3. The Laplace Transform

- **Concept of Transforms**

An integral of the form

$$\int_a^b K(s,t)f(t)dt$$ Is called integral transform of f(t).

The function K(s, t) is called kernel of the transform. The parameter (s) belongs to some domain on the real line or in the complex plane.

Choosing different kernels and different values of a and b, we get different integral transforms

Ex.: Laplace, Fourier, Hankel and Mellin transforms.

(i) For $K(s,t)=e^{-st}, a=0, b=\infty$, the improper integral

$$\int_0^\infty e^{-st}f(t)dt$$ **is called Laplace transform of f(t).**

(ii) If we set $K(s,t)=K(\omega,t)=e^{-i\omega t}, a=-\infty, b=\infty$, then

$$\int_{-\infty}^\infty e^{-i\omega t}f(t)dt$$ **is called Fourier transform of f(t).**

Applications: Solution of

- Integral and integro-differential equations
- Ordinary and partial differential equations.

LAPLACE TRANSFORM

$$F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$$ **"Laplace transform"**

$$f(t)=\mathcal{L}^{-1}F(s)=\frac{1}{i\,2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)e^{st}ds$$ **"Complex Inversion formula"**

Ex. Find Laplace transform for the function f(t) = 1

$$F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{1\}=\int_0^\infty e^{-st}.1dt=\left.\frac{e^{-st}}{-s}\right|_0^\infty=\frac{-1}{s}[0-1]=\frac{1}{s}$$

Table (1) Elementary Laplace transforms

f(t)	F(s)	f(t)	F(s)
a	$\frac{a}{s}$	e^{at}	$\frac{1}{s-a}$
t^n	$\begin{cases}\frac{\Gamma(n+1)}{s^{n+1}} & n>-1 \\ \frac{n!}{s^{n+1}} & \text{n a positive integer}\end{cases}$		
$\sin at$	$\frac{a}{s^2+a^2}$	$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$f_1(t)\mp f_2(t)$	$F_1(s)\mp F_2(s)$	$\int_a^t f(t)dt$	$\frac{1}{s}F(s)+\frac{1}{s}\int_a^0 f(t)dt$
$e^{at}f(t)$	$F(s-a)$	$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$

- **Laplace Transform of Derivative**

$$\mathcal{L}\{f'(t)\}=\int_0^\infty e^{-st}.f'(t)dt=e^{-st}f(t)\Big|_0^\infty-\int_0^\infty f(t)(-s)e^{-st}dt$$

$$0-f(0)+s\int_0^\infty f(t)e^{-st}dt=s\mathcal{L}\{f(t)\}-f(0)$$

$$\therefore \mathcal{L}\{f'(t)\}=sF(s)-f(0)$$

Generally:

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots\cdots f^{(n-1)}(0)$$

- **Laplace Transform for integral**

$$\mathcal{L}\left\{\int_a^t f(t)dt\right\} = \int_0^\infty \left(\int_a^t f(t)dt\right)e^{-st}\,dt = \left[\int_a^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty f(t)\left(\frac{e^{-st}}{-s}\right)dt$$

$$= \cancel{\int_a^\infty f(t)dt\left(\frac{e^{-\infty}}{-s}\right)}^{=0} - \int_a^0 f(t)dt\left(\frac{e^0}{-s}\right) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}\int_a^0 f(t)dt + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$\therefore \mathcal{L}\left\{\int_a^t f(t)dt\right\} = \frac{1}{s}F(s) - \frac{1}{s}\int_0^a f(t)dt$$

This can be extended to double, triple and higher integration.

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s)$$

Ex. Find the Laplace transform for the functions

$$1- f(t) = \sin^2 t$$
$$2- f(t) = t^2 e^{2t}$$

1. $f(t) = \sin^2 t \quad , f'(t) = 2\sin t\cos t = \sin 2t$

$$\therefore f'(t) = \sin 2t, \qquad \mathcal{L}\{f'(t)\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \qquad \because f(0) = \sin 0 = 0$$

$$\therefore sF(s) = \frac{2}{s^2+4} \quad \rightarrow \quad \therefore F(s) = \mathcal{L}\{f(t)\} = \frac{2}{s(s^2+4)}$$

$$Or \qquad \sin^2 t = \frac{1}{2}(1-\cos 2t) = f(t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}$$

2. $f(t) = t^2 e^{2t}$

$$\because \mathcal{L}(t^2)=\frac{2!}{s^3} \qquad \rightarrow \qquad \therefore \mathcal{L}\{f(t)\}=\frac{2!}{(s-2)^3}$$

$$Or \ \because \mathcal{L}(e^{2t})=\frac{1}{s-2} \qquad \rightarrow \qquad \mathcal{L}\{t^2e^{2t}\}=(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s-2}\right)=\frac{2}{(s-2)^3}$$

Ex. Find the Laplace transform for $\int_0^t \cos 4t \, dt$

$$\because \ \mathcal{L}\{\cos 4t\}=\frac{s}{s^2+16}$$

$$\therefore \mathcal{L}\left\{\int_0^t \cos 4t \, dt\right\}=\frac{1}{s}\cdot\frac{s}{s^2+16}=\frac{1}{s^2+16}$$

$$Or \ \int_0^t \cos 4t \, dt=\frac{\sin 4t}{4} \qquad \rightarrow \qquad \mathcal{L}\left\{\frac{\sin 4t}{4}\right\}=\frac{1}{4}\cdot\frac{4}{s^2+16}=\frac{1}{s^2+16}$$

Ex. Find the inverse Laplace transform for $F(s)=\frac{1}{s(s^2-4)}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-4)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{2}\cdot\frac{2}{s^2-4}\right\}$$

$$=\int_0^t \frac{1}{2}\sinh 2t \, dt=\frac{1}{2}\left[\frac{\cosh 2t}{2}\right]_0^t=\frac{1}{4}(\cosh 2t-1)$$

Note: if case of ($1/s^2$) using double integral; become $\int_0^t\left(\int_0^t \frac{1}{2}\sinh 2t \, dt\right)$

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H.W (1) Find Laplace transform for the following function;

$1-f(t)=t\sin\omega t$	$4-f(t)=e^{2t}\sin 4t$
$2-f(t)=t\cosh at$	$5-f(t)=\sin 2t-t^2e^{-2t}$
$3-f(t)=t^2\cos 2t$	$6-\int_2^t \sin t \, dt$

(2) Find Inverse Laplace transform for;

$$1-F(s)=\frac{1}{s(s+1)}$$

$$2-F(s)=\frac{3}{s^2+4}$$

$$3-F(s)=\frac{3}{s(s^2+16)}$$

$$4-F(s)=\frac{2}{s^2(s+4)}$$

$$5-F(s)=\frac{1}{s}\cdot\left(\frac{s-1}{s+1}\right)$$

* First shifting theorem, s-shifting

If f(t) has the transform F(s) (where s >k), then $e^{at}f(t)$ has the transform F(s - a) (where s - a >k).

$$\mathcal{L}\{e^{at}f(t)\}=F(s-a)$$

Proof: $$\mathcal{L}\{e^{at}f(t)\}=\int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt=\int_0^\infty e^{-(s-a)t}f(t)dt=F(s-a)$$

We can show that

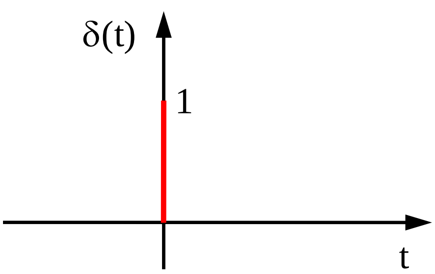
$$\mathcal{L}\left(e^{-at}\cos bt\right)=\frac{s+a}{(s+a)^2+b^2}\qquad,\ \mathcal{L}\left(e^{-at}\sin bt\right)=\frac{b}{(s+a)^2+b^2}$$

$$\mathcal{L}\left(e^{-at}t^n\right)=\frac{n!}{(s+a)^{n+1}}$$

*Unit Step Function (Heaviside Function):-

Unit impulse function

$$\delta(t)=\begin{cases}1 & t=0\\ 0 & t\neq 0\end{cases}$$

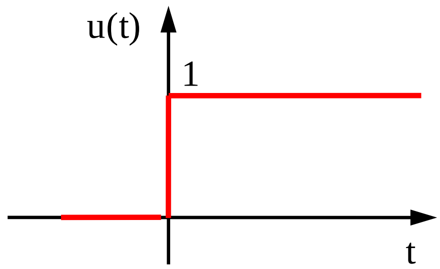


$\Delta(t)=A\delta(t)$ **Dirac delta function**

$$\Delta(t)=\begin{cases}A & t=0\\ 0 & t\neq 0\end{cases}$$

$$\mathcal{L}(\delta(t))=1\qquad\rightarrow\quad \mathcal{L}(\Delta(t))=A$$

Unit step function, u(t)



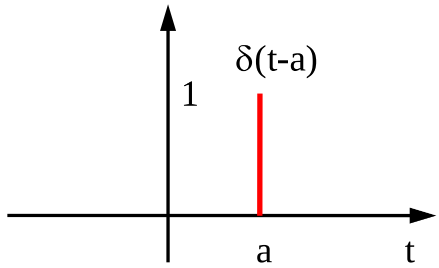
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \le 0 \end{cases}$$

Step Function (Heaviside function)

$$H(t) = Au(t) = \begin{cases} 0 & t < 0 \\ A & t \le 0 \end{cases}$$

$$\mathcal{L}\left(u(t)\right) = \int_0^\infty 1.e^{-st}dt = \frac{1}{s} \qquad \therefore \mathcal{L}\left(H(t)\right) = \frac{A}{s}$$

Shifting (Translated) functions



$$\delta(t-a) = \begin{cases} 1 & t = a \\ 0 & t \ne a \end{cases}$$

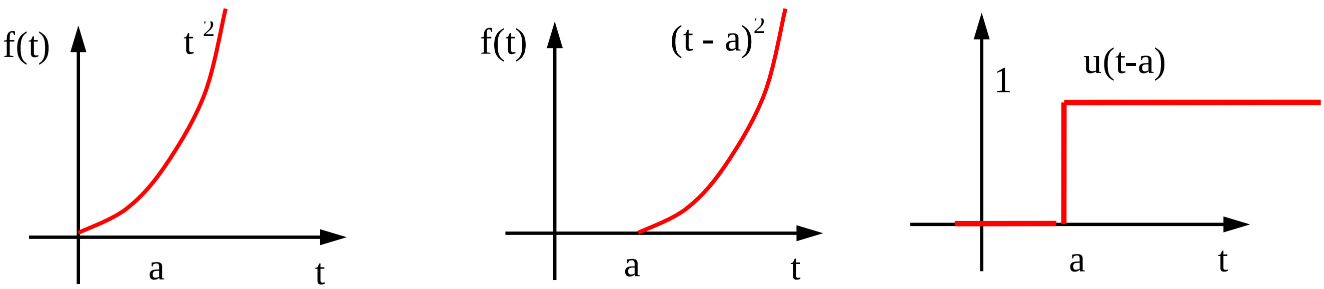
$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \ge a \end{cases}$$

$$\mathcal{L}\left(u(t-a)\right) = \int_0^\infty u(t-a)e^{-st}dt = \int_0^a 0.dt + \int_a^\infty 1.e^{-st}dt$$

$$= \frac{-1}{s}\left[e^{-st}\right]_a^\infty = \frac{-1}{s}\left[0 - e^{-as}\right] = \frac{e^{-as}}{s}$$

Laplace transform of shifting functions

For example;



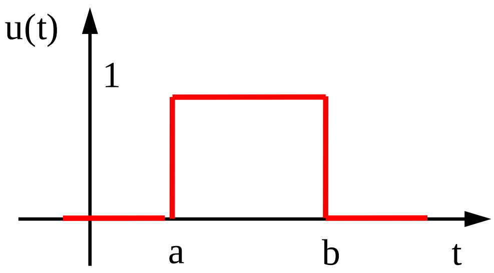
If we take the expression:

$$(t-a)^2 u(t-a) = \begin{cases} 0 & t > a \\ \left(t-a\right)^2 & t > a \end{cases}$$

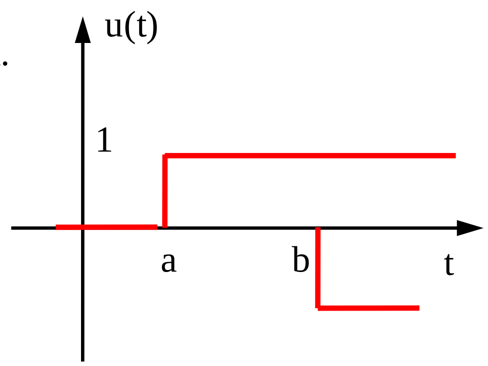
More generally , the expression

$$f\left(t-a\right)u(t-a)$$

Ex. What is the equation of the function whose graphs in shown below;

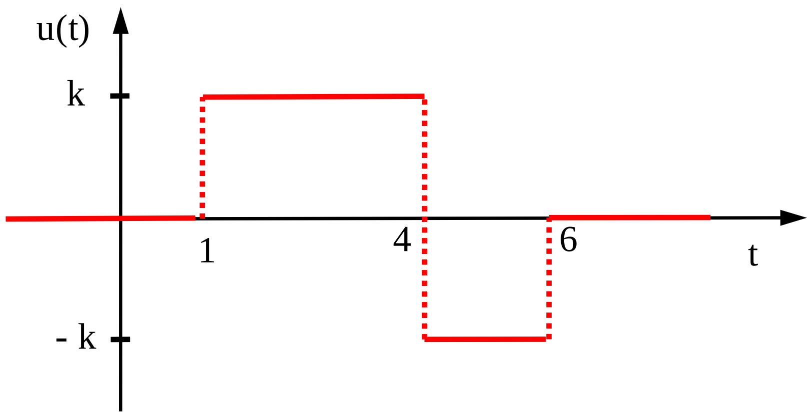


Sol.



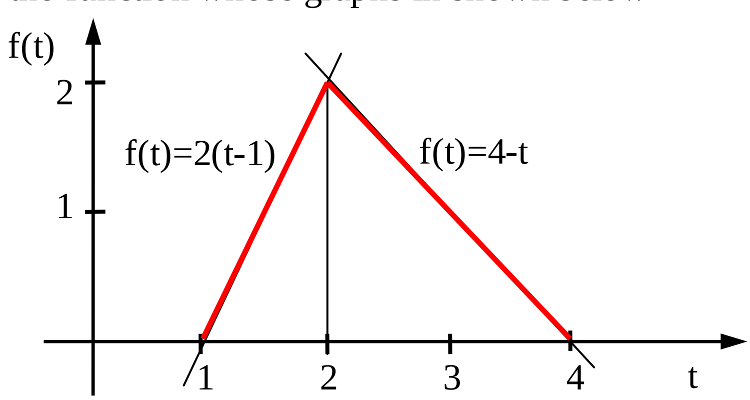
$$\therefore u(t-a)-u(t-b)$$

Ex.



$$k\left[u(t-1)-2u(t-4)+u(t-6)\right]$$

Ex. What is the equation of the function whose graphs in shown below



The unit step function between 1 and 2 is $u(t-1)-u(t-2)$

$\therefore$ The function between 1 and 2 ;

$$\therefore 2(t-1)\left[u(t-1)-u(t-2)\right]$$

The unit step function between 2 and 4 is $u(t-2)-u(t-4)$

$\therefore$ The function between 2 and 4 ;

$$\therefore (-t+4)\left[u(t-2)-u(t-4)\right]$$

The complete representation of the function is therefore;

$$2(t-1)\left[u(t-1)-u(t-2)\right]+(-t+4)\left[u(t-1)-u(t-2)\right]$$
$$=2(t-1)u(t-1)-3(t-2)u(t-2)+(t-4)u(t-4)$$

Time Shifting (t-Shifting): Replacing t by t - a in f(t)

if $\mathcal{L}\{f(t)\}=F(s)$

$\mathcal{L}\left\{e^{at}f(t)\right\}=F(s-a)$ **(s-Shifting) "first shifting theorem"**

$\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)$ **(t-Shifting) "Second shifting theorem"**

$$\mathcal{L}\{f(t)u(t-a)\}=e^{-as}\mathcal{L}\{f(t+a)\}$$

Or, we can write

$$f(t-a)u(t-a)=\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}$$

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\}=\int_0^\infty f(t-a)u(t-a)e^{-st}dt=\int_a^\infty f(t-a)e^{-st}dt$$

Now let $t-a=T, \quad dt=dT$, then

$$\int_0^\infty f(T)e^{-s(T+a)}dT=e^{-as}\int_0^\infty f(T)e^{-sT}dT=e^{-as}F(s)$$

Ex. Find $\mathcal{L}\{\cos(t-1)u(t-1)\}$

$$\mathcal{L}\{\cos(t-1)u(t-1)\} = e^{-s}\frac{s}{s^2+1} \qquad \text{where } f(t)=\cos t \to F(s)=\frac{s}{s^2+1}$$

Ex. Find $\mathcal{L}\{\sin(2t-2)u(t-1)\}$

$$F(t)=\sin(2t-2)u(t-1)=\sin 2(t-1)u(t-1)$$

$$\therefore \mathcal{L}\{F(t)\}=e^{-s}\frac{2}{s^2+4} \qquad where\ \mathcal{L}\{\sin 2t\}=\frac{2}{s^2+4}$$

Ex. Find $F(t)=\cos(t-1)u(t-2)$

$$F(t)=\cos(t-2+1)u(t-2)$$
$$=\left[\cos(t-2)\cos 1-\sin(t-2)\sin 1\right]u(t-1)$$

$$\therefore \mathcal{L}\{F(t)\}=\left[\frac{s\cos 1}{s^2+1}-\frac{\sin 1}{s^2+1}\right]e^{-2s}$$

Ex. Write the following using unit step functions and find its transform

$$f(t)=\begin{cases} 2 & if\ 0<t<1 \\ \frac{1}{2}t^2 & if\ 1<t<\frac{1}{2}\pi \\ \cos t & if\ \ t>\frac{1}{2}\pi \end{cases}$$

Sol. Step (1) In terms of unit step functions

$$F(t)=2[1-u(t-1)]+\frac{1}{2}t^2\left[u(t-1)-u(t-\frac{1}{2}\pi)\right]+(\cos t)u(t-\frac{1}{2}\pi)$$

Step (2) Find inverse Laplace transform

$$\therefore \mathcal{L}\{2[1-u(t-1)]\}=2\left(1-e^{-s}\right)/s$$

Using the formula $\mathcal{L}\{f(t)u(t-a)\}=e^{-as}\mathcal{L}\{f(t+a)\}$

$$\therefore \mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\}=e^{-2}\mathcal{L}\left\{\frac{1}{2}t^2+t+\frac{1}{2}\right\}=e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)$$

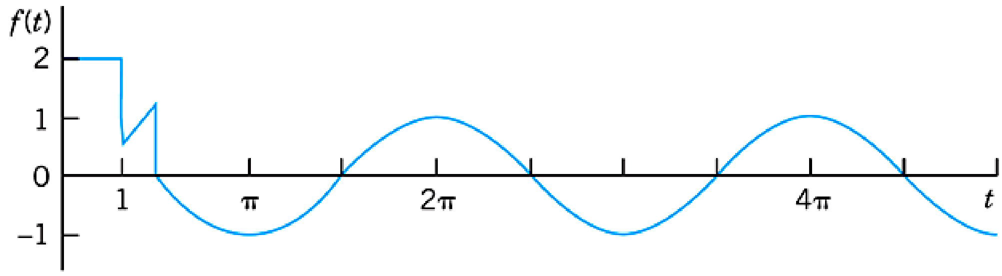
Similarly;

$$\therefore \mathcal{L}\left\{\frac{1}{2}t^2u(t-\frac{\pi}{2})\right\} = e^{-\pi s/2}\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)$$

$$\mathcal{L}\left\{\cos t u(t-\frac{\pi}{2})\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin(t)\} = -e^{-\pi s/2}\left(\frac{1}{s^2+1}\right)$$

Finally;

$$\therefore \mathcal{L}\{F(t)\} = \frac{2}{s}-\frac{2}{s}e^{-s}+e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)+e^{-\pi s/2}\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)-e^{-\pi s/2}\left(\frac{1}{s^2+1}\right)$$



* Differentiation of Transforms ($t^n f(t)$)

$$\mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^nF(s)}{ds^n}$$

Proof: $F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt \quad \Rightarrow \quad F'(s) = -\int_0^\infty f(t)te^{-st}dt$

Consequently, then

$\mathcal{L}(tf(t)) = -F'(s)$ hence $\mathcal{L}^{-1}(F'(s)) = -tf(t)$

Ex. Find the inverse transform of $\ln\left(1+\frac{b^2}{s^2}\right)$

Sol. $F(s) = \ln\left(1+\frac{b^2}{s^2}\right) = \ln\left(\frac{s^2+b^2}{s^2}\right)$

$$F'(s) = \frac{d}{ds}\left[\ln\left(s^2 - b^2\right) - \ln s^2\right] = \frac{2s}{s^2 + b^2} - \frac{2s}{s^2}$$

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2 + b^2} - \frac{2}{s}\right\} = 2\cos bt - 2 = -tf(t)$$

$$\therefore f(t) = \frac{2}{t}(1 - \cos bt)$$

* Integration of Transforms.

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds \qquad hence \qquad \mathcal{L}^{-1}\left\{\int_s^\infty F(s)ds\right\} = \frac{f(t)}{t}$$

Proof: $\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds$

We may reverse the order of integration, that is,

$$\int_s^\infty F(s)ds = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)ds\right] dt = \int_0^\infty \left[f(t)\int_s^\infty e^{-st} ds\right] dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Ex. What is $\mathcal{L}\left\{\frac{\sin kt}{t}\right\}$?

Sol. $\because f(t) = \sin kt$

$$\mathcal{L}\left\{\frac{\sin kt}{t}\right\} = \int_s^\infty \mathcal{L}\{\sin kt\} ds = \int_s^\infty \frac{k}{s^2 + k^2} ds = \tan^{-1}\frac{k}{s}\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{k}{s} = \cot^{-1}\frac{k}{s}$$

Ex. What is $y \text{ if } \mathcal{L}\{y\} = \frac{s}{(s^2-1)^2}$?

$$y = t\,\mathcal{L}^{-1}\left\{\int_s^\infty \frac{s}{(s^2-1)^2}ds\right\} = t\,\mathcal{L}^{-1}\left\{\left.\frac{-1}{2(s^2-1)}\right|_s^\infty\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\left(0+\frac{1}{s^2-1}\right)\right\} = t\,\mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\right\}$$

$$= \frac{t}{4}\left(e^t - e^{-t}\right) = \frac{t\sinh t}{2}$$

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*Transforms of periodic functions:-

If f(t) is a piecewise regular function is periodic with period a, that is

$$F(t+a) = F(t) \qquad t > 0$$

Then its direct Laplace transform is given by;

$$\mathcal{L}\{f(t)\} = \frac{\int_0^a f(t)e^{-st}dt}{1-e^{-as}}$$

Proof:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^a f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt + \int_{2a}^{3a} f(t)e^{-st}dt + \cdots$$

Now, in the 2nd integral, let t = T + a; in the 3rd integral let t = T +2a, in general let t = T + na, and dt = dT, then

$$\mathcal{L}\{f(t)\} = \int_0^a f(T)e^{-sT}dT + \int_0^a f(T+a)e^{-s(T+a)}dT + \int_0^a f(T+2a)e^{-s(T+2a)}dT + \cdots$$

$$= \int_0^a f(T)e^{-sT}dT + e^{-as}\int_0^a f(T+a)e^{-sT}dT + e^{-2as}\int_0^a f(T+2a)e^{-sT}dT + \cdots$$

But $f(T+a) = f(T+2a) = \cdots = f(T+na) = \cdots = f(T)$ for all values of T

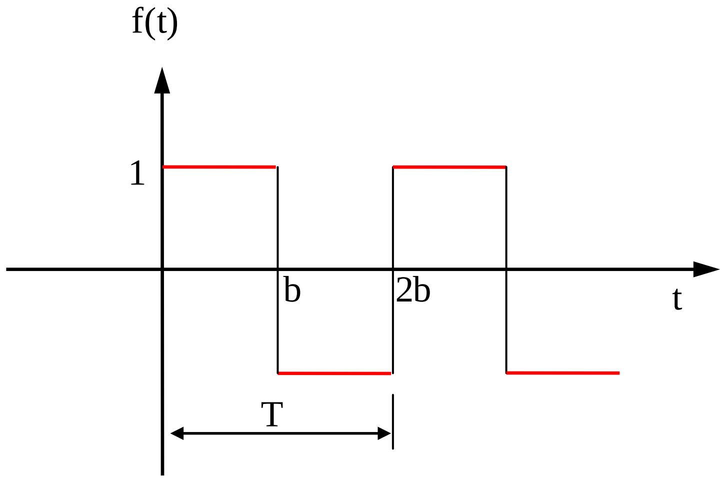
$$\mathcal{L}\{f(t)\}=\int_0^a f(T)e^{-sT}dT+e^{-as}\int_0^a f(T)e^{-sT}dT+e^{-2as}\int_0^a f(T)e^{-sT}dT+\cdots$$

$$=(1+e^{-as}+e^{-2as}+\cdots)\int_0^a f(T)e^{-sT}dT$$

The infinite geometric series $(1+e^{-as}+e^{-2as}+\cdots)$ have the sum $S=1/(1-r)$, where the ratio r is e^{-as}

$$\mathcal{L}\{f(t)\}=\left(\frac{1}{1-e^{-as}}\right)\int_0^a f(t)e^{-st}dt$$

Ex. Find the transform of the rectangular wave shown in below fig.



Sol.

The period is 2b

$$\mathcal{L}\{f(t)\}=\left(\frac{1}{1-e^{-as}}\right)\int_0^a f(t)e^{-st}dt$$

$$=\left(\frac{1}{1-e^{-2bs}}\right)\int_0^{2b} f(t)e^{-st}dt$$

$$=\left(\frac{1}{1-e^{-2bs}}\right)\left[\int_0^b 1.e^{-st}dt+\int_b^{2b} -1.e^{-st}dt\right]$$

$$=\left(\frac{1}{1-e^{-2bs}}\right)\left[\frac{1-2e^{-bs}+e^{-2bs}}{s}\right]=\frac{\left(1-e^{-bs}\right)^2}{s\left(1-e^{-bs}\right)\left(1+e^{-bs}\right)}=\frac{1-e^{-bs}}{s\left(1+e^{-bs}\right)}$$

H.W. Problems in P269 in "Wylie"

Partial Fraction method:

$$F(s)=\frac{Y(s)}{Q(s)}$$

In condition Y(s) has less degree then Q(s)

1. Unrepeated or simple root (s - a)

$$F(s)=\frac{Y(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots$$

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=A_1e^{a_1t}+A_2e^{a_2t}+\cdots$$

$$A_1=\lim_{s\to a_1}\left\{(s-a_1)\frac{Y(s)}{Q(s)}\right\}=\left[(s-a_1)\frac{Y(s)}{Q(s)}\right]_{s=a_1}$$

$$A_2=\lim_{s\to a_2}\left\{(s-a_2)\frac{Y(s)}{Q(s)}\right\}=\left[(s-a_2)\frac{Y(s)}{Q(s)}\right]_{s=a_2}$$

Ex. Find $\mathcal{L}^{-1}$ $for \quad F(s)=\dfrac{s-2}{s^2+5s+6}$

Sol. $F(s)=\dfrac{s-2}{s^2+5s+6}=\dfrac{s-2}{(s+2)(s+3)}=\dfrac{A}{s+2}+\dfrac{B}{s+3}$

$$A=\lim_{s\to -2}\left\{\cancel{(s+2)}\frac{s-2}{\cancel{(s+2)}(s+3)}\right\}=\left[\frac{s-2}{s+3}\right]_{s=-2}=-4$$

$$A=\lim_{s\to -2}\left\{\cancel{(s+3)}\frac{s-2}{(s+2)\cancel{(s+3)}}\right\}=5$$

$$\therefore f(t)=-4e^{-2t}+5e^{-3t}$$

2. Repeated roots $(s - a)^m$, where m positive integer

$$F(s)=\frac{Y(s)}{Q(s)}=\frac{A_m}{(s-a)^m}+\frac{A_{m-1}}{(s-a)^{m-1}}+\frac{A_{m-2}}{(s-a)^{m-2}}+\cdots+\frac{A_1}{(s-a)}$$

$$f(t)=e^{at}\left[A_m\frac{t^{m-1}}{(m-1)!}+A_{m-1}\frac{t^{m-2}}{(m-2)!}+A_{m-2}\frac{t^{m-3}}{(m-3)!}+\cdots+A_1\right]$$

$$A_m=\lim_{s\to a}\left\{(s-a)^m\frac{Y(s)}{Q(s)}\right\}=R_a(s)\Big|_{s=a_1}$$

$$A_{m-1}=\frac{1}{1!}\lim_{s\to a}\left\{\frac{d}{ds}R_a(s)\right\} \qquad \therefore R_a(s)=(s-a)^m\frac{Y(s)}{Q(s)}$$

* For General:

$$A_{m-k} = \frac{1}{k!}\lim_{s\to a}\left\{\frac{d^k}{ds^k}R_a(s)\right\} \qquad where \quad k = 1,\ 2,\ 3,\cdots(m-1$$

Ex. Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2(s-2)}\right\}$

Sol. $F(s) = \frac{s+2}{(s+1)^2(s-2)} = \frac{A_2}{(s+1)^2} + \frac{A_1}{(s+1)} + \frac{B}{s-2}$

$$R_a(s) = \cancel{(s+1)^2}\,\frac{s+2}{\cancel{(s+1)^2}\,(s-2)} = \frac{s+2}{s-2}$$

$$A_2 = \lim_{s\to -1}\{R_a(s)\} = \left[\frac{s+2}{s-2}\right]_{s=-1} = \frac{-1}{3}$$

$$A_1 = \frac{1}{1!}\lim_{s\to -1}\left\{\frac{d}{ds}R_a(s)\right\} = \left[\frac{d}{ds}\left(\frac{s+2}{s-2}\right)\right]_{s=-1} = \left[\frac{(s+2)-s-2}{(s-2)^2}\right]_{s=-1} = \frac{-4}{9}$$

$$B = \lim_{s\to 2}\left\{\cancel{(s-2)}\,\frac{s+2}{(s+1)^2\,\cancel{(s-2)}}\right\} = \left[\frac{s+2}{(s+1)^2}\right]_{s=-1} = \frac{4}{9}$$

$$\therefore F(s) = \frac{-1/3}{(s+1)^2} - \frac{4/9}{(s+1)} + \frac{4/9}{s-2}$$

$$f(t) = e^{-t}\left(\frac{-1}{3}\frac{t}{1!} - \frac{4}{9}\right) + \frac{4}{9}e^{2t}$$

3. Unrepeated complex root $(s-a)(s-\bar{a})$

where a is a complex number $a = \alpha + i\beta$, $\bar{a} = \alpha - i\beta$

$$F(s)=\frac{Y(s)}{Q(s)}=\frac{As+B}{(s-a)(s-\bar{a})}$$

$$f(t)=\frac{e^{\alpha t}}{\beta}[C_a\cos\beta t+D_a\sin\beta t]$$

where

C_a *is imaginary part of* $R_a(s)|_{s=a}$

D_a *is real part of* $R_a(s)|_{s=a}$

$$\because R_a(s)=\left[(s-a)(s-\bar{a})\frac{Y(s)}{Q(s)}\right]$$

Ex. Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+2s+5)(s+1)}\right\}$

s^2+2s+5

$s=\frac{-2\mp\sqrt{4-4*5}}{2}$

$s=-1\mp i2$

$\therefore \alpha=-1$

$\beta=2$

Sol. $F(s)=\frac{As+B}{s^2+2s+5}+\frac{C}{s+1}$

$$F(s)=\frac{As+B}{(s+1-i2)(s+1+i2)}+\frac{C}{s+1}$$

$$R_a(s)|_{s=-1+i2}=\left[\frac{s+2}{s-1}\right]_{s=-1+i2}=\frac{1+i2}{i2}=1-\frac{i}{2}$$

$$\therefore C_a=\frac{-1}{2}\quad,\quad D_a=1$$

$$C=\lim_{s\to-1}\left\{\frac{s+2}{s^2+2s+5}\right\}=R_c(s)|_{s=-1}=\frac{1}{4}$$

$$\therefore f(t)=\frac{e^{-t}}{2}(-0.5\cos 2t+\sin 2t)+\frac{1}{4}e^{-t}$$

H.W. Find $\mathcal{L}^{-1}$ for the following functions:

1. $F(s)=\frac{s+9}{2s^2+s-1}$
2. $F(s)=\frac{s+1}{s^3(s-1)}$
3. $F(s)=\frac{s^2+3}{(s-1)(s+1)^2}$
4. $F(s)=\frac{s^2+2s+3}{(s^2+2s+6)(s-3)}$