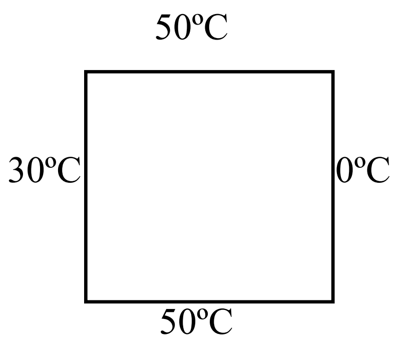
$$u(x,t)=X(x).Y(y)=\left(A\cos\lambda x+B\sin\lambda x\right)\left(E\cosh\lambda y+F\sinh\lambda y\right)$$

H.W. if taking the positive constant $=\lambda^2$ find the solution.

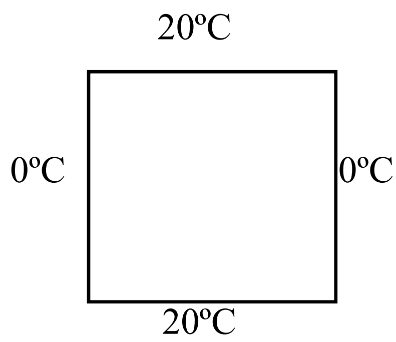
Answer $u(x,t)=\left(A\cosh\lambda x+B\sinh\lambda x\right)\left(E\cos\lambda y+F\sin\lambda y\right)$



Symmetric about x

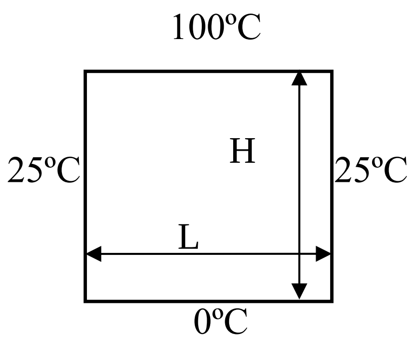


about y



about x or y

Ex. Find the steady state temperature distribution for the following plate shown.



$T_{xx}+T_{yy}=0$

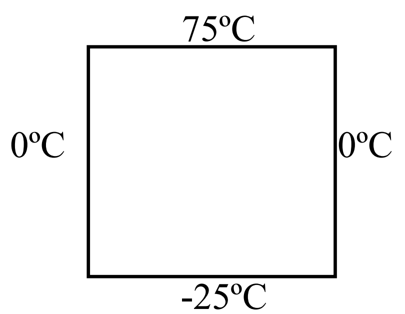
$\phi(x,y)=T(x,y)-25$ satisfy Laplace eq.

$\phi_{xx}+\phi_{yy}=0$

B.C.

(1) $\phi(0,y)=0$, (2) $\phi(L,y)=0$

(3) $\phi(x,0)=-25$, (4) $\phi(x,H)=75$



Assume $\phi(x,t)=X(x).Y(y)$

$\phi_{xx}=X''.Y$, $\phi_{yy}=X.Y''$

$X''.Y+X.Y''=0$ $\quad\%\,X.Y$

$$\frac{X''}{X}=-\frac{Y''}{Y}=\text{constant}=-\lambda^2$$ **because symmetric about x**

$$X'' + \lambda^2 X = 0 \quad \Rightarrow \quad X(x) = A\cos\lambda x + B\sin\lambda x$$

$$Y'' - \lambda^2 Y = 0 \quad \Rightarrow \quad Y(y) = E\cosh\lambda y + F\sinh\lambda y$$

$$\therefore \phi(x,t) = (A\cos\lambda x + B\sin\lambda x)(E\cosh\lambda y + F\sinh\lambda y)$$

Apply the B.C.

B.C (1) $\phi(0,y) = 0$

$$0 = X(0).Y(y) \qquad \therefore Y(y) \neq 0$$

$$\therefore X(0) = 0 = A\cancel{\cos(0)}^{=1} + B\cancel{\sin(0)}^{=0} \quad \Rightarrow \quad \therefore A = 0$$

$$\phi(x,t) = B\sin\lambda x(E\cosh\lambda y + F\sinh\lambda y)$$

$$\therefore \phi(x,t) = \sin\lambda x\left(E^*\cosh\lambda y + F^*\sinh\lambda y\right) \qquad where\ E^* = BE\ ,\ F^* = BF$$

B.C. (2) $\phi(L,y) = 0$

$$0 = \sin\lambda L\left(E^*\cosh\lambda y + F^*\sinh\lambda y\right) \qquad \because \left(E^*\cosh\lambda y + F^*\sinh\lambda y\right) \neq 0$$

$$\therefore \sin\lambda L = 0 \quad \rightarrow \quad \lambda L = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{L}$$

$$\therefore \lambda_n = \frac{n\pi}{L}$$ where n= 1, 2, 3, ...

$$\therefore \phi(x,t) = \sum_{n=1}^{\infty} \sin\lambda_n x\left(E_n^*\cosh\lambda_n y + F_n^*\sinh\lambda_n y\right)$$

B. C. (3) $\phi(x,0) = -25$

$$-25 = \sum_{n=1}^{\infty} \sin\lambda_n x\left(E_n^*\cancel{\cosh(0)}^{=1} + F_n^*\cancel{\sinh(0)}^{=0}\right)$$

$$-25 = \sum_{n=1}^{\infty} E_n^* \sin\frac{n\pi}{L}x \qquad \frac{n\pi}{L} = \omega_n = \frac{n\pi}{p} \quad \rightarrow \therefore p = L \quad \text{(Fourier series)}$$

$$\therefore E_n^* = \frac{2}{L}\int_0^L (-25)\sin\frac{n\pi}{L}x\,dx = \frac{-50}{\cancel{L}}\left[\frac{-\cos\frac{n\pi}{L}x}{\frac{n\pi}{\cancel{L}}}\right]_0^L = \frac{50}{n\pi}[\cos n\pi - 1]$$

$$\therefore E_n^* = \begin{cases} 0 & n: even \\ \dfrac{-100}{n\pi} & n: odd \end{cases}$$

B. C. (4) $\phi(x,H)=75$

$$75=\sum_{n=1}^{\infty}\left(E_n^{*}\cosh\lambda_n H+F_n^{*}\sinh\lambda_n H\right)\sin\frac{n\pi}{L}x \qquad \text{Apply Fourier series}$$

$$\therefore\left(E_n^{*}\cosh\lambda_n H+F_n^{*}\sinh\lambda_n H\right)=\frac{2}{L}\int_0^L(75)\sin\frac{n\pi}{L}x\,dx$$

$$=\frac{150}{n\pi}\left(1-\cos n\pi\right)$$

$$\therefore F_n^{*}=\frac{1}{\sinh\lambda_n H}\left[\frac{150}{n\pi}\left(1-\cos n\pi\right)-\overbrace{\frac{50}{n\pi}\left[\cos n\pi-1\right]}^{E_n^{*}}\cosh\lambda_n H\right]$$

$$\therefore F_n^{*}=\begin{cases}0 & n:even\\ \dfrac{1}{\sinh\lambda_n H}\left[\dfrac{300}{n\pi}+\dfrac{100}{n\pi}\cosh\lambda_n H\right] & n:odd\end{cases}$$

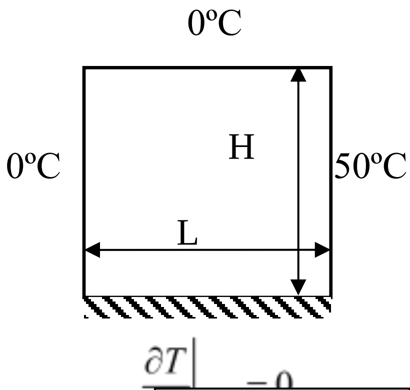
$$\therefore\phi(x,t)=\sum_{n=1}^{\infty}\sin\lambda_n x\frac{1}{n\pi}\left(-100\cosh\lambda_n y+\frac{1}{\sinh\lambda_n H}\left(300+100\cosh\lambda_n H\right)\sinh\lambda_n y\right)$$

$$T(x,y)=\phi(x,y)+25$$

Ex2. Find the steady state temperature distribution for the following information:-

A thin square plate $0 \le x \le L$, $0 \le y \le H$ has its faces insulated. The edges $x = 0,\; y = H,$ maintained at zero temperature. The edge $y = 0$ is insulated and the edge $x = L$ is kept at a temperature 50ºC. (An insulated edge implies that the normal derivative of temperature is zero there).

Sol.



$$T_{xx} + T_{yy} = 0$$

Assume $T(x,t) = X(x).Y(y)$

$$T_{xx} = X''.Y \quad , \quad T_{yy} = X.Y''$$

$$X''.Y + X.Y'' = 0 \qquad \% \; X.Y$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = \lambda^2$$ **because symmetric about y**

B.C

(1) $T(0, y) = 0$

(2) $T(L, y) = 50$

(3) $\dfrac{\partial T(x,0)}{\partial y} = 0$

(4) $T(x, H) = 0$

$$X'' - \lambda^2 X = 0 \qquad \Rightarrow \quad X(x) = A\cosh\lambda x + B\sinh\lambda x$$

$$Y'' + \lambda^2 Y = 0 \qquad \Rightarrow \quad Y(y) = E\cos\lambda y + F\sin\lambda y$$

$$\therefore T(x,t) = \left(A\cosh\lambda x + B\sinh\lambda x\right)\left(E\cos\lambda y + F\sin\lambda y\right)$$

Apply the B.C.

B.C (1) $T(0, y) = 0$

$$0 = X(0).Y(y) \qquad \therefore Y(y) \ne 0$$

$$\therefore X(0) = 0 = A\overbrace{\cosh(0)}^{=1} + B\overbrace{\sinh(0)}^{=0} \qquad \Rightarrow \quad \therefore A = 0$$

$$T(x,t) = B\sinh\lambda x\left(E\cos\lambda y + F\sin\lambda y\right)$$

$$\therefore T(x,t) = \sinh\lambda x\left(E^*\cos\lambda y + F^*\sin\lambda y\right) \qquad \textit{where } E^* = BE\,,\; F^* = BF$$

B.C. (3) $T_y(x, 0) = 0$

$$\frac{\partial T}{\partial y} = \sinh\lambda x\left(-\lambda E^*\sin\lambda y + \lambda F^*\cos\lambda y\right)$$

$$0 = \sinh\lambda x\left(-\lambda E^*\overbrace{\sin(0)}^{=0} + \lambda F^*\overbrace{\cos(0)}^{=1}\right)$$

$$\sinh\lambda x \ne 0 \qquad \therefore \lambda F^* = 0 \qquad \lambda \ne 0 \qquad \therefore F^* = 0$$

$$\therefore T(x,t) = E^*\sinh\lambda x.\cos\lambda y$$

B. C. (4) $T(x,H)=0$

$$0=E^{*}\sinh\lambda x.\cos\lambda H$$

$$E^{*}\neq 0,\qquad \sinh\lambda x\neq 0,\qquad \therefore \cos\lambda H=0\quad \rightarrow \lambda H=\frac{n\pi}{2}\qquad n:odd$$

$$\therefore \lambda_n=\frac{n\pi}{2H}\qquad n:odd\ \ only$$

$$\therefore T(x,t)=\sum_{n=1}^{\infty}E^{*}\sinh\lambda_n x\ \cos\lambda_n y$$ n:odd only

B. C. (2) $T(L,y)=50$

$$50=\sum_{n=1}^{\infty}\left(E^{*}\sinh\lambda_n L\right)\ \cos\frac{n\pi}{2H}y\qquad \omega_n=\frac{n\pi}{p}=\frac{n\pi}{2H}\qquad \therefore p=2H$$

Apply Fourier series

$$\therefore E_n^{*}\sinh\lambda_n L=\frac{\not 2}{\not 2H}\int_0^{2H}(50)\cos\lambda_n y\,dy$$ $Note:\int_0^{2H}=2\int_0^{H}$

$$\therefore E_n^{*}\sinh\lambda_n L=\frac{2\times 50}{H}\int_0^{H}\cos\lambda_n y\,dy=\frac{100}{\lambda_n H}\left(\sin\lambda_n H-0\right)=\frac{200}{n\pi}\sin\frac{n\pi}{2}$$

$$\therefore E_n^{*}=\frac{1}{\sinh\lambda_n L}\frac{200}{n\pi}\sin\frac{n\pi}{2}\qquad n:odd$$

$$\therefore T(x,t)=\sum_{n=1}^{\infty}\frac{200}{n\pi}\sin\frac{n\pi}{2}\frac{\sinh\lambda_n x}{\sinh\lambda_n L}\cos\lambda_n y\qquad \sin\frac{n\pi}{2}=(-1)^n\quad n:odd$$

$$\therefore T(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\sinh\lambda_n x}{\sinh\lambda_n L}\cos\lambda_n y$$

Separation of Variables for 1D Diffuse Equation

$$u_{xx}=\frac{1}{\alpha}u_t$$

Assume $u(x,t)=X(x).T(t)$

$$u_{xx}=X''.T \qquad u_t=X.T'$$

Subs. in the diff. eq.

$X''.T+X.T'=0$ divided by $X.T$

$$\frac{X''(x)}{X(x)}=\frac{1}{\alpha}\frac{T'(t)}{T(t)}=\text{constant}\begin{cases}\lambda^2 & \textit{positive con.}\\ 0 & \textit{zero con.}\\ -\lambda^2 & \textit{negative con.}\end{cases}$$

Taking negative constant

$$\frac{X''}{X}=-\lambda^2 \quad \Rightarrow X''+\lambda^2X=0 \;\rightarrow\; (D^2+\lambda^2)X=0$$

$$(D-i\lambda)(D+i\lambda)X=0 \quad \Rightarrow \therefore X(x)=c_1e^{i\lambda x}+c_2e^{-i\lambda x}$$

$$\therefore X(x)=A\cos\lambda x+B\sin\lambda x$$

$$\frac{1}{\alpha}\frac{T'}{T}=-\lambda^2 \quad \Rightarrow T'+\lambda^2\alpha T=0 \;\rightarrow\; (D^2+\alpha\lambda^2)T=0 \;\rightarrow \therefore T(t)=Ce^{-\alpha\lambda^2t}$$

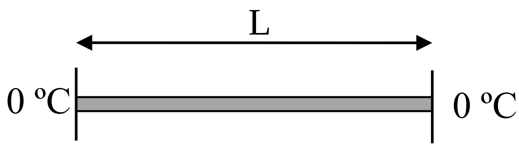
In this case the solution

$$u(x,t)=X.T=\left(A\cos\lambda x+B\sin\lambda x\right)\left(Ce^{-\alpha\lambda^2t}\right)$$

$$\therefore u(x,t)=\left(A^*\cos\lambda x+B^*\sin\lambda x\right)e^{-\alpha\lambda^2t}$$

Ex1. A bar of length L is subjected to the following boundary and initial conditions, find the unsteady state temperature distribution.

$$u_{xx}=\frac{1}{\alpha}u_t$$



B.C	**I.C.**
(1) $u(0,t)=0$ (2) $u(L,t)=0$	$u(x,0)=f(x)=\begin{cases}50 & \frac{L}{4}<x<\frac{L}{2}\\ 0 & otherwise\end{cases}$

From the previous section get the analytic solution as;

$$\therefore u(x,t)=\left(A^*\cos\lambda x+B^*\sin\lambda x\right)e^{-\alpha\lambda^2 t}$$

Apply the B.C.

$$(1)\ u(0,t)=0=\left(A^*\cos(0)+B^*\sin(0)\right)e^{-\alpha\lambda^2 t}=A^*e^{-\alpha\lambda^2 t}$$

$$e^{-\alpha\lambda^2 t}\neq 0 \quad\Rightarrow\quad \therefore A^*=0$$

$$\therefore u(x,t)=B^*\sin\lambda x.e^{-\alpha\lambda^2 t}$$

$$(2)\,u(L,t)=0=B^*\sin\lambda L.e^{-\alpha\lambda^2 t}\qquad\rightarrow B^*\neq 0\quad ,e^{-\alpha\lambda^2 t}\neq 0$$

$$\therefore\sin\lambda L=0\ \rightarrow\ \lambda L=n\pi\quad\Rightarrow\ \therefore\lambda=\frac{n\pi}{L}\quad\rightarrow\lambda_n=\frac{n\pi}{L}\quad n=1,\ 2,\cdots$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty}B_n^*\sin\lambda_n x.e^{-\alpha\lambda_n^2 t}$$

Apply the I.C.

$$\therefore u(x,0)=f(x)=\sum_{n=1}^{\infty}B_n^*\sin\lambda_n x.e^{0}$$

$$f(x)=\sum_{n=1}^{\infty}B_n^*\sin\frac{n\pi}{L}x.$$

Apply Fourier series $B_n^*=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

$$B_n^* = \frac{100}{L}\int_{L/4}^{L/2} \sin\frac{n\pi x}{L}\,dx = \frac{-100}{L}\left[\frac{\cos\frac{n\pi x}{L}}{n\pi / L}\right]_{L/4}^{L/2} = \frac{100}{n\pi}\left[\cos\frac{n\pi}{4} - \cos\frac{n\pi}{2}\right]$$

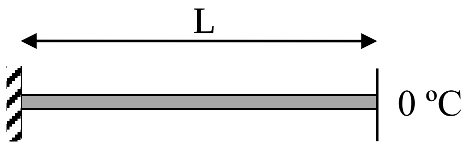
$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{100}{n\pi}\left[\cos\frac{n\pi}{4} - \cos\frac{n\pi}{2}\right] \sin\lambda_n x . e^{-\alpha\lambda_n^2 t}$$

Examples:

B.C.

(1) $u_x(0,t) = 0$

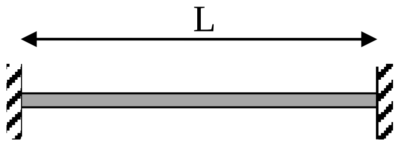
(2) $u(L,t) = 0$



B.C

(1) $u_x(0,t) = 0$

(2) $u_x(L,t) = 0$



Where $u_x = \dfrac{\partial u}{\partial x}$

Ex. The temperature $u(x,t)$ in a strip of metal of width L is governed by the heat equation $\alpha u_{xx} = u_t$ for $0 \le x \le L$ *and* $t > 0$. Use separation of variables method to find the unsteady state temperature distribution in the strip given that the initial condition is $u(x,0) = x$ and the boundary conditions $u_x(0,t) = u(L,t) = 0$ *for* $t > 0$. The condition $u_x(0,t) = 0$ expresses mathematically the constraint that no heat flows through the left boundary (insulated end condition).

-H.W. Problems P.327 "wylie" Q.5,7,8