2. Fourier Series and Transform

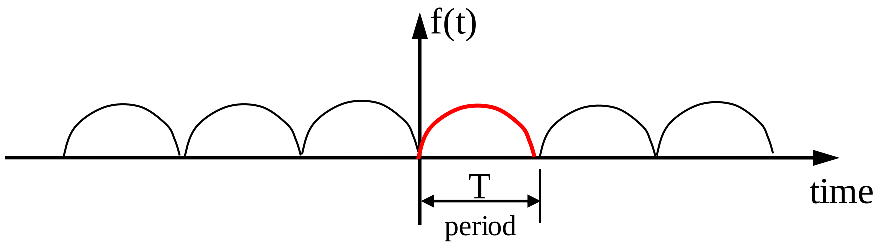
- **To know Fourier series should be known three points:**
  - **I. Defined of Periodic function.**
  - **II. Types of Periodic function.**
  - **III. What formula using to calculated constants.**

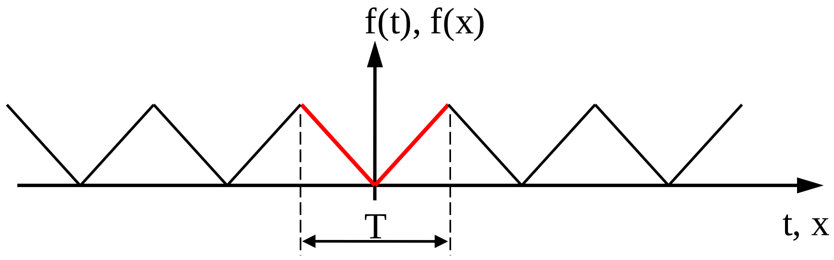
I. **Periodic function**: The function, which repeats itself each "T" second, where "T" is called period.

هي دوال بتكرر نفسها (نفس الشكل او الصورة) كل فترة معينة من الزمن (تسمى Period ويرمز لها "T")

ويعبر عن Periodic function بشكل رياضي $f(t+T) = f(t)$

Example: The function sin(x) has period 2π , since $\sin(x+2\pi) = \sin(x)$.

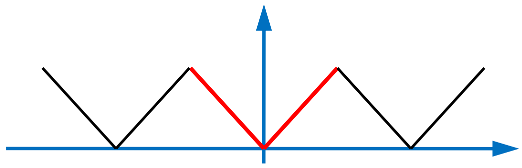


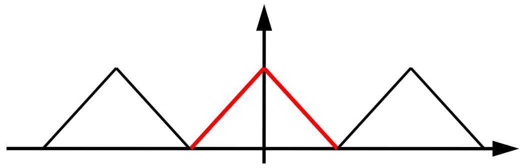


II. Types of Periodic Function

<u>ملاحظة مهمة</u>: يجب ان نرسم الدالة الدورية لكي نحدد نوعها

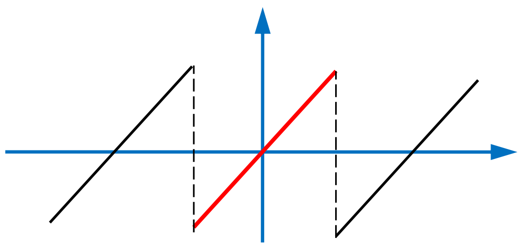
(1) Even Function $f(-t) = f(t)$

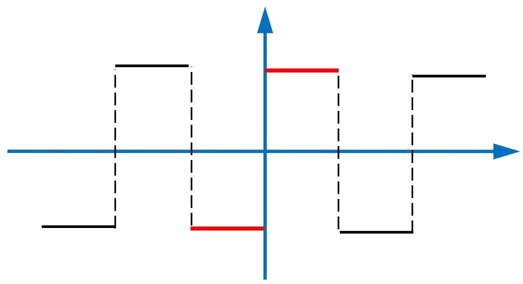




الدالة الزوجية : متماثلة حول محور y

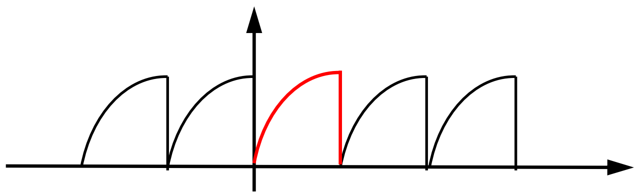
(2) Odd Function $f(-t) = -f(t)$

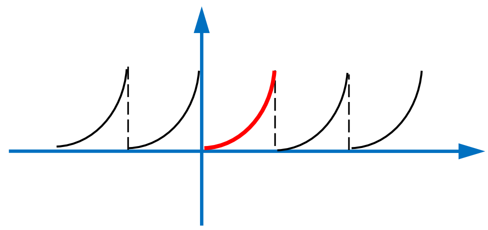




الدالة الفردية : متماثلة حول نقطة الاصل

(3) Neither even nor odd

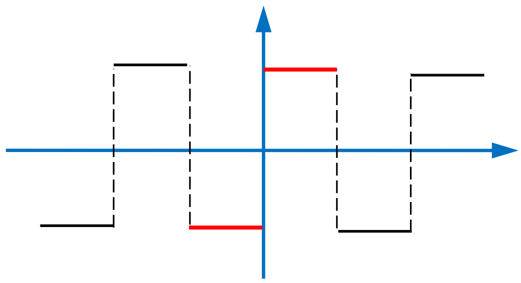


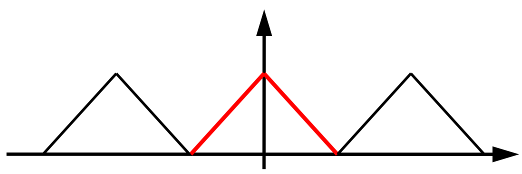


Ex. Draw the periodic functions whose definition in one period as:

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$

$$f(t) = \begin{cases} 1+t & -1 < t < 0 \\ 1-t & 0 < t < 1 \end{cases}$$





III. **Fourier Series**: Any periodic function f(t) can be rewritten as a sum of sines and cosines components as follows:-

$$f(t)=\underbrace{\frac{a_0}{2}}_{\substack{\text{constant part}\\ \text{mean, D.C. value}}}+\sum_{n=1}^{\infty}\underbrace{a_n\cos\omega_n t}_{\text{even part}}+\underbrace{b_n\sin\omega_n t}_{\text{odd part}}$$

is Called as **Fourier Series** or **Fourier Expansion**

where : $\omega_n=\frac{2n\pi}{T}$: radian frequency (rad/s) , $f_n=\frac{n}{T}=\frac{\omega_n}{2\pi}$: frequency (Hz)

$p=\frac{T}{2}$, half period. $\therefore \omega_n=\frac{n\pi}{p}$

a_0, a_n & b_n **:- are constants to be evaluated (ثوابت يتم حسابها)**

$$a_0=\frac{1}{p}\int_d^{d+2p} f(t)dt$$

$$a_n=\frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n t dt$$

$$b_n=\frac{1}{p}\int_d^{d+2p} f(t)\sin\omega_n t dt$$

ملاحظة: جميع التكاملات تتم خلال فترة واحدة (T)

H.W. Proof Fourier theorem utilizing the following:-

$$\int_d^{d+T}\sin\omega_n t dt=0 \quad , \quad \int_d^{d+T}\cos\omega_n t dt=0$$

$$\int_d^{d+2p}\sin\omega_n t\cos\omega_m t dt=0$$

$$\int_d^{d+2p}\sin\omega_n t\sin\omega_m t dt=\begin{cases}0 & n\neq m\\ p & n=m\end{cases}$$

$$\int_d^{d+2p}\cos\omega_n t\cos\omega_m t dt=\begin{cases}0 & n\neq m\\ p & n=m\end{cases}$$

For calculated the constants are dependent the type of periodic function as:

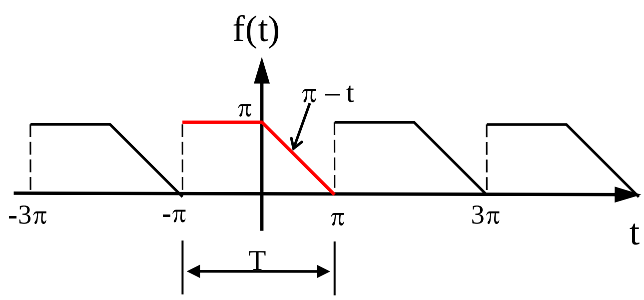
Odd function	Even function	Neither even nor odd
$a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{p}\int_d^{d+p} f(t)\sin\omega_n t\,dt$	$b_n = 0$ $a_0 = \frac{2}{p}\int_d^{d+p} f(t)dt$ $a_n = \frac{2}{p}\int_d^{d+p} f(t)\cos\omega_n t\,dt$	$a_0 = \frac{1}{p}\int_d^{d+2p} f(t)dt$ $a_n = \frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n tdt$ $b_n = \frac{1}{p}\int_d^{d+2p} f(t)\sin\omega_n tdt$
Fourier sine series	Fourier cosine series	

خطوات الحل: 1. ارسم   2. كرر   3. حدد النوع   4. أحسب الثوابت

Ex.(1) Find the Fourier series expansion of the periodic function whose definition in one period is;

$$f(t) = \begin{cases} \pi & -\pi < t < 0 \\ \pi - t & 0 < t < \pi \end{cases} \qquad \therefore f(t+2\pi) = f(t)$$

$\therefore$ **Neither even nor odd**



$\therefore$ Should be calculated all Constants

$$\therefore T = 2\pi,\ p = \pi,\ \omega_n = \frac{n\pi}{p} = n$$

$$a_0 = \frac{1}{p}\int_d^{d+2p} f(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi dt + \int_0^{\pi} (\pi - t)dt\right] = \frac{1}{\pi}\left[\pi t\Big|_{-\pi}^{0} + \pi t\Big|_0^{\pi} - \frac{t^2}{2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi(0+\pi) + \pi(\pi - 0) - \frac{1}{2}(\pi^2 - 0)\right] = \frac{3\pi}{2}$$

$$a_n = \frac{1}{p}\int_d^{d+2p} f(t)\cos\omega_n tdt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi \cos(nt)\,dt + \int_{0}^{\pi} (\pi - t)\cos(nt)\,dt\right]$$

Using udv or Tabular Integration

π - t	cos(nt)
$\pi - t$ (+)	$\cos(nt)$
-1 (−)	$\frac{1}{n}\sin(nt)$
0	$-\frac{1}{n^2}\cos(nt)$

$$a_n = \frac{1}{\pi}\left[\left.\frac{\pi \sin(nt)}{n}\right|_{-\pi}^{0} + \left[\frac{(\pi - t)}{n}\sin(nt) - \frac{1}{n^2}\cos(nt)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\{\sin(0) - \sin(-n\pi)\} + \left\{\left(0 - \frac{1}{n^2}\cos n\pi\right) - \left(\frac{(\pi - 0)}{n}\sin(0) - \frac{1}{n^2}\cos(0)\right)\right\}\right]$$

$$a_n = \frac{1}{\pi}\left[0 + \left\{\left(0 - \frac{1}{n^2}\cos n\pi\right) - \left(0 - \frac{1}{n^2}\right)\right\}\right] = \frac{1}{\pi n^2}(1 - \cos n\pi)$$

$$\therefore a_n = \frac{1}{n\pi^2}\left[1 - (-1)^n\right] = \begin{cases} \dfrac{2}{n\pi^2} & n : odd \\ 0 & n : even \end{cases}$$

Note :−
$\sin(n\pi) = 0$
$\cos(n\pi) = (-1)^n$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi \sin(nt)\,dt + \int_{0}^{\pi} (\pi - t)\sin(nt)\,dt\right]$$

Home work

Ans. $\therefore b_n = \dfrac{(-1)^n}{n}$

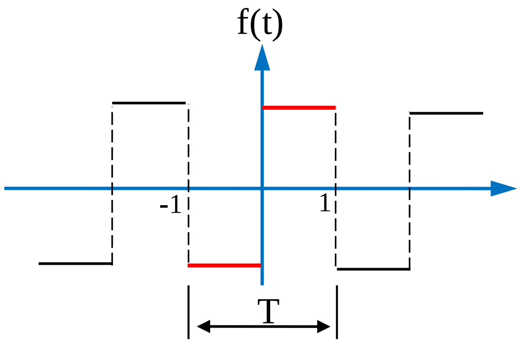
$$\because f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$

$$\therefore f(t)=\frac{3\pi}{4}+\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n\pi^2}\cos(nt)+\frac{(-1)^n}{n}\sin(nt)$$

$$\therefore f(t)=\frac{3\pi}{4}+\frac{2}{\pi^2}\cos(t)-\sin(t)+\frac{1}{2}\sin(2t)+\frac{2}{3\pi^2}\cos(3t)-\frac{1}{3}\sin(3t)+\cdots$$

Ex.(2) Find the Fourier expansion of the periodic function whose definition in one period as:

$$f(t)=\begin{cases}-1 & -1<t<0\\ 1 & 0<t<1\end{cases}$$



$\therefore$ **Odd function**

$$\therefore a_0=0 \quad \& \quad a_n=0$$

$$\therefore T=2,\ p=1,\ \omega_n=\frac{n\pi}{p}=n\pi$$

$$b_n=\frac{2}{p}\int_d^{d+p} f(t)\sin\omega_n t\,dt=\frac{2}{1}\int_0^1 f(t)\sin n\pi t\,dt$$

$$=2\int_0^1 \sin n\pi t\,dt=2\left[\frac{-\cos n\pi t}{n\pi}\right]_0^1$$

$$=-2\left[\frac{\cos n\pi}{n\pi}-\frac{1}{n\pi}\right]$$

$$\therefore b_n=\frac{2}{n\pi}(1-\cos n\pi)=\frac{2}{n\pi}\left[1-(-1)^n\right]=\begin{cases}4/n\pi & n:odd\\ 0 & n:even\end{cases}$$

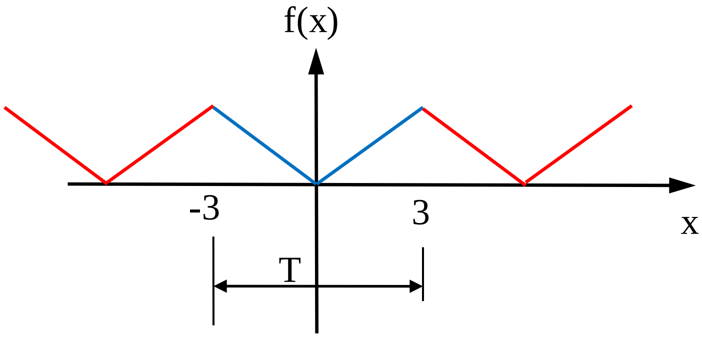
$$\therefore f(t)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\sin n\pi t \qquad ,\quad n:odd\ \ only$$

$$\therefore f(t)=\frac{4}{\pi}\sin\pi t+\frac{4}{3\pi}\sin 3\pi t+\frac{4}{5\pi}\sin 5\pi t+\frac{4}{7\pi}\sin 7\pi t+\cdots$$

Ex. (3) Find the Fourier expansion of the periodic function whose definition in one period is;

$$f(x)=|x| \quad -3<x<3 \quad \text{can be rewritten as} \quad f(t)=\begin{cases}-x & -3<x<0\\ x & 0<x<3\end{cases}$$

$\therefore$ **Even function**



$\therefore b_n=0$

$$\therefore T=6,\ p=3,\ \omega_n=\frac{n\pi}{p}=\frac{n\pi}{3}$$

$$a_0=\frac{2}{p}\int_d^{d+p} f(x)dx=\frac{2}{3}\int_0^3 xdx=\frac{2}{3}\left.\frac{x^2}{2}\right|_0^3=3$$

$$a_n=\frac{2}{p}\int_d^{d+p} f(x)\cos\omega_n x dx=\frac{2}{3}\int_0^3 x\cos\frac{n\pi}{3}xdx$$

x	cos(nπx/3)
x (+)	$\frac{3}{n\pi}\sin\frac{n\pi}{3}x$
1 (−)	$-\frac{9}{n^2\pi^2}\cos\frac{n\pi}{3}x$
0	

Using udv or Tabular intergration

$$\therefore a_n=\frac{2}{3}\left[\frac{3x}{n\pi}\sin\frac{n\pi}{3}x+\frac{9}{n^2\pi^2}\cos\frac{n\pi}{3}x\right]_0^3$$

$$=\frac{2}{3}\left[\left(\frac{9}{n\pi}\sin n\pi+\frac{9}{n^2\pi^2}\cos n\pi\right)-\left(0+\frac{9}{n^2\pi^2}\cos(0)\right)\right]$$

$$=\frac{2}{3}\left[\frac{9}{n^2\pi^2}\cos n\pi-\frac{9}{n^2\pi^2}\right]=\frac{6}{n^2\pi^2}(\cos n\pi-1)=\frac{6}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$a_n=\frac{6}{n^2\pi^2}\left[(-1)^n-1\right]=\begin{cases}\dfrac{-12}{n^2\pi^2} & n:odd\\ 0 & n:even\end{cases}$$

Substitute in Fourier series

$$\therefore f(t)=\frac{3}{2}-\frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{3}x \qquad ,\ n:odd \quad only$$

Or $$f(t)=\frac{3}{2}-\frac{12}{\pi^2}\left(\frac{1}{1}\cos\frac{\pi}{3}x+\frac{1}{9}\cos\frac{3\pi}{3}x+\frac{1}{25}\cos\frac{5\pi}{3}x+\cdots\cdots\right)$$

2.2 Half Range Expansion

هذا النوع هو يحدد لك نوع الدالة يعني انت تتعامل مع الدالة اما زوجية او فردية بغض النظر عن نوعها (لا يحتاج الى الرسم والتكرار لمعرفة نوعها) (تعطى الدالة لنصف الفترة فقط اي p). وهي ينقسم الى نوعين:-

(i) **Fourier sine expansion**

يتم التعامل مع الدالة على انها فردية بغض النظر عن نوعها

$$a_0 = 0 \quad \& \quad a_n = 0$$

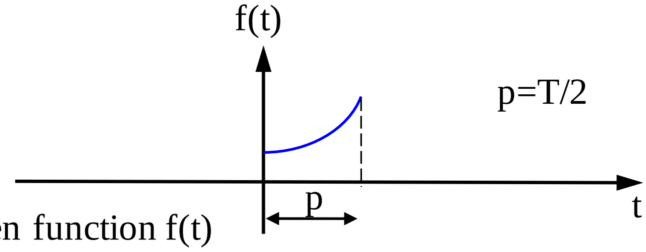
$$b_n = \frac{2}{p}\int_{d}^{d+p} f(t)\sin\omega_n t\, dt$$

(ii) **Fourier cosine expansion**

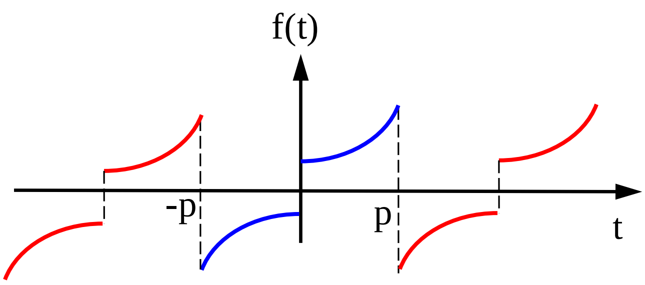
يتم التعامل مع الدالة على انها زوجية بغض النظر عن نوعها

$$b_n = 0$$

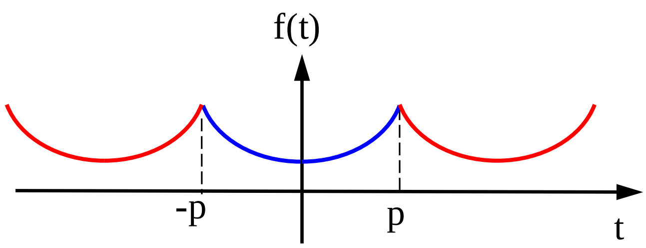
$$a_0 = \frac{2}{p}\int_{d}^{d+p} f(t)dt \quad \& \quad a_n = \frac{2}{p}\int_{d}^{d+p} f(t)\cos\omega_n t\, dt$$



Given function f(t)



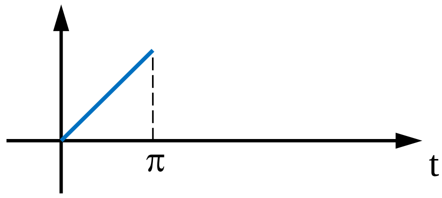
Sine extension (odd function)



Cosine extension (even function)

Ex. Find the Fourier sine and Fourier cosine expansions for the given function as;

$$f(t) = t \qquad 0 < t < \pi$$



(a) **For Fourier sine expansions (odd extension)**

Using this relations $\quad a_0 = 0 \;,\; a_n = 0 \;\;\&\;\; b_n = \dfrac{2}{p}\displaystyle\int_d^{d+p} f(t)\sin\omega_n t\, dt$

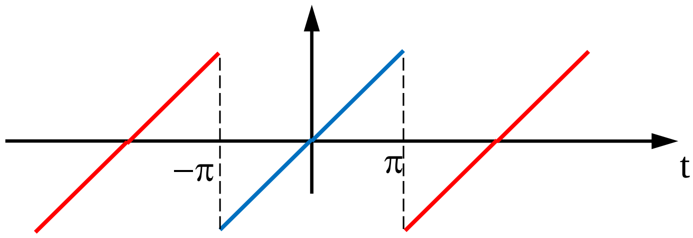
$$\therefore p = \pi,\; \omega_n = \frac{n\pi}{p} = n \qquad \therefore b_n = \frac{2}{\pi}\int_0^{\pi} t\sin(nt)dt$$

t	sin(nt)
1	$-\frac{1}{n}\cos(nt)$ (+)
0	$-\frac{1}{n^2}\sin(nt)$ (−)

$$\therefore b_n = \frac{2}{\pi}\left[\frac{-t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(\pi t)\right) - \left(0 + \frac{1}{n^2}\sin(0)\right)\right]$$

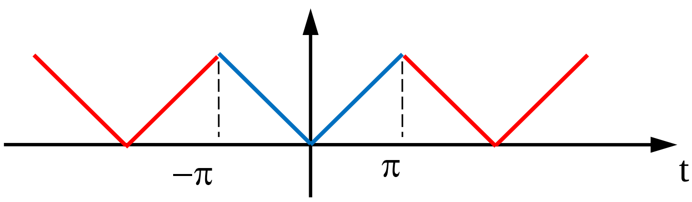
$$bn = \frac{-2}{n}\cos(n\pi)$$



(b) **For Fourier cosine expansions (even extension)**

Using this relations $\quad a_0 = \dfrac{2}{p}\displaystyle\int_d^{d+p} f(t)dt \;,\; a_n = \dfrac{2}{p}\displaystyle\int_d^{d+p} f(t)\cos\omega_n t\, dt \quad \& \; b_n = 0$

Home work



2.3 Complex Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} \right) + b_n \left(\frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{i\omega_n t} + \left(\frac{a_n + ib_n}{2} \right) e^{-i\omega_n t}$$

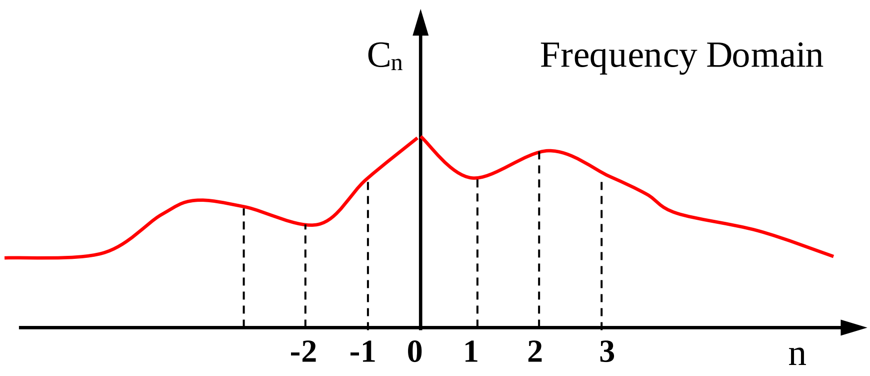
$$\therefore f(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{i\omega_n t} + C_{-n} e^{-i\omega_n t}$$

where: $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n - ib_n}{2}$, $C_{-n} = \frac{a_n + ib_n}{2}$, $C_n = \overline{C_{-n}}$

We can rewritten as:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{i\omega_n t}$$

where: $$C_n = \frac{1}{2p} \int_{d}^{d+2p} f(t) e^{-i\omega_n t} dt$$



Proof:

- when n = 0

$$C_0 = \frac{a_0}{2} = \frac{1}{2p} \int_{d}^{d+2p} f(t) dt$$

- when n is positive

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t\,dt - i\,\frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t\,dt\right]$$

$$= \frac{1}{2p}\int_{d}^{d+2p} f(t)\left(\cos\omega_n t - i\,\sin\omega_n t\right)dt$$

$$\therefore C_n = \frac{1}{2p}\int_{d}^{d+2p} f(t)e^{-i\omega_n t}\,dt$$

- when n is negative

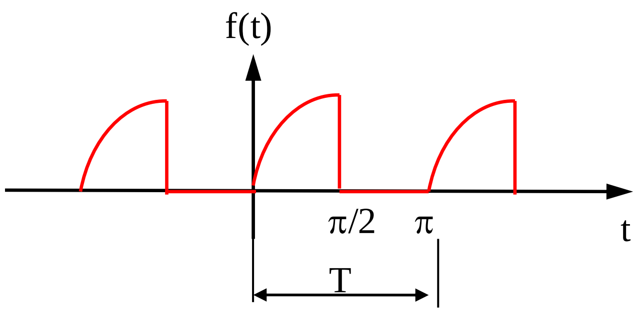
$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_{d}^{d+2p} f(t)\cos\omega_n t\,dt + i\,\frac{1}{p}\int_{d}^{d+2p} f(t)\sin\omega_n t\,dt\right]$$

$$= \frac{1}{2p}\int_{d}^{d+2p} f(t)\left(\cos\omega_n t + i\,\sin\omega_n t\right)dt$$

$$\therefore C_{-n} = \frac{1}{2p}\int_{d}^{d+2p} f(t)e^{i\omega_n t}\,dt$$

Ex. Find the complex Fourier series for the following function defined in one period:

$$f(t) = \begin{cases} \sin t & 0 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases}$$



$$T = \pi,\; p = \frac{\pi}{2},\; \omega_n = \frac{n\pi}{p} = 2n$$

$$C_n = \frac{1}{2p}\int_{d}^{d+T} f(t)e^{-i\omega_n t}dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} \sin t\, e^{-i\omega_n t}dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0 * e^{-i\omega_n t}dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2}\left(\frac{e^{it} - e^{-it}}{2i}\right)e^{-i\omega_n t}dt = \frac{1}{2\pi i}\int_{0}^{\pi/2}\left(e^{i(1-2n)t} - e^{-i(1-2n)t}\right)dt$$

$$C_n = \frac{1}{2\pi i}\left[\frac{e^{i(1-2n)t}}{i(1-2n)} + \frac{e^{-i(1+2n)t}}{i(1+2n)}\right]_0^{\pi/2}$$

$$= \frac{-1}{2\pi}\left\{\left[\frac{e^{i(1-2n)\pi/2}}{(1-2n)} + \frac{e^{-i(1+2n)\pi/2}}{(1+2n)}\right] - \left[\frac{1}{(1-2n)} + \frac{1}{(1+2n)}\right]\right\}$$

$$= \frac{-1}{2\pi}\left[\frac{i\cos n\pi}{(1-2n)} + \frac{-i\cos n\pi}{(1+2n)} - \frac{2}{(1-4n^2)}\right]$$

$$= \frac{-1}{2\pi}\left[\frac{i\,4n\cos n\pi}{1-4n^2} - \frac{2}{1-4n^2}\right]$$

$$\therefore C_n = \frac{-1}{2\pi}\left[\frac{i\,4n(-1)^n - 2}{1-4n^2}\right]$$

Note:

$$e^{i(1-2n)\pi/2} = e^{i\left(\frac{\pi}{2}-n\pi\right)} = e^{i\frac{\pi}{2}}.e^{-in\pi}$$

$$= \left(\cos(\pi/2)^{=0} + i\sin(\pi/2)^{=1}\right)\left(\cos n\pi - i\sin n\pi^{=0}\right)$$

$$= i\cos n\pi$$

$$\therefore f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{i\omega_n t} = \sum_{n=-\infty}^{n=\infty} \frac{-1}{2\pi}\left[\frac{i\,4n(-1)^n - 2}{1-4n^2}\right] e^{i\omega_n t}$$

For Check at n = 0

$$C_0 = \frac{1}{\pi} \qquad , C_0 = \frac{a_0}{2} = \frac{1}{2p}\int_d^{d+T} f(t)dt$$

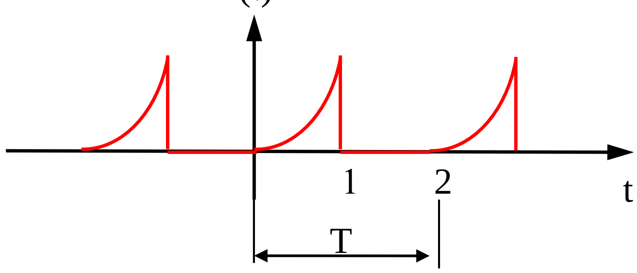
$$C_0 = \frac{1}{\pi}\int_0^{\pi/2} \sin t\, dt = \frac{-1}{\pi}[\cos t]_0^{\pi/2} = \frac{1}{\pi}$$

هناك حالات تحدث في المقام $(1-n^2)$ فلا يمكن حساب C_1 , C_{-1} نذهب الى التعريف الاساسي وهكذا

$$C_1 = \frac{1}{2\pi}\int f(t) e^{-i\omega_1 . t} dt$$

Ex. Find the complex Fourier series for the following function defined in one period:

$$f(t) = \begin{cases} t^2 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$



$$T = 2,\ p = 1,\ \omega_n = \frac{n\pi}{p} = n\pi$$

$$C_n=\frac{1}{2p}\int_{d}^{d+T} f(t)e^{-i\omega_n t}dt=\frac{1}{2}\int_0^1 t^2e^{-in\pi t}dt+0$$

$$=\frac{1}{2}\left[t^2\frac{e^{-in\pi t}}{-in\pi}-2t\frac{e^{-in\pi t}}{(-in\pi)^2}+2\frac{e^{-in\pi t}}{(-in\pi)^3}\right]_0^1=\frac{1}{2}\left[t^2\frac{e^{-in\pi t}}{-in\pi}+2t\frac{e^{-in\pi t}}{n^2\pi^2}+2\frac{e^{-in\pi t}}{in^3\pi^3}\right]_0^1$$

$$=\frac{1}{2n\pi}\left\{\left[ie^{-in\pi}+2\frac{e^{-in\pi}}{n\pi}-2i\frac{e^{-in\pi}}{n^2\pi^2}\right]-\left(0+0-\frac{i2}{n^2\pi^2}\right)\right\}$$

$$=\frac{1}{2n\pi}\left\{\left[i\cos n\pi+\frac{2\cos n\pi}{n\pi}-\frac{i2\cos n\pi}{n^2\pi^2}\right]+\frac{i2}{n^2\pi^2}\right\}$$

$$\therefore C_n=\frac{1}{2n\pi}\left\{\cos n\pi\left(i+\frac{2}{n\pi}-\frac{2i}{n^2\pi^2}\right)+\frac{i2}{n^2\pi^2}\right\}$$

Note:
$e^{-in\pi}=\left(\cos n\pi-i\,\sin n\pi^{=0}\right)=\cos n\pi$

$$\therefore f(t)=\sum_{n=-\infty}^{n=\infty}C_n e^{i\omega_n t}=\sum_{n=-\infty}^{n=\infty}\frac{1}{2n\pi}\left\{\cos\left(i+\frac{2}{n\pi}-\frac{2i}{n^2\pi^2}\right)+\frac{i2}{n^2\pi^2}\right\}e^{i\omega_n t}$$

For Check at n = 0

$$C_0=\frac{1}{2p}\int_{d}^{d+T} f(t)dt\ \frac{1}{2}\int_0^1 t^2dt=\frac{1}{6}$$

H.W. Problems P.200 in "Wylie"

2.4 Fourier transform

يستخدم للدوال الغير دورية

يحول عبارة عملية تحويل دالة او اشارة تدعى f(t) من time domain الى Frequency domain تسمى G(ω)

المدى من ∞- الى ∞+ عكس لابلاس من 0 الى ∞

Fourier transform is a tool for signal processing and Laplace transform is mainly applied to controller design

- Fourier series;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos w_n t + b_n \sin w_n t$$

Complex Fourier series;

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{i\omega_n t} \qquad \text{where:} \quad C_n = \frac{1}{2p}\int_d^{d+2p} f(t) e^{-i\omega_n t} dt$$

$$f(t) = \sum_{n=-\infty}^{n=\infty} \left[\frac{1}{2p}\int_d^{d+2p} f(t) e^{-i\omega_n t} dt \right] e^{i\omega_n t} \qquad \textbf{... (1)}$$

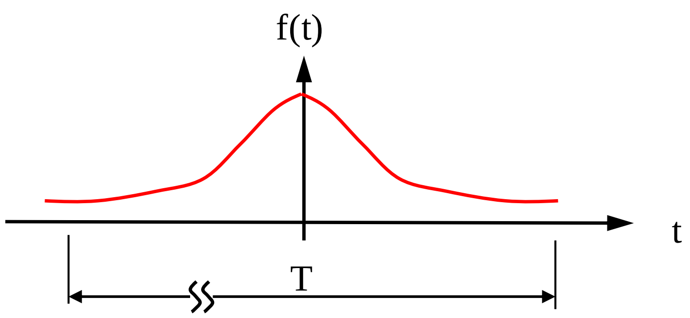
$$At \quad T = -\infty \to \infty$$

$$p \to \infty$$

$$d + 2p \to \infty$$

$$\Delta\omega_n = \omega_{n+1} - \omega_n$$

$$= \frac{\pi(n+1)}{p} - \frac{\pi n}{p} = \frac{\pi}{p}$$



$$T = \infty, \; p = \infty, \; \omega_n = \frac{n\pi}{p}$$

* If times & divided eq.(1) in $\Delta\omega_n$

$$f(t) = \sum_{-\infty}^{\infty} \left[\frac{1}{2p}\int_d^{d+2p} f(t) e^{-i\omega_n t} dt \right] e^{i\omega_n t} \; . \frac{\Delta\omega_n}{\Delta\omega_n}$$

$$= \sum_{-\infty}^{\infty} \frac{\not{p}}{2\not{p}\pi} \int_d^{d+2p} f(t) e^{-i\omega_n t} dt . e^{i\omega_n t} \; .\Delta\omega_n$$

$$as: \begin{matrix} p \to \infty \\ \Delta\omega \to 0 \end{matrix} \quad then \quad \Delta\omega \to dw \;\; \& \sum \to \int \;\; \& \; \omega_n = \omega$$

$$f(t)=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt\right].e^{i\omega t}.d\omega$$

$$f(t)=\int_{-\infty}^{\infty}G(\omega)e^{i\omega t}\,d\omega \qquad \text{"Inverse Fourier transform"}$$

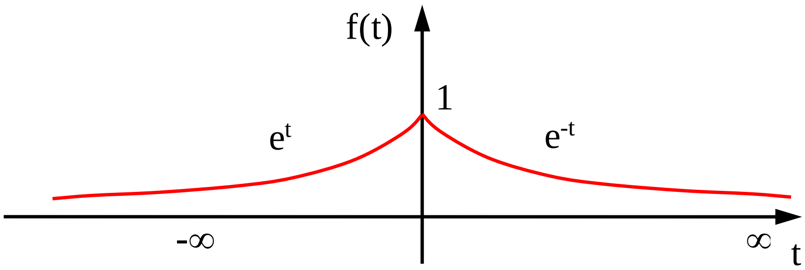
$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt \qquad \text{"Fourier Transform"}$$

بالنسبة الى (1/2π) يمكن ان تقسم الى جزئين او تكتب مع G(ω) او f(t).

Ex.1. Find Fourier transform for the following function: $f(t)=e^{-|t|}$

Sol.

$$\therefore f(t)=\begin{cases}e^{t} & t<0\\ e^{-t} & t>0\end{cases}$$



$$f(t)=\int_{-\infty}^{\infty}G(\omega).e^{i\omega t}d\omega$$

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t).e^{i\omega t}dt$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{0}e^{t}.e^{-i\omega t}dt+\int_{0}^{\infty}e^{-t}.e^{-i\omega t}dt\right]$$

Note:
$e^{-\infty}=0$

$$=\frac{1}{2\pi}\left[\frac{e^{(1-i\omega)t}}{1-i\omega}\Bigg|_{-\infty}^{0}+\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\Bigg|_{0}^{\infty}\right]$$

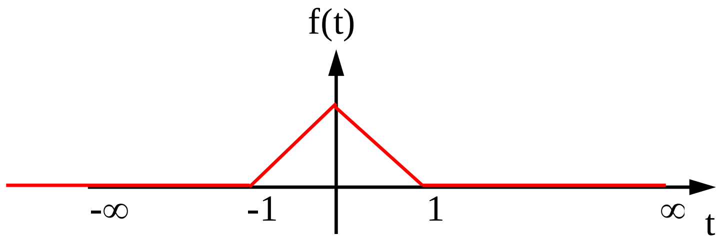
$$=\frac{1}{2\pi}\left[\frac{1}{1-i\omega}(1-0)-\frac{1}{1+i\omega}(0-1)\right]=\frac{1}{2\pi}\left[\frac{1}{1-i\omega}+\frac{1}{1+i\omega}\right]$$

$$G(\omega)=\frac{1}{2\pi}\left[\frac{1+i\omega+1-i\omega}{1+\omega^2}\right]=\frac{1}{\pi\left(1+\omega^2\right)}$$

$$\therefore f(t)=\int_{-\infty}^{\infty}G(\omega)e^{i\omega t}d\omega=\int_{-\infty}^{\infty}\frac{1}{\pi\left(1+\omega^2\right)}.e^{i\omega t}d\omega$$

Ex.2. Find Fourier transform for the following function:

$$f(t)=\begin{cases}1+t & -1<t<0\\ 0 & otherwise\\ 1-t & 0<t<1\end{cases}$$



$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{-1} 0.dt+\int_{-1}^{0}(1+t)e^{-i\omega t}dt+\int_{0}^{1}(1-t)e^{-i\omega t}dt+\int_{1}^{\infty}0.dt+\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{e^{-i\omega t}}{-i\omega}+\frac{te^{-i\omega t}}{-i\omega}-\frac{e^{-i\omega t}}{i^2\omega^2}\right)_{-1}^{0}+\left(\frac{e^{-i\omega t}}{-i\omega}-\frac{te^{-i\omega t}}{-i\omega}-\frac{e^{-i\omega t}}{-i^2\omega^2}\right)_{0}^{1}\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{1}{-i\omega}+\frac{0}{-i\omega}+\frac{1}{\omega^2}\right)-\left(\frac{e^{i\omega}}{-i\omega}+\frac{e^{i\omega}}{i\omega}+\frac{e^{i\omega}}{\omega^2}\right)\right.$$

$$\left.+\left(\frac{e^{-i\omega}}{-i\omega}+\frac{e^{-i\omega}}{i\omega}-\frac{e^{-i\omega}}{\omega^2}\right)-\left(\frac{1}{-i\omega}+\frac{0}{i\omega}-\frac{1}{\omega^2}\right)\right]$$

$$=\frac{1}{2\pi}\left[\frac{2}{\omega^2}+\cancel{\frac{e^{i\omega}}{i\omega}}-\cancel{\frac{e^{i\omega}}{i\omega}}-\frac{e^{i\omega}}{\omega^2}-\cancel{\frac{e^{-i\omega}}{i\omega}}+\cancel{\frac{e^{-i\omega}}{\omega}}-\frac{e^{-i\omega}}{\omega^2}\right]$$

$$G(\omega)=\frac{1}{2\pi}\left[\frac{2}{\omega}-\frac{2\cos\omega}{\omega^2}\right]=\frac{1}{\pi\omega^2}(1-\cos\omega)$$

> Note:
> $e^{i\omega}+e^{-i\omega}$
> $(\cos\omega+i\sin\omega)+(\cos\omega-i\sin\omega)=2\cos\omega$

$$f(t)=\int_{-\infty}^{\infty}G(\omega).e^{i\omega t}d\omega=\int_{-\infty}^{\infty}\frac{1}{\pi\omega^2}(1-\cos\omega).e^{i\omega t}d\omega$$

H.W. Problems P.200 & P. 220 "Wylie"