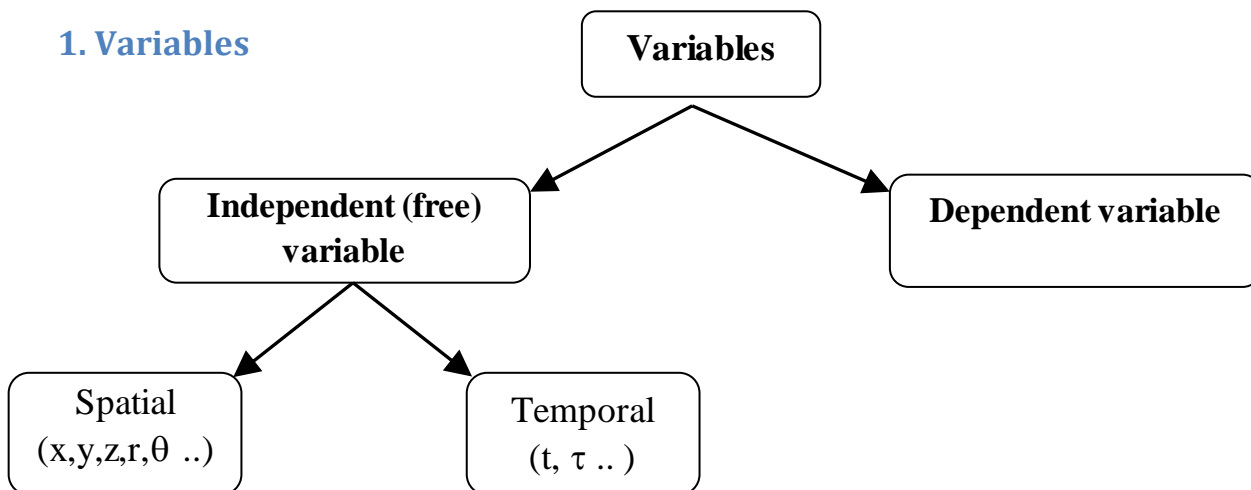


4. Ordinary Differential Equation (ODE)

Differential equation (DE): from defined have derivative and equal sign

Definitions:-

1. Variables



Independent (free) variable: The variable that differential equation is differentiated with respect to. such as x, t, r, θ, ..

Dependent variable: The differentiated variable. such as y, z, u, v

Ex. $\frac{dy}{dx} + y = \sin x$ **y: dependent variable**

$y'' + y' + 2y = e^x$ **x : Independent (free) variable**

ODE: The equation contains one dependent variable & one independent variable.

2. Classification of ODE:

(i) Order: The highest derivative of the dependent variable.

$y'' + 2y' + 3y = 0$ **2nd order**

$\frac{d^3u}{dt^3} + \frac{d^2u}{dt^2} + 3\frac{du}{dt} = \sin t$ **3rd order**

(ii) Degree: The power of the highest derivative of the dep. variable.

$y'' + 2(y')^2 + 3y = e^x$ **2nd order, first degree**

$(y'')^2 + y' + 3y = 0$ **2nd order, second degree**

(iii) Linear: if all terms are 1st or zero degree in the dependent variable.

$y'' + 2y' + x^2y = \sin x$ **2nd order, 1st degree, linear**

$$y'' + 5y' + y^2 = x \quad \text{2nd order, 1st degree, non-linear}$$

(iv) Homogenous: if all terms are the same degree in the dep. variable.

$$y'' + 2y' + x^2y = 0 \quad \text{2nd order, 1st degree, linear, homogenous}$$

$$y'' + 2y' + x^2y = \sin x \quad \text{2nd order, 1st degree, linear, Inhomogenous}$$

Types of Solution of ODE

1. **Particular solution:** A solution that does not contain arbitrary parameters.

2. **General solution:** A solution containing arbitrary parameters.

ex. $y' = 2 \quad \therefore y = 2x$ particular sol.

$$y = 2x + c \quad \text{general sol.} \quad \text{Where } c: \text{arbitrary constant.}$$

Note: The number of arbitrary parameters is equal to the order of DE in complete solution.

3. **Singular solution:** A particular solution that cannot be obtained from a general solution.

4. **Complete solution:** A solution containing all possible solutions including singular ones.

ex. $y'' = 2 \quad \therefore y = x^2$ particular sol.

$$y = x^2 + cx \quad \text{general sol.}$$

$$y = x^2 + c_1x + c_2 \quad \text{complete sol.}$$

1. D-Operator method

$$D = \frac{d}{dx} \text{ or } \frac{d}{dt}, D^2 = \frac{d^2}{dx^2} \text{ or } \frac{d^2}{dt^2}$$

Ex. $y'' + 4y' + y = 0 \quad \Rightarrow \quad (D^2 + 4D + 1)y = 0$ "Auxiliary eq."

$$y'' = \sin x \quad \Rightarrow \quad (D^2)y = \sin x$$

(i) If $(D - m)y = 0 \quad \Rightarrow \quad y = ce^{mx}$

$$y' - my = 0 \quad \rightarrow \quad Dy - my = 0 \quad \rightarrow \quad (D - m)y = 0 \quad \Rightarrow \therefore y = ce^{mx}$$

Ex. $y' + y = 0 \quad \rightarrow \quad (D + 1)y = 0 \quad \rightarrow \quad y = ce^{-x}$ complete sol.

(ii) If $(D - m_1)(D - m_2)y = 0$

$$\text{either } (D - m_1)y = 0 \quad \rightarrow \quad y_1 = c_1e^{m_1x} \quad \text{"general sol."}$$

$$\text{Or } (D - m_2)y = 0 \quad \rightarrow \quad y_2 = c_2e^{m_2x} \quad \text{"general sol."}$$

$$\therefore y = y_1 + y_2 = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{Complete sol.}$$

Ex. $y'' - 3y' - 4y = 0 \rightarrow (D^2 - 3D - 4)y = 0$

$$(D + 1)(D - 4)y = 0 \Rightarrow y = c_1 e^{-x} + c_2 e^{4x}$$

c_1, c_2 can be found from boundary conditions.

(iii) If $(D - a)(D - \bar{a})y = 0$ "complex root"

where $a = \alpha + i\beta, \bar{a} = \alpha - i\beta$

$$\therefore y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

Ex. $\ddot{y} + 4\dot{y} + 5y = 0 \rightarrow (D^2 + 4D + 5)y = 0$

$$m = \frac{-4 \mp \sqrt{16 - 20}}{2} = -2 \mp i \Rightarrow \therefore \alpha = -2, \beta = 1$$

$$\therefore y = e^{-2x} (c_1 \cos x + c_2 \sin x)$$

(iv) If $(D - m)^k y = 0 \Rightarrow y = e^{mx} (c_{k-1} x^{k-1} + c_{k-2} x^{k-2} + \dots + c_0)$

$$(D - m)^2 y = 0 \rightarrow y = e^{mx} (c_1 x + c_0)$$

Ex. $\ddot{y} + 2\dot{y} + y = 0 \rightarrow (D^2 + 2D + 1)y = 0$

$$(D + 1)^2 y = 0 \rightarrow y = e^{-t} (c_1 t + c_0) \quad \text{general sol.}$$

Ex. $y''' - 3y' - 2y = 0 \rightarrow (D^3 - 3D - 21)y = 0$

$$(D - 2)(D + 1)^2 y = 0 \rightarrow y = c_2 e^{2x} + e^{-x} (c_1 t + c_0) \quad \text{general sol.}$$

***Note:** the number of arbitrary parameters is equal to the order of the ODE

D-operator for Inhomogeneous ODE

(I) If $(D - m)y = f(x) \Rightarrow y = y_h + y_p$

y_h : homogenous solution = ce^{mx} , y_p : particular sol.

$$y_p = \frac{1}{(D - m)} f(x) = e^{mx} \int e^{-mx} f(x) dx$$

Ex. $y' - y = e^{2x}$, given that $y(0) = 1$

$$(D-1)y = e^{2x} \quad \Rightarrow \quad y_h = ce^x$$

$$y_p = \frac{1}{(D-1)} e^{2x} = e^x \int e^{-x} e^{2x} dx = e^{2x}$$

$$\therefore y = y_h + y_p = ce^x + e^{2x}$$

$$y(0) = 1 = c + 1 \quad \Rightarrow \quad c = 0$$

$$\therefore y = e^{2x}$$

(II) If $(D - m_1)(D - m_2)y = f(x)$

$$\begin{aligned} y_p &= \frac{1}{(D - m_1)(D - m_2)} f(x) = \frac{1}{(D - m_1)} \left[\frac{1}{D - m_2} f(x) \right] \\ &= \frac{1}{D - m_1} \left[e^{m_2 x} \int e^{-m_2 x} f(x) dx \right]_{=g(x)} \\ &= e^{m_1 x} \int e^{-m_1 x} g(x) dx \end{aligned}$$

Ex. Solve $y'' - 2y' - 3y = x$

$$(D^2 - 2D - 3)y = x \quad \Rightarrow \quad (D + 1)(D - 3)y = x$$

$$y_h = c_1 e^{-x} + c_2 e^{3x}$$

$$y_p = \frac{1}{(D + 1)(D - 3)} f(x) = \frac{1}{(D + 1)} \left[\frac{1}{(D - 3)} x \right] = \frac{1}{(D + 1)} \left[e^{3x} \int e^{-3x} x dx \right]$$

$$y_p = \frac{1}{(D + 1)} \left[e^{3x} \left(\frac{x e^{-3x}}{-3} - \frac{e^{-3x}}{9} \right) \right] = \frac{1}{(D + 1)} \left[\frac{-x}{3} - \frac{1}{9} \right]$$

$$y_p = e^{-x} \int e^x \left(\frac{-x}{3} - \frac{1}{9} \right) dx$$

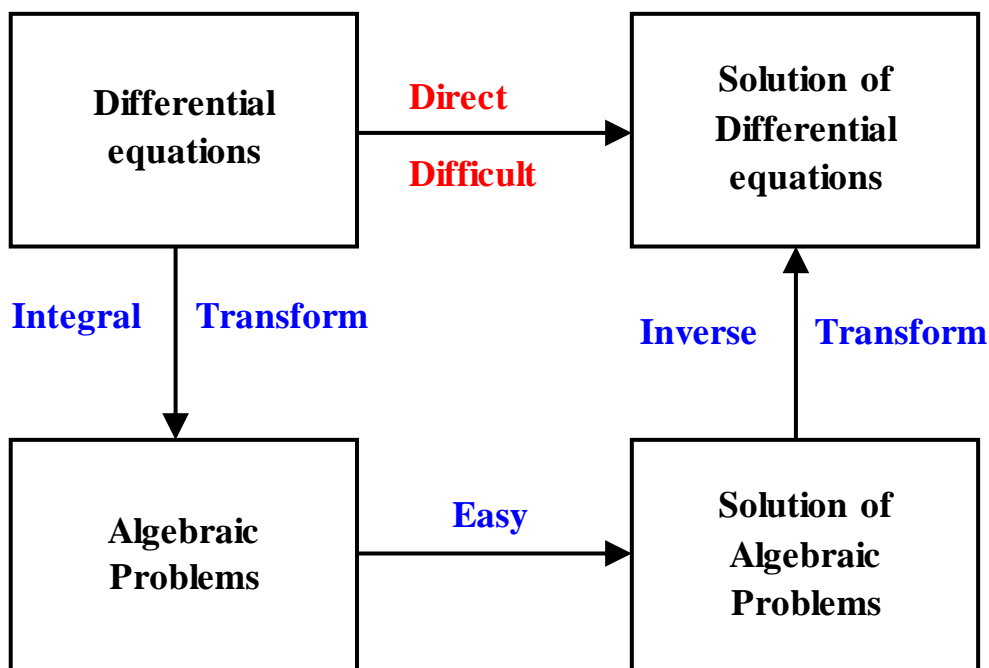
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$$\therefore y_p = \frac{2}{9} - \frac{x}{3}$$

$$\therefore y = y_h + y_p = c_1 e^{-x} + c_2 e^{3x} + \frac{2}{9} - \frac{x}{3}$$

2- Laplace Transform method

Basic Idea of Solving Differential Equations



Steps of Solving Differential Equations

- Step (1):** Take Laplace transform for both sides of the equation
- Step (2):** Apply the given Initial Conditions.
- Step (3):** Rearrange the equation to make $Y(s)$ the subject.
- Step (4):** Take inverse Laplace transform to find $y(t)$

Ex. solve $\ddot{y} + 2\dot{y} + y = \sin 2t$, given that $y(0) = 1$, $\dot{y}(0) = 0$

sol.

$$\left[s^2 Y(s) - s \cancel{y(0)}^{-1} - \cancel{\dot{y}(0)}^{-0} \right] + 2 \left[s Y(s) - \cancel{y(0)}^{-1} \right] + Y(s) = \frac{2}{s^2 + 4}$$

$$\left[s^2 Y(s) - s \right] + 2 \left[s Y(s) - 1 \right] + Y(s) = \frac{2}{s^2 + 4}$$

$$Y(s)(s^2 + 2s + 1) - s - 2 = \frac{2}{s^2 + 4}$$

$$Y(s) = \frac{s + 2}{s^2 + 2s + 1} + \frac{2}{(s^2 + 4)(s^2 + 2s + 1)} = \frac{(s + 2)(s^2 + 4) + 2}{(s^2 + 4)(s^2 + 2s + 1)}$$

$$Y(s) = \frac{(s+2)(s^2+4)+2}{(s^2+4)(s+1)^2}$$

$$= \frac{As+B}{(s-i2)(s+i2)} + \frac{C_2}{(s+1)^2} + \frac{C_1}{s+1}$$

$$(s^2+4) = (s-i2)(s+i2)$$

$$a = \alpha + i\beta$$

$$\therefore \alpha = 0 ; \beta = 2$$

Using the formula for complex root $f(t) = \frac{e^{\alpha t}}{\beta} [C_a \cos \beta t + D_a \sin \beta t]$

$$\therefore R_a(s) = \left[\frac{(s+2)(s^2+4)+2}{(s+1)^2} \right]_{s=i2} = \frac{(i2+2)(-4+4)+2}{(i2+1)^2} = \frac{2}{-3+i4}$$

$$= \frac{2(-3-i4)}{25} = \frac{-6}{25} - i \frac{8}{25}$$

$$\therefore Da = \frac{-6}{25}, \quad Ca = \frac{-8}{25}$$

$$C_2 = \left[\frac{(s+2)(s^2+4)+2}{s^2+4} \right]_{s=-1} = \frac{7}{5}$$

$$C_1 = \frac{1}{1!} \frac{d}{ds} \left[\frac{(s+2)(s^2+4)+2}{s^2+4} \right]_{s=-1} = \frac{29}{25}$$

Taking inverse Laplace transform

$$\therefore y(t) = \frac{1}{2} \left(\frac{-8}{25} \cos 2t - \frac{6}{25} \sin 2t \right) + e^t \left(\frac{7}{5} t + \frac{29}{25} \right)$$

Ex. Find the solution of the equation $\dot{y} + 3y + 2 \int_0^t y dt = 2u(t-1)$ for which $y(0)=1$

Step (1) taking Laplace transform for both sides

$$\left[sY(s) - \cancel{y(0)}^{-1} \right] + 3Y(s) + \frac{2}{s} Y(s) = 2 \frac{e^{-s}}{s} \quad \times s$$

$$Y(s)(s^2 + 3s + 2) = s + 2e^{-s}$$

$$\therefore Y(s) = \frac{s + 2e^{-s}}{(s^2 + 3s + 2)} = \frac{s}{(s+1)(s+2)} + \frac{2e^{-s}}{(s+1)(s+2)}$$

Using Partial fraction for first term

$$\begin{aligned} \frac{s}{(s+1)(s+2)} &= \frac{A}{s+1} + \frac{B}{s+2} \\ &= \frac{-1}{s+1} + \frac{2}{s+2} = \frac{2}{s+2} - \frac{1}{s+1} \end{aligned}$$

$A = \frac{s}{s+2} \Big _{s=-1} = -1$
$B = \frac{s}{s+1} \Big _{s=-2} = 2$

But for the second term

$$\begin{aligned} \frac{1}{(s+1)(s+2)} &= \frac{C}{s+1} + \frac{D}{s+2} \\ &= \frac{1}{s+1} - \frac{1}{s+2} \end{aligned}$$

$C = \frac{1}{s+2} \Big _{s=-1} = 1$
$D = \frac{1}{s+1} \Big _{s=-2} = -1$

$$\therefore Y(s) = \frac{2}{s+2} - \frac{1}{s+1} + 2e^{-s} \left(\frac{1}{s+1} - \frac{1}{s+2} \right)$$

Taking inverse Laplace, we have

$$\therefore y(t) = 2e^{-2t} - e^{-t} + 2(e^{-(t-1)} - e^{-2(t-1)})u(t-1)$$

Ex. Solve the following differential equation using Laplace transform method.

$$y'' + y' - 2y = f(t) \quad \text{given that } y(0) = 0, \quad y'(0) = 0$$

$$\text{Where } f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

Sol.

$$f(t) = e^t [1 - u(t-1)] = e^t - e^t u(t-1) = e^t - e^1 e^{(t-1)} u(t-1)$$

$$F(s) = \frac{1}{s-1} - \frac{e^1 e^{-s}}{s-1}$$

$$y'' + y' - 2y = f(t)$$

Taking Laplace for both sides

$$\left[s^2 Y(s) - s y(0) - y'(0) \right] + \left[s Y(s) - y(0) \right] + 2Y(s) = \frac{1}{s-1} - \frac{e^1 e^{-s}}{s-1}$$

$$Y(s)(s^2 + s - 2) = \frac{1}{s-1} - \frac{e^1 e^{-s}}{s-1}$$

$$\therefore Y(s) = \frac{1}{(s-1)(s^2 + s - 2)} - \frac{e^1 e^{-s}}{(s-1)(s^2 + s - 2)}$$

Therefore $(s^2 + s - 2) = (s - 1)(s + 2)$

$$Y(s) = \frac{1}{(s-1)^2(s+2)} - \frac{e^1 e^{-s}}{(s-1)^2(s+2)}$$

Using Partial fraction

$$\begin{aligned} \frac{1}{(s-1)^2(s+2)} &= \frac{A_2}{(s-1)^2} + \frac{A_1}{s-1} + \frac{B}{s+2} \\ &= \frac{1/3}{(s-1)^2} - \frac{1/9}{s-1} + \frac{1/9}{s+2} \end{aligned}$$

$$\begin{aligned} A_2 &= \left. \frac{1}{(s+2)} \right|_{s=1} = \frac{1}{3} \\ A_1 &= \left[\frac{d}{ds} \frac{1}{(s+2)} \right] \Big|_{s=1} = \frac{-1}{9} \\ B &= \left. \frac{1}{(s-1)^2} \right|_{s=-2} = \frac{1}{9} \end{aligned}$$

$$\therefore Y(s) = \frac{1/3}{(s-1)^2} - \frac{1/9}{s-1} + \frac{1/9}{s+2} - e^1 e^{-s} \left(\frac{1/3}{(s-1)^2} - \frac{1/9}{s-1} + \frac{1/9}{s+2} \right)$$

Taking inverse Laplace, we have

$$\therefore y(t) = \frac{1}{3}te^t - \frac{1}{9}e^t + \frac{1}{9}e^{-2t} - e^1 \left(\frac{1}{3}(t-1)e^{(t-1)} - \frac{1}{9}e^{(t-1)} + \frac{1}{9}e^{-2(t-1)} \right) u(t-1)$$

H.W. Problems in P269 in "Wylie" Q(11-13)

P253 Q(43-47)

P241 Q(6-8)