



University of Basrah - College of Engineering
Department of Mechanical Engineering



Subject: *Engineering Analysis*

Stage: *Third*

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Syllabus of Engineering Analysis

1. Complex Variables and Functions
2. Fourier Series and Transforms.
3. Laplace Transform of Special Functions and Applications
4. Solution of Ordinary Differential Equations
5. Bessel and Legendre Functions
6. Solution of Partial Differential Equations
7. Probability and Statistics

Textbook:

Advanced Engineering Mathematics, Wylie, McGraw Hill Books Company.

References:

1. Advanced Engineering Mathematics, Kreyszig, Jon Wylie and Sons.
2. Advanced Engineering Mathematics, Jeffery, Academic Press.
2. Mathematical Methods for Engineers and Scientists, K. T. Tang

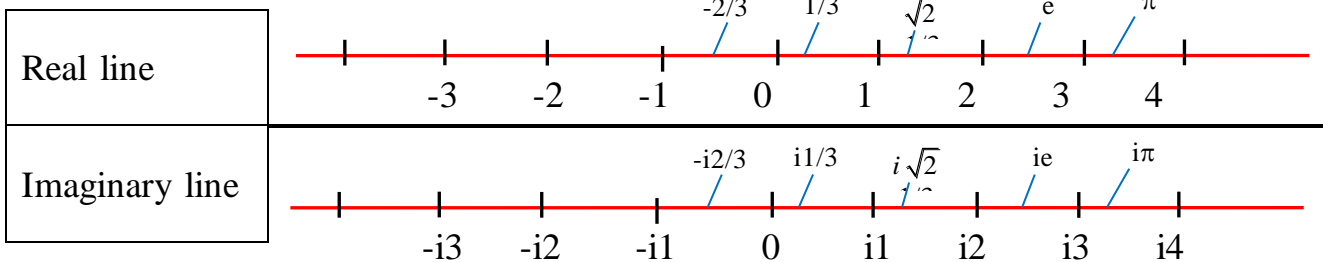
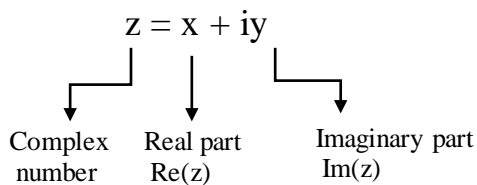
1. Complex Variables and Functions

	-ve 0 +ve	
Real Numbers	$\dots -3, -2, -1, 0, 1, 2, 3 \dots$	Integers No.
	$\dots -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$	Rational No.
	$\dots, \frac{-\sqrt{3}}{5}, -\sqrt{3}, 0, \sqrt{3}, \frac{\sqrt{3}}{5}, \sqrt{7} \dots$	Irrational No.

$$\sqrt{-4} = \sqrt{-1}\sqrt{4} = \pm i 2 \quad \text{Imaginary number}$$

Complex number = Real number + Imaginary number

$$= \text{Real} + i \text{ Real}$$

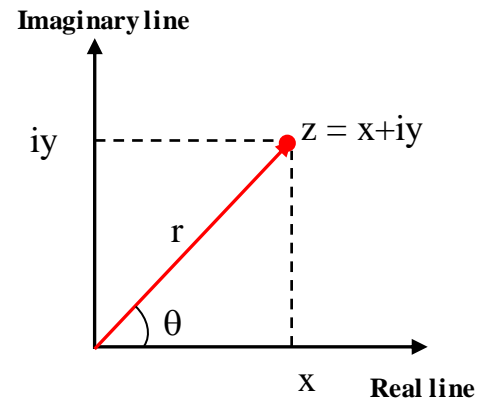


∴ The complex number may be represented as a point on the complex plane.

$$z = x + iy, (x, iy), (x, y) \quad \text{Cartesian Form}$$

Or

$$z = r \angle \theta, (r, \theta) \quad \text{Polar form}$$



where;

$$x = \text{Re}(z)$$

$$y = \text{Im}(z)$$

r: Amplitude or modulus, $r = \sqrt{x^2 + y^2}$, $r = |z|$, $r = \text{Amp}(z)$

θ: Phase or Angle, $\theta = \tan^{-1} \frac{y}{x}$, $\theta = \text{Arg}(z)$

$$\therefore x = r \cos \theta, \quad y = r \sin \theta$$

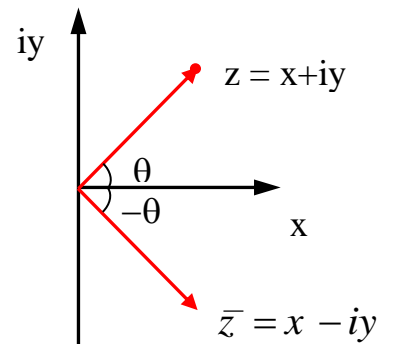
$$z = x + iy = r \cos \theta + ir \sin \theta$$

$$z = r(\cos \theta + i \sin \theta)$$

Definitions:

(1) Conjugate of z is denoted by \bar{z}

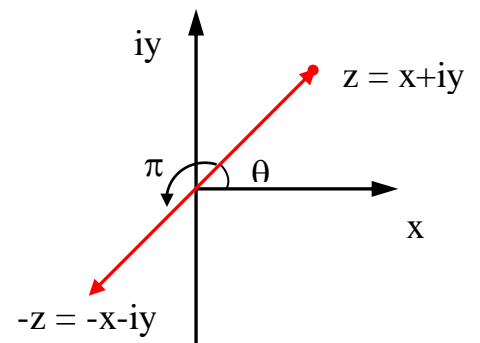
$$z = r \angle \theta \rightarrow \bar{z} = r \angle -\theta$$



(2) Reverse of z is written as -z

$$z = x + iy \rightarrow -z = -x - iy$$

$$z = r \angle \theta \rightarrow -z = r \angle \theta + \pi$$



Note: All angles are measured in **radians** and positive in the counterclockwise sense.

(3) Inverse of z. It is simply $\frac{1}{z} = ?$

Operations on Complex numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

(1) Addition and Subtraction

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

(2) Multiplication

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i^2y_1y_2 + ix_1y_2 + iy_1x_2 \end{aligned}$$

$$\therefore z_1 \cdot z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Note: $z\bar{z} = x^2 + y^2 = r^2 = |z|^2$

$$i^2 = i \cdot i = -1$$

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = 1$$

$$\vdots \quad \vdots$$

(3) Division $\frac{z_1}{z_2}$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2}$$

$$\therefore \frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \quad , r^2 = x^2 + y^2$$

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2} \right) = (x_1 + iy_1) \left(\frac{x_2 - iy_2}{x_2^2 + y_2^2} \right) = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}$$

$$\therefore \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2 + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

*** In polar form:**

$$\begin{aligned} \therefore z_1 \cdot z_2 &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \\ &= (r_1 \cos \theta_1 r_2 \cos \theta_2 - r_1 \sin \theta_1 r_2 \sin \theta_2) + i(r_1 \cos \theta_1 r_2 \sin \theta_2 + r_2 \cos \theta_2 r_1 \sin \theta_1) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

$$\therefore z_1 z_2 = r_1 r_2 \underline{(\theta_1 + \theta_2)}$$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{r \underline{-\theta}}{r \underline{\theta} * r \underline{-\theta}} = \frac{r \underline{-\theta}}{r^2}$$

$$\therefore \frac{1}{z} = \frac{1}{r} \underline{-\theta}$$

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2} \right) = r_1 \underline{\theta_1} \left(\frac{1}{r_2 \underline{-\theta_2}} \right)$$

$$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{\theta_1 - \theta_2}$$

Properties of Operations:

- (1) $z_1 + z_2 = z_2 + z_1$, $z_1 \cdot z_2 = z_2 \cdot z_1$
- (2) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
- (3) $\bar{z}_1 + \bar{z}_2 = \overline{(z_1 + z_2)}$, $\bar{z}_1 \cdot \bar{z}_2 = \overline{z_1 \cdot z_2}$

$$\begin{aligned} \bar{z}_1 &= \overline{(z_1)} \\ \bar{z}_2 &= \overline{(z_2)} \end{aligned}$$

Ex. Given $z_1 = -3 - i4$, $z_2 = 1 - i2$

Find \bar{z}_1 , z_2^{-1} , $z_1 + z_2$, $z_2 - z_1$, $z_1 z_2$, and $\frac{z_1}{z_2}$

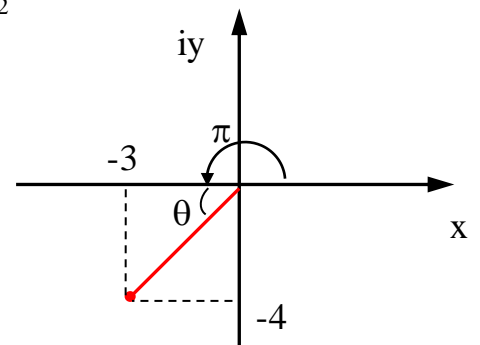
Sol.

$$z_1 = -3 - i4, \quad r = \sqrt{(-3)^2 + (-4)^2} = 5,$$

$$\theta = \tan^{-1} \frac{-4}{-3} = 0.927 + \pi = 4.06 \text{ rad}$$

$$\therefore z_1 = 5 \underline{4.06}$$

$$z_2 = 1 - i2, \quad r = \sqrt{1^2 + (-2)^2} = \sqrt{5}, \quad \theta = \tan^{-1} \frac{-2}{1} = -1.107 \text{ rad}$$



(a) $\bar{z}_1 = -3 + i4$

(b) $z_2^{-1} = \frac{1}{z_2} = \frac{x_2 - iy_2}{x_2^2 + y_2^2} = \frac{1 + i2}{5} = \frac{1}{5} + i \frac{2}{5}$

(c) $z_1 + z_2 = (-3 + 1) + i(-4 - 2) = -2 - i6$

(d) $z_2 - z_1 = (1 + 3) + i(-2 + 4) = 4 + i2$

(e) $z_1 z_2 = (-3 - i4)(1 - i2) = (-3 - 8) + i(6 - 4) = -11 + i2$

Or Using Polar formula

$$z_1 z_2 = r_1 r_2 \angle \theta_1 + \theta_2 = 5\sqrt{5} \angle 2.953$$

(f) $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = (-3 - i4)(0.2 + i0.4) = 2.2 - i2$

Or

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2 = \frac{5}{\sqrt{5}} \angle 5.167$$

H.W. Given $z_1 = -2 + i4$, $z_2 = 5 + i3$, $z_3 = 2 - 6$

find $(z_1 + z_3) \frac{z_1}{z_2 - z_3}$, z_1^2 , $\left(\frac{z_2}{z_3}\right)^2$

Note: ex. $z = -3 + i$

*if $x < 0 \rightarrow \theta = \theta_{calc.} + \pi$

$$** \text{ if } x = 0 \begin{cases} y > 0 \Rightarrow \theta = \frac{\pi}{2} \\ y < 0 \Rightarrow \theta = \frac{3\pi}{2} \end{cases}$$

Simple function of complex variables;

(1) Power functions $z^n = ?$

when n : +ve integer

$$z = r \angle \theta$$

$$z^2 = (r \angle \theta)(r \angle \theta) = r^2 \angle 2\theta$$

$$z^3 = z^2 \cdot z = (r^2 \angle 2\theta)(r \angle \theta) = r^3 \angle 3\theta$$

$$\vdots \quad \vdots$$

$$z^n = r^n \angle n\theta \quad \text{or} \quad z^n = r^n [\cos n\theta + i \sin n\theta]$$

when n :-ve

$$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n \angle n\theta} = \frac{1}{r^n} \angle -n\theta = r^{-n} \angle -n\theta$$

$$\therefore z^{-n} = r^{-n} \angle -n\theta$$

Ex. Find $(3 - i4)^3$

Sol.

$$z = 3 - i4 = 5 \angle -0.927$$

$$z^3 = 5^3 \angle 3 * (-0.927) = 125 \angle -2.781$$

(2) Root function $z^{1/n} = ?$

let $w = z^{1/n}$, and $w = R \angle \Phi$

$$w^n = z \Rightarrow R^n \angle n\Phi = r \angle \theta$$

$$\therefore R^n = r \Rightarrow R = r^{1/n}$$

$$n\Phi = \theta \Rightarrow \Phi = \frac{\theta}{n}$$

$$w = R \angle \Phi = z^{1/n} = r^{1/n} \angle \frac{\theta}{n}$$

$$z^{1/n} = r^{1/n} \angle \frac{\theta}{n} \quad \text{" Principle value"}$$

Generally: $z^{1/n} = r^{1/n} \left| \frac{\theta + 2\pi k}{n} \right|$ where $k = 0, 1, 2, \dots, n-1$

Or $z^{1/n} = r^{1/n} \left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right) \right]$ n: numbers of roots

Ex. Find $\sqrt[3]{(5+i4)}$

Sol. $z = 5+i4 = \sqrt{41} \angle 0.67 = 6.4 \angle 0.67$

$$z^{1/3} = (6.4)^{1/3} \left| \frac{0.67 + 2\pi k}{3} \right| \quad \text{where } k = 0, 1, 2$$

$$\begin{aligned} z^{1/3} &= 1.85 \rightarrow \underline{0.223} \quad k = 0 \\ &\rightarrow \underline{2.317} \quad k = 1 \\ &\rightarrow \underline{4.41} \quad k = 2 \end{aligned}$$

H.W. Find $\sqrt{-1}$, and $\sqrt{1}$

(3) Exponential function: $w = e^z$

$$e^z = e^{(x+iy)} = e^x e^{iy}$$

$$\begin{aligned} &= e^x \left[1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \right] \\ &= e^x \left[\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots \right) + i \left(\frac{y}{1!} - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right) \right] \end{aligned}$$

$$e^z = e^x (\cos y + i \sin y) = e^x \cos y + i e^x \sin y$$

Or

$$e^z = e^x \underline{y} \quad |e^z| = e^x, \quad \text{Arg}(e^z) = y$$

Generally:-

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{"Euler's Formula"}$$

$$z = r \angle \theta = r(\cos \theta + i \sin \theta) \quad \Rightarrow \quad \underline{\therefore z = r e^{i\theta}} \quad \text{"Exponential Form"}$$

Results:-

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\frac{\quad}{\quad} +$$
$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta$$

$$\therefore \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Subtracting $\Rightarrow e^{i\theta} - e^{-i\theta} = i \sin \theta$

$$\therefore \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{i 2}$$

$$\therefore \cos i\theta = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh \theta$$

$$\sin i\theta = \frac{e^{-\theta} - e^{\theta}}{i 2} = -i \frac{e^{-\theta} - e^{\theta}}{2} = i \frac{e^{\theta} - e^{-\theta}}{2} = i \sinh \theta$$

Notes:- $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = -i$

$$\therefore \cosh i\theta = \frac{e^{-i\theta} + e^{i\theta}}{2} = \cos \theta$$

$$\sinh i\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = -i \frac{e^{-\theta} - e^{\theta}}{2} = i \frac{e^{\theta} - e^{-\theta}}{2} = i \sin \theta$$

Also:

$$\cos iz = \cosh z \quad ; \quad \cosh iz = \cos z$$

$$\sin iz = i \sinh z \quad ; \quad \sinh iz = i \sin z$$

Ex. Find $\sin(1+i3)$

$$\sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$$

$$\therefore \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\therefore \sin(1+i3) = \sin 1 \cosh 3 + i \cos 1 \sinh 3$$

Ex. Find $\sin^{-1}(1-2i)$

$$z = \sin^{-1}(1-2i) \Rightarrow 1-2i = \sin z = \sin(x+iy)$$

$$1-2i = \sin x \cosh y + i \cos x \sinh y$$

$$1 = \sin x \cosh y \quad \dots(1) \quad \Rightarrow \sin x = \frac{1}{\cosh y} \quad (3)$$

$$-2 = \cos x \sinh y \quad \dots(2)$$

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{1}{\cosh^2 y}} \quad \dots(4)$$

Substitute eq.(4) in eq.(2) we have

$$-2 = \sqrt{1 - \frac{1}{\cosh^2 y}} \sinh y \Rightarrow 4 = \left(1 - \frac{1}{\cosh^2 y}\right) \sinh^2 y$$

$$\therefore 4 = \left(1 - \frac{1}{\cosh^2 y}\right) (\cosh^2 y - 1)$$

$$4 = \cosh^2 y - 1 - 1 + \frac{1}{\cosh^2 y} = \cosh^2 y - 2 + \frac{1}{\cosh^2 y}$$

Let $t = \cosh^2 y$

$$4 = t - 2 + \frac{1}{t} \Rightarrow t^2 - 6t + 1 = 0$$

$$t = \frac{6 \mp \sqrt{32}}{2} = \begin{cases} 5.828 \\ 0.17 \end{cases} \quad \text{Im possible because } < 1$$

$$\cosh y = \sqrt{t} = \sqrt{5.828} = 2.414$$

$$\therefore y = \cosh^{-1} 2.414 = 1.528$$

$$\sin x = \frac{1}{2.414} = 0.414 \Rightarrow x = \sin^{-1} 0.414$$

$$\therefore x = 0.427$$

$$\therefore \sin^{-1}(1-2i) = 0.427 + i 1.528$$

(4) Logarithmic function: $w = \ln z$

let $w = \ln z$ and $w = u + iv$, $z = re^{i\theta}$

$$e^w = z \Rightarrow e^{u+iv} = re^{i\theta}$$

$$\therefore e^u = r \Rightarrow u = \ln r$$

$$e^{iv} = e^{i\theta} \Rightarrow v = \theta$$

$$\therefore w = \ln z = \ln r + i\theta$$

"Principle value"

In general;

$$\ln z = \ln r + i(\theta + 2\pi k)$$

$$k = 0, 1, 2, \dots, \infty$$

Ex. Find $\ln(1+i)$

$$\ln(1+i) = \ln(\sqrt{2} \angle \pi/4)$$

$$= \ln \sqrt{2} + i \left(\frac{\pi}{4} + 2\pi k \right)$$

$$= \frac{1}{2} \ln 2 + i \left(\frac{\pi}{4} + 2\pi k \right) \quad k = 0, 1, 2, \dots, \infty$$

$$= \frac{1}{2} \ln 2 + i \frac{\pi}{4}$$

"principle value $k=0$ "

Ex. Find z for $\ln z = 1 - i\pi$

$$\therefore z = e^{1-i\pi} = e^1 e^{-i\pi} = e^1 (\cos^{-1}(-\pi) - i \sin(-\pi))$$

$$\therefore z = -e^1$$

H.W.

(1) Evaluate the following functions

$$1. \ln(3-i6) \quad , \quad 2. e^{(3|0.52)}$$

$$3. \cos^{-1}(1+i3) \quad , \quad 4. \tanh^{-1}(2-i5)$$

(2) Find all the value of $\sin^{-1} 2$

(3) Prove that

$$(a) \sin^2 z + \cos^2 z = 1 \quad , \quad (b) \sin(-z) = -\sin z$$
$$(c) \ln e^z = z \quad , \quad (d) \cos(-z) = \cos z$$

(4) Find all values of z for which;

$$(a) e^{3z} = 1 \quad ; \quad (b) e^z = 1 - i \quad ; \quad (c) e^{4z} = i$$

(5) Find all roots of

$$1. \sqrt[3]{1+i} \quad ; \quad 2. \sqrt[3]{8i} \quad ; \quad 3. \sqrt[8]{1} \quad ; \quad 4. \sqrt{-7+24i}$$

Limit of Complex function;

A function $f(z)$ is said to have the limit ℓ as z approaches a point z_0 , written

$$\lim_{z \rightarrow z_0} f(z) = \ell$$

Ex. Find $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ (a) along the x-axis. (b) along the y-axis.

Sol. (a) along the x-axis, $y = 0$

$$\therefore z = x, \quad \bar{z} = x \quad , \text{so that} \quad \frac{\bar{z}}{z} = 1$$

$$\lim_{\substack{y=0 \\ x \rightarrow 0}} \frac{\bar{z}}{z} = 1$$

(b) along the y-axis, $x = 0$

$$\therefore z = iy, \quad \bar{z} = -iy$$

$$\lim_{\substack{x=0 \\ y \rightarrow 0}} \frac{\bar{z}}{z} = \lim_{\substack{x=0 \\ y \rightarrow 0}} \frac{-iy}{iy} = -1$$

Ex. Find $\lim_{z \rightarrow 3+i} (z^2 - z)$

$$\lim_{z \rightarrow 3+i} (z^2 - z) = ((3+i)^2 - (3+i)) = 5 + i5$$

Continuity of Complex function:

The complex function $f(z)$ is defined to be continuous at z_0 if

(a) $f(z_0)$ exists, and

(b) $\lim_{z \rightarrow z_0} f(z)$ exists and is equal to $f(z_0)$.

Ex (1). In the function $f(z) = z^2$ continuous every where?

assume the function is to be tested as z_0 where z_0 is an arbitrary point.

$$f(z) = z^2 = (x^2 - y^2) + i(2xy)$$

$$f(z_0) = (x_0^2 - y_0^2) + i(2x_0y_0)$$

(a) A long horizontal lines pass through z_0

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= \lim_{\substack{y=y_0 \\ x \rightarrow x_0}} [(x^2 - y^2) + i(2xy)] = \lim_{x \rightarrow x_0} [(x^2 - y_0^2) + i2xy_0] \\ &= (x_0^2 - y_0^2) + i2x_0y_0 = f(z_0) \end{aligned}$$

(b) A long vertical path, $x = x_0$, $y \rightarrow y_0$

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= \lim_{\substack{x=x_0 \\ y \rightarrow y_0}} [(x^2 - y^2) + i(2xy)] = \lim_{y \rightarrow y_0} [(x_0^2 - y^2) + i2x_0y] \\ &= (x_0^2 - y_0^2) + i2x_0y_0 = f(z_0) \end{aligned}$$

(c) A long and straight line passes through z_0 , $y = ax + b$

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= \lim_{\substack{y=ax+b \\ x \rightarrow x_0}} [(x^2 - y^2) + i2xy] = \lim_{x \rightarrow x_0} [(x^2 - (ax + b)^2) + i2x(ax + b)] \\ &= (x_0^2 - (ax_0 + b)^2) + i2x_0(ax_0 + b) \\ &= (x_0^2 - y_0^2) + i2x_0y_0 = f(z_0) \end{aligned}$$

Since the limit exist along any path passes through z_0 and equal $f(z_0)$, then the function $f(z) = z^2$ is continuous everywhere.

Ex.(2) where does the function $f(z) = \frac{x}{z}$ fail to be continuous?

Sol. $f(z) = \frac{x}{x + iy}$

if we select a path on the line $y=ax$

$$\lim_{z \rightarrow z_0} f(z) = \lim_{\substack{y=ax \\ x \rightarrow x_0}} f(z) = \lim_{x \rightarrow x_0} \left[\frac{x}{x + iax} \right] = \frac{1}{1 + ia}$$

Since the limit value depend on the path, then the function fail to be continuous on any line $y = ax$.

Ex.(3) where does the function $f(z) = \frac{x^2}{x^2 + y}$ fail to be analytic?

Answer: where $y = ax^2$

Ex.(4) where does the function $f(z) = \frac{x^2}{x^2 + 2xy}$ fail to be analytic?

Answer: where $y = ax$

Note:

- (1) Polynomials are continuous functions everywhere except at $\pm \infty$ perhaps, thus $\sin z, \cos z, e^z, \dots$ etc. are continuous functions.
- (2) Addition, subtraction, multiplication and division of continuous functions are also continuous except the denominator is zero.
- (3) The continuous function of a continuous function is also continuous.

Ex. $\tan z$ is continuous function, except when $\cos z = 0 \implies z = n\pi/2$ where n is odd.

Ex.

$$f(z) = z^2, \quad g(z) = \cos z$$

$$f[g(z)] = (\cos z)^2 = \cos^2 z \quad \text{continuous}$$

$$g[f(z)] = \cos z^2 \quad \text{continuous}$$

H.W. Solve 6 problems about continuity. In Wylie books

Differentiation of Complex Function;

The function $f(z)$ is said to be differentiable at $z = z_0$ if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

OR Put $z - z_0 = \Delta z$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Exist along all possible paths.

Note: Every differentiable function is continuous.

$$f(z) = \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) + f(z_0)$$

$$\text{Take limit as } z \rightarrow z_0 \Rightarrow \lim_{z \rightarrow z_0} f(z) = f'(z) \times 0 + f(z_0) \Rightarrow \lim_{z \rightarrow z_0} f(z) = f(z_0)$$

$\therefore f(z)$ is continuous

Note: The convers need not be true i.e:

If $f(z)$ is continuous at $z = z_0$ then it may Or may not be differentiable at $z = z_0$.

Ex. The function $f(z) = \bar{z}$ at the point $z=0$

$$\lim_{z \rightarrow 0} f(z) = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} (x - iy) = 0$$

It is continuous at $z = 0$

$$\text{But } \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x - iy}{x + iy} \quad \text{---(1)}$$

$$\text{(a) along horizontal path, } y = 0, x \rightarrow 0, \text{ eq(1) } \lim_{\substack{x \rightarrow 0 \\ y=0}} \frac{x - iy}{x + iy} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\text{(a) along vertical path, } x = 0, y \rightarrow 0, \text{ eq (1) } \lim_{\substack{x=0 \\ y \rightarrow 0}} \frac{x - iy}{x + iy} = \lim_{y \rightarrow 0} \frac{-iy}{iy} = -1$$

The limit of (1) does not exist since depends on the path, then the function $f(z) = \bar{z}$ is not differentiable at $z = 0$.

Ex. Show that the function $f(z) = |z|^2$ is differentiable at $z = 0$ & No where else.

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \rightarrow z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} \\ &= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) - z_0 \bar{z}_0}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{z_0 \bar{z}_0 + z_0 \overline{\Delta z} + \bar{z}_0 \Delta z + \Delta z \overline{\Delta z} - z_0 \bar{z}_0}{\Delta z} = \lim_{\Delta z \rightarrow 0} \left(\frac{z_0 \overline{\Delta z} + \bar{z}_0 \Delta z}{\Delta z} + \overline{\Delta z} \right) \\ &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{z_0(\Delta x - i \Delta y) + \bar{z}_0(\Delta x + i \Delta y)}{\Delta x + i \Delta y} + 0 \end{aligned}$$

(Case I) along vertical path, $\Delta x = 0$, $\Delta y \rightarrow 0$

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \rightarrow 0}} \frac{z_0(\Delta x - i \Delta y) + \bar{z}_0(\Delta x + i \Delta y)}{\Delta x + i \Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-z_0 i \Delta y + \bar{z}_0 i \Delta y}{i \Delta y} = (-z_0 + \bar{z}_0)$$

(Case II) along horizontal path, $\Delta y = 0$, $\Delta x \rightarrow 0$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y = 0}} \frac{z_0(\Delta x - i \Delta y) + \bar{z}_0(\Delta x + i \Delta y)}{\Delta x + i \Delta y} = \lim_{\Delta x \rightarrow 0} \frac{z_0 \Delta x + \bar{z}_0 \Delta x}{\Delta x} = (z_0 + \bar{z}_0)$$

$\therefore \lim_{\Delta z \rightarrow 0} |z|^2$ does not exist.

So, for $z \neq 0$, the function $f(z) = |z|^2$ is not differentiable.

However, if $z = 0$, then $f(z) = |z|^2$ is diff.

Note: All the rules of differentiation to real functions can be used for complex functions;-

$$\begin{aligned} \frac{d}{dz} (z^n) &= n z^{n-1}, & \frac{d}{dz} (\sin z) &= \cos z \\ \frac{d}{dz} (g(z) \cdot f(z)) &= g(z) f'(z) + f(z) g'(z) \\ &\vdots & &\vdots \end{aligned}$$

H.W. Problems P.649 continuity. & Problems P.656 differentiation.

Analytic function

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is differentiable at all points of D . The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0 (not merely at z_0 itself).

Ex: The function $f(z) = |z|^2$ is differentiable at $z=0$ & Nowhere else.

So, it is not analytic at $z = 0$.

Ex: The $f(z) = e^z, \sin z \dots etc$ are diff. everywhere in domain. That is analytic functions also.

Analytic function also known as **holomorphic** function.

Cauchy - Riemann Conditions (C.R.C)

Theorem: (Necessary & sufficient conditions for a function to be analytic)

Necessary: suppose $f(z) = u(x, y) + iv(x, y)$ is continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then the first-order partial derivatives of u and v exist and satisfy the equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (1) \quad \text{Called (C.R.E.)}$$

at the point z .

Hence, if $f(z)$ is analytic in the domain D , then partial derivative

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \text{and } \frac{\partial v}{\partial y}$ exist and satisfy the C.R. Equations

$$u_x = v_y, \quad u_y = -v_x$$

Proof: Since $f(z)$ is differentiable at z , So

$$f'(z) = \lim_{\Delta z \rightarrow 0} \left\{ \frac{f(z + \Delta z) - f(z)}{\Delta z} \right\} \quad \text{exists}$$

$$f'(z) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left\{ \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \right\}$$

Consider two particular paths.

(a) along horizontal path, $\Delta y = 0$, $\Delta x \rightarrow 0$

$$f'(z) = \lim_{\Delta x \rightarrow 0} \left\{ \frac{[u(x + \Delta x, y) + iv(x + \Delta x, y)] - u(x, y) + iv(x, y)}{\Delta x} \right\}$$

$$= \lim_{\Delta x \rightarrow 0} \left\{ \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right\}$$

i.e. $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} \quad \dots(1)$

(a) along vertical path, $\Delta x = 0$, $\Delta y \rightarrow 0$. Leave it to you to show that

$$f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{1}{i} \frac{\partial f}{\partial y} \quad \dots(2)$$

By comparing the real and Imaginary parts of Eqs. (1) & (2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{"CRC"}$$

Ex. Show that $f(z) = z^2$ is analytic using Cauchy-Riemann Conditions.

$$f(z) = z^2 = x^2 - y^2 + i 2xy$$

$$\therefore u = x^2 - y^2 \quad , v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad , \frac{\partial v}{\partial y} = 2x \quad \Rightarrow \quad \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y \quad , \frac{\partial v}{\partial x} = 2y \quad \Rightarrow \quad \therefore \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore f(z)$ is analytic.

Ex. Is the function $f(z) = x^2 + y^2 - i 2xy$ analytic?

$$\therefore u = x^2 + y^2 \quad , v = -2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad , \frac{\partial v}{\partial y} = -2x \quad \Rightarrow \quad \therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$\therefore f(z)$ is not analytic.

Results:-

- (1) If u and v satisfy C.R.C., then the function $f(z) = u + iv$ is analytic.
- (2) If u and v satisfy C.R.C., then the lines $u = c$, $v = k$ (c and k are constants) are orthonormal.

Proof of Results (3):

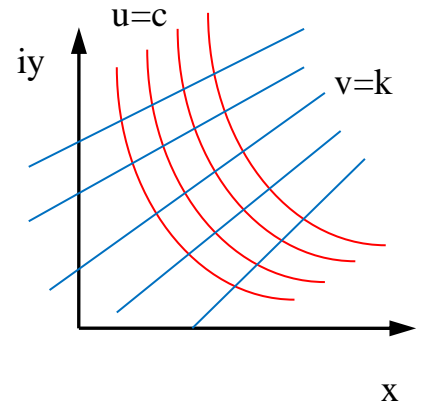
$$u = c, \quad du = 0 = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \text{slop}|_u \quad \dots(1)$$

$$v = k, \quad dv = 0 = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \text{slop}|_v \quad \dots(2)$$

$$\text{slop}|_u \times \text{slop}|_v = \left(\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \right) = -1$$



\therefore The lines $u = c$ and $v = k$ are orthonormal.

Harmonic function

A real valued function $\phi(x, y)$ that has continuous 2nd order partial derivatives in a domain D and satisfies the Laplace equations:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Is known as Harmonic function

Theorem: If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D , then its real & imaginary part will satisfy Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

i.e. u & v are Harmonic functions.

Proof: Suppose $f(z)$ is analytic in some region.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{diff. w. r. t. } x \quad \rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (1)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{diff. w. r. t. } y \quad \rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (2)$$

Adding eq.(1) to eq.(2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Or} \quad \nabla^2 u = 0$$

In a similar way we find that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{Or} \quad \nabla^2 v = 0$$

We call u and v the harmonic functions.

So, If $f(z) = u + iv$ is analytic, then its real & imaginary part are Harmonic functions.

v is said to be the conjugate harmonic of u .

Remark: The converse is not true in general.

i.e. if u & v are any two harmonic functions, then $f(z) = u + iv$ **is not** necessarily analytic..

Ex. Consider $u = x^2 - y^2$ & $v = 3x^2y - y^3$

Sol.

$$u = x^2 - y^2 \quad \rightarrow \quad \left. \begin{array}{l} \frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2 \\ \frac{\partial u}{\partial y} = -2y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2 \end{array} \right\} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$v = 3x^2y - y^3 \quad \rightarrow \quad \left. \begin{array}{l} \frac{\partial v}{\partial x} = 6xy \Rightarrow \frac{\partial^2 v}{\partial x^2} = 6y \\ \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 v}{\partial y^2} = -6y \end{array} \right\} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

u & v are harmonic function.

Check the $f(z) = u + iv = (x^2 - y^2) + i(3x^2 - y^3)$ is analytic function

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \Rightarrow \quad \therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 6xy \quad \Rightarrow \quad \therefore \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

Is $f(z)$ not analytic function.

Results:-

(1) If u and v satisfy C.R.C., then both u and v satisfy Laplace equation in 2-D.

Ex. Verify that $u = x^2 - y^2 - y$ is harmonic and find a conjugate harmonic function v of u . Hence express $u + iv$ as an analytic function of z .

Sol. $u = x^2 - y^2 - y$

$$u_x = 2x \quad \rightarrow \quad u_{xx} = 2$$

$$u_y = -2y - 1 \quad \rightarrow \quad u_{yy} = -2$$

$$\therefore u_{xx} + u_{yy} = 0 \quad \therefore u \text{ is harmonic}$$

To find v by using C.R.E.

$$v_y = u_x = 2x \quad \dots(1)$$

$$v_x = -u_y = 2y + 1 \quad \dots(2)$$

Integration eq.(1) with respect to y

$$v = 2xy + h(x) \quad \xrightarrow{\text{Diff. w.r.t } x} \quad v_x = 2y + h'(x) \quad \dots(3)$$

Comparison eq.(3) with eq.(2)

$$2y + h'(x) = 2y + 1 \quad \rightarrow \quad h'(x) = 1 \quad \xrightarrow{\text{Integral}} \quad \therefore h(x) = x + c$$

$$\therefore v = 2xy + x + c$$

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = (x^2 - y^2 - i2xy) + (-y + ix) + ic$$

$$\therefore f(z) = z^2 + iz + ic$$

C.R.E. in Polar form.

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r} \frac{\partial v}{\partial \theta} = -\frac{\partial u}{\partial r}$$

$$f(z) = u(x, y) + iv(x, y) \\ = u(r, \theta) + iv(r, \theta)$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} \quad , \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

Hint:

$$x = r \cos \theta$$

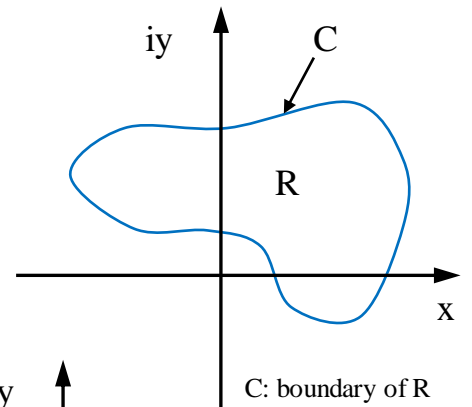
$$y = r \sin \theta$$

H.W. Proof That.

H.W. Write Laplace equations for u and v in polar form.

H.W. Solve 8 problems about differentiation and analytic function.

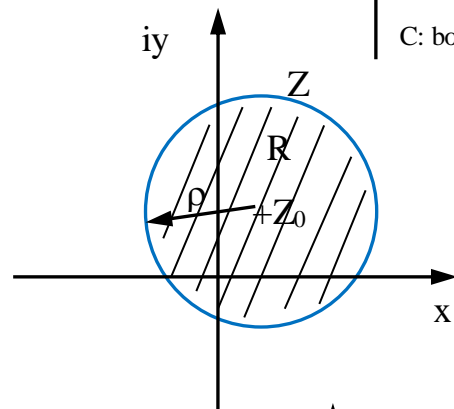
Regions in Complex Plane



(1) Open region:

In this region the boundaries don't belong to the region. $C \notin R$

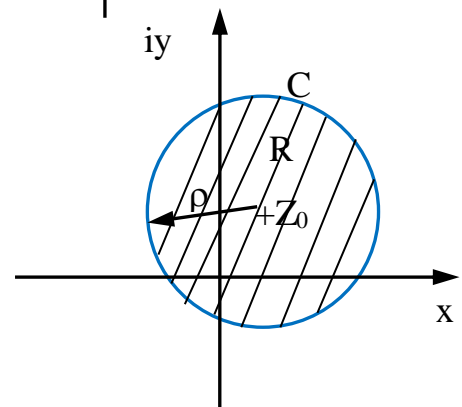
$$R = \{z : |z - z_0| < \rho\}$$



(2) Close region:

The boundaries points belong to the region. $C \in R$

$$R = \{z : |z - z_0| \leq \rho\}$$



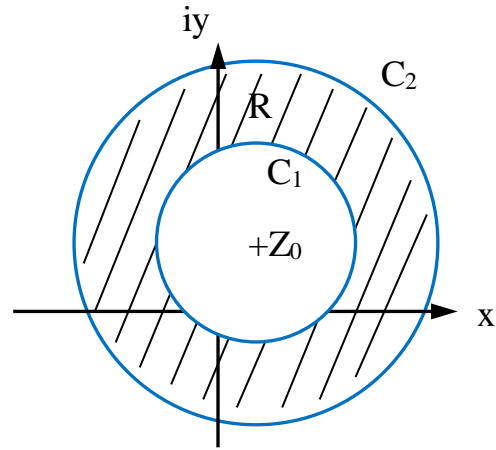
(3) Neither open nor closed:

In this region part of the boundary does not belong to the region.

$$R = \{z : C_1 \leq |z - z_0| < C_2\}$$

$$C_1 \in R$$

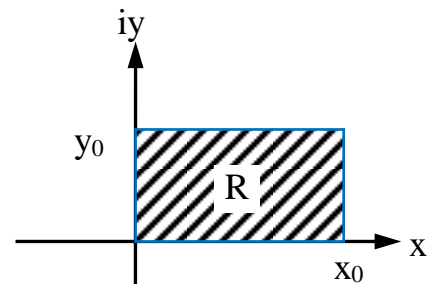
$$C_2 \notin R$$



(4) Bounded region:

In this region, each two points can be connected by a line of finite length.

$$R = \{z : 0 \leq \text{Re}(z) \leq x_0, 0 \leq \text{Im}(z) \leq y_0\}$$



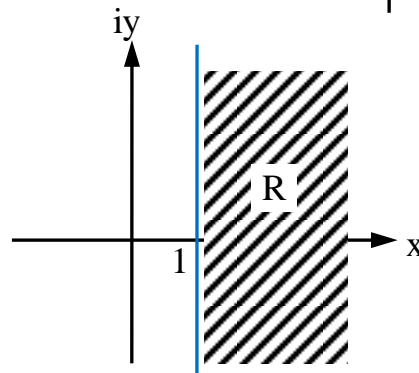
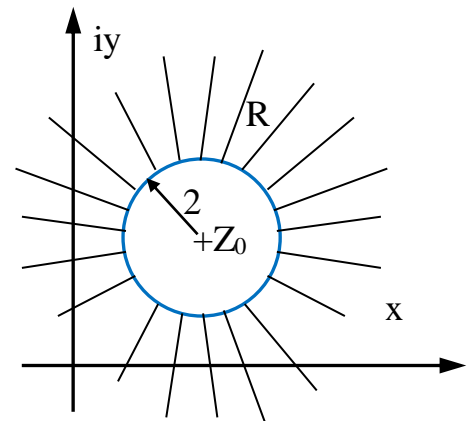
(5) Unbounded region:

In this region, there exist at least two points that not be connected by a finite length.

$$R = \{z : |z - z_0| \geq 2\}$$

or

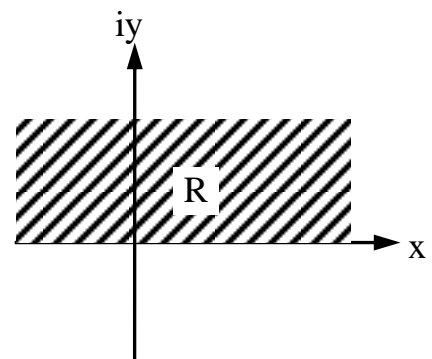
$$R = \{z : \text{Re}(z) > 1\}$$



(6) Connected region:

In that region, each two points can be connected by a path all of its points belong to region.

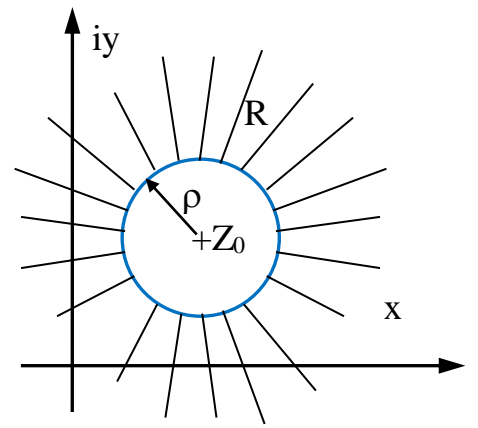
Ex. $R = \{z : \text{Im}(z) \geq 0\}$



(a) Simple connected there is no holes.

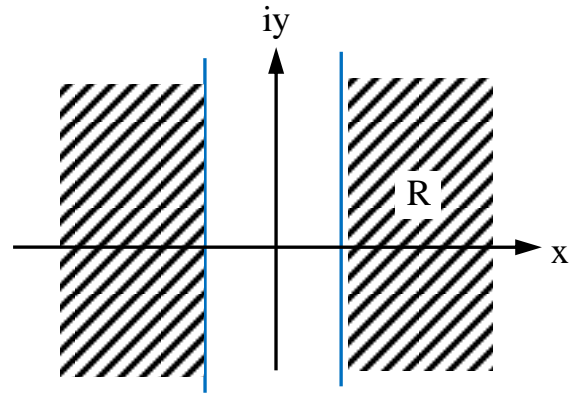
(b) multiply connected there is one or more holes.

$$R = \{z : |z - z_0| \geq \rho\}$$



(7) Unconnected region:

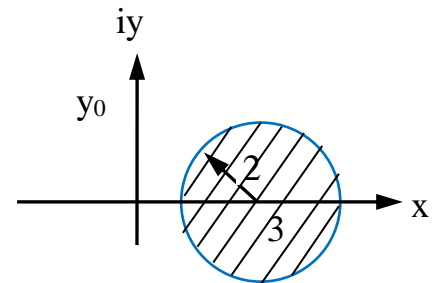
A region in which there exist two or more points that not can be connected by a line all of its points belong to region.



Describe and draw the following regions:

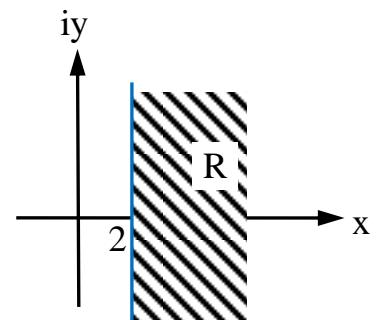
(1) $R = \{z : |z - 3| \leq 2\}$

close, bounded, simply connected region.



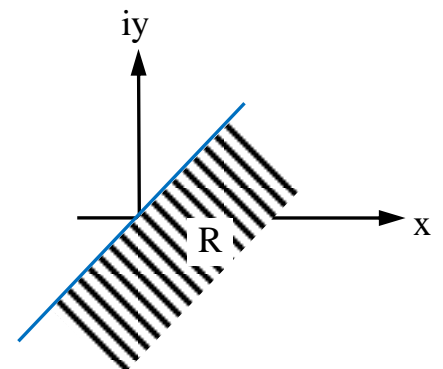
(2) $R = \{z : \text{Re}(z) \geq 2\}$

Neither open nor close, unbounded, simply connected region.



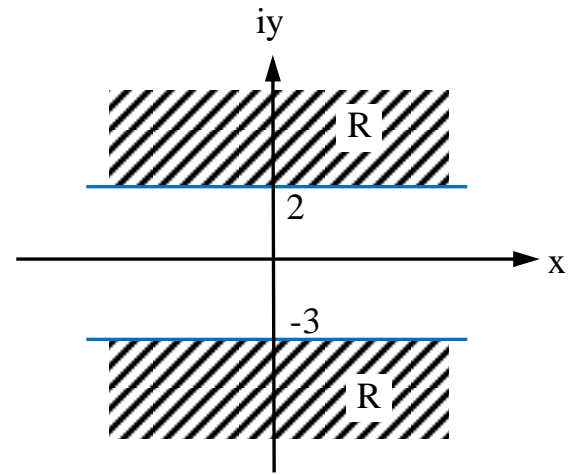
(3) $R = \{z : \text{Re}(z) \geq \text{Im}(z)\}$

Neither open nor close, unbounded, simply connected region.



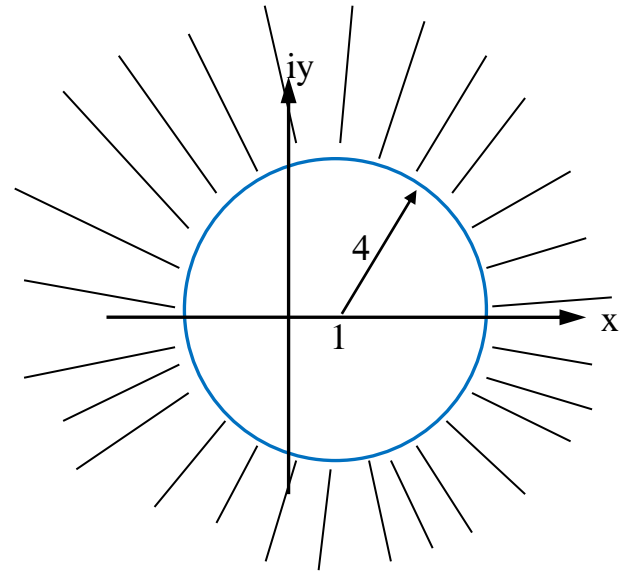
$$(4) R = \{z : \operatorname{Im}(z) \geq 2, \operatorname{Im}(z) \leq -3\}$$

Neither open nor close, unbounded, disconnected region.



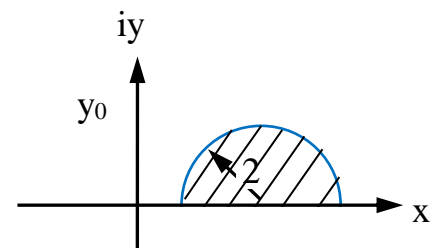
$$(5) R = \{z : |z - 1| > 4\}$$

open, unbounded, multiply connected region.



$$(6) R = \{z : 0 \leq \operatorname{Arg}(z) \leq \pi, |z| \leq 2\}$$

close, bounded, simply connected region.



Complex Integrals:-

If $f(z)$ is a single-value, continuous function in some region R , then we define the integral of $f(z)$ along path C in R as:-

$$f(z) = u + iv, \quad dz = dx + idy$$

$$\int_c f(z) dz = \int_c (u + iv)(dx + idy)$$

$$\int_c f(z) dz = \int_c (u dx - v dy) + i \int_c (v dx + u dy)$$

Ex. Find $\int_{1+i}^{2+i2} z dz$ along the following paths-

(a) horizontally from $1 + i$ to $2 + i$ then vertically to $2 + i2$.

(b) vertically from $1 + i$ to $1 + i2$ then horizontally to $2 + i2$.

(c) along line from $1 + i$ directly to $2 + i2$.

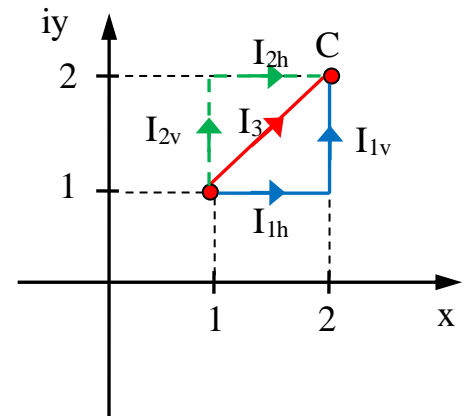
Sol. (a) $I_1 = I_{1h} + I_{1v}$

a long I_{1h} ; $y = 1, \quad dy = 0$
 $dx : 1 \rightarrow 2$

$$\begin{aligned} I_{1h} &= \int_{1+i}^{2+i} (u dx - v dy) + i \int_{1+i}^{2+i} (v dx + u dy) \\ &= \int_1^2 x dx + i \int_1^2 y dx = \left. \frac{x^2}{2} \right|_1^2 + iyx \Big|_1^2 \\ &= \frac{3}{2} + i [1 * (2-1)] = \frac{3}{2} + i \end{aligned}$$

a long I_{1v} ; $x = 2, \quad dx = 0$
 $dy : 1 \rightarrow 2$

$$\begin{aligned} I_{1v} &= \int_{1+i}^{2+i2} (u dx - v dy) + i \int_{1+i}^{2+i2} (v dx + u dy) \\ &= \int_1^2 -y dy + i \int_1^2 x dy = -\left. \frac{y^2}{2} \right|_1^2 + i xy \Big|_1^2 \\ &= -\frac{3}{2} + i [2 * (2-1)] = -\frac{3}{2} + i2 \end{aligned}$$



$$\therefore I_1 = \left(\frac{3}{2} + i\right) + \left(-\frac{3}{2} + i2\right) = i3$$

(b) $I_2 = I_{2v} + I_{2h}$

a long I_{2v} ; $x = 1, dx = 0$
 $dy : 1 \rightarrow 2$

$$I_{2v} = \int_1^2 -y dy + i \int_1^2 x dy = -\frac{y^2}{2} \Big|_1^2 + i xy \Big|_1^2 = -\frac{3}{2} + i [1*(2-1)] = -\frac{3}{2} + i1$$

a long I_{2h} ; $y = 2, dy = 0$
 $dx : 1 \rightarrow 2$

$$I_{2h} = \int_1^2 x dx + i \int_1^2 y dx = \frac{x^2}{2} \Big|_1^2 + i y x \Big|_1^2$$

$$= \frac{3}{2} + i [2*(2-1)] = \frac{3}{2} + i2$$

$$\therefore I_2 = \left(-\frac{3}{2} + i\right) + \left(\frac{3}{2} + i2\right) = i3$$

(c) Along the line (1, 1) ---> (2, 2) the line has the equation

$y = x$
 $dy = dx$ <-----using $\frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(y - y_1)}{(x - x_1)}$

$$I_3 = \int_{1+i}^{2+i} (u dx - v dy) + i \int_{1+i}^{2+i} (v dx + u dy)$$

$$= \int_{1+i}^{2+i} (x dx - y dy) + i \int_{1+i}^{2+i} (y dx + x dy)$$

$$= i \int_{1+i}^{2+i} 2x dx = ix^2 \Big|_{1+i}^{2+i} = i3$$

$\therefore I_1 = I_2 = I_3$

Note: The integration of an analytic function does not depend on the path.

$$\therefore \int_{1+i}^{2+i2} z dz = \frac{z^2}{2} \Big|_{1+i}^{2+i2} = \frac{(2+i2)^2}{2} - \frac{(1+i)^2}{2} = i3$$

This is coincidence if f(z) analytic only.

H.W. Evaluate $\int_{1+i2}^{2+i3} \bar{z} dz$ along the following paths-

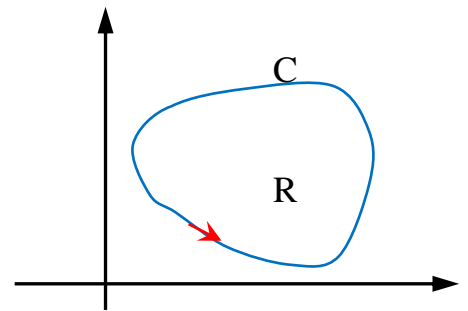
- (a) horizontally from $1 + i2$ to $2 + i2$ then vertically to $2 + i3$.
- (b) vertically from $1 + i$ to $1 + i3$ then horizontally to $2 + i3$.

Contour Integration

When the integration starts and ends at the same point along a closed path "C", it is called "Contour Integration".

Positive direction: moving along C such that R is to your left.

It is denoted: \oint



Pole: The value of z which makes the combination $\frac{f(z)}{z - z_0}$ to not analytic; i.e. the pole is z_0 . (the denominator is zero)

Theorems:-

(1) Cauchy-Goursat Theorem:- If $f(z)$ is analytic in a simply connected bounded region R then $\oint_C f(z) dz = 0$ for every simple closed path C lying in the region R.

Proof:-

$$\oint_C f(z) dz = \oint_C (u dx - v dy) + i \oint_C (v dx + u dy)$$

Using Green's Theorem

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C (Q dy - P dx)$$

$$\oint_C f(z) dz = -\oint_C (v dy - u dx) + i \oint_C (u dy + v dx)$$

$$= -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

Using C.R.E.

$$\therefore \oint_C f(z) dz = 0$$

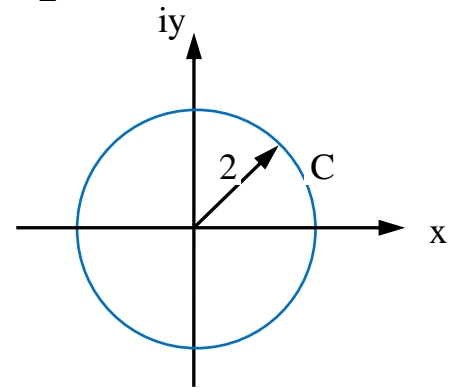
Ex. Find $\oint_C f(z) dz$, where $f(z) = z$ and C is the circle $|z| = 2$

$$z = re^{i\theta} \rightarrow dz = rie^{i\theta} d\theta$$

$$\oint_C z dz = \int_0^{2\pi} (re^{i\theta})(rie^{i\theta}) d\theta$$

$$= ir^2 \int_0^{2\pi} e^{i2\theta} d\theta = \frac{r^2}{2} [e^{i2\theta}]_0^{2\pi}$$

$$= \frac{r^2}{2} [\cos 2\pi + i \sin 2\pi]_0^{2\pi} = 0$$



(2)

$$\oint_C \frac{dz}{(z - z_0)^{n+1}} = \begin{cases} i 2\pi & \text{if } n=0, \text{ and } z_0 \text{ inside } C \\ 0 & \text{if } n \neq 0, \text{ or } z_0 \text{ outside } C \end{cases}$$

if $n=0$, and z_0 inside C

if $n \neq 0$, or z_0 outside C

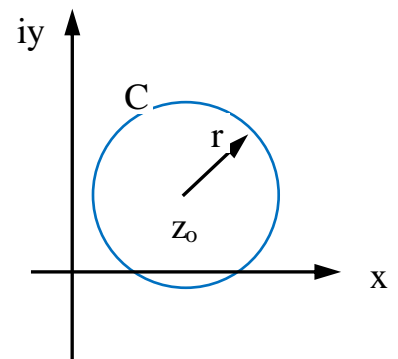
Proof:

$$z - z_0 = re^{i\theta}$$

$$dz = ire^{i\theta} d\theta$$

$$\oint_C \frac{dz}{(z - z_0)^{n+1}} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{r^{n+1} e^{i(n+1)\theta}} = \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta} d\theta$$

$$= \frac{i}{r^n} \int_0^{2\pi} (\cos n\theta - i \sin n\theta) d\theta$$



When $n = 0$

$$= \frac{i}{r^0} \int_0^{2\pi} (1 - 0) d\theta = i 2\pi$$

if $n \neq 0$

$$= \frac{i}{r^n} \int_0^{2\pi} (\cos n\theta - i \sin n\theta) d\theta = 0$$

If z_0 outside C , then $\frac{1}{(z - z_0)^{n+1}}$ will be analytic inside C hence as integral around C

is zero.

Ex. Evaluate $\oint \frac{dz}{(z-i)}$ around the following paths:-

a. the circle $|z-i|=2$

b. the circle $|z-1|=1$

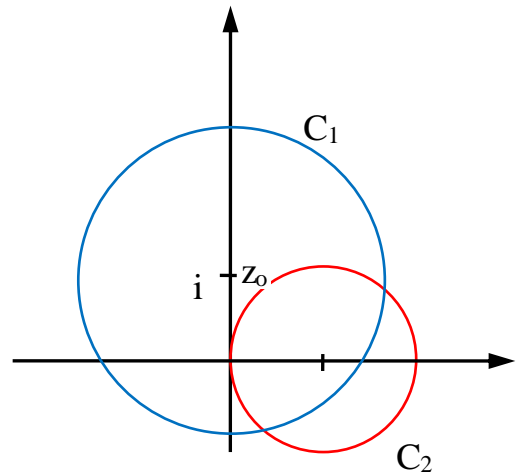
Sol. $n = 0$, Pole = $i = z_0$

a. along C_1

$$\oint \frac{dz}{(z-i)} = i2\pi \text{ because } z_0 \text{ inside } C_1$$

b. along C_2

$$\oint \frac{dz}{(z-i)} = 0 \text{ because } z_0 \text{ outside } C_2$$



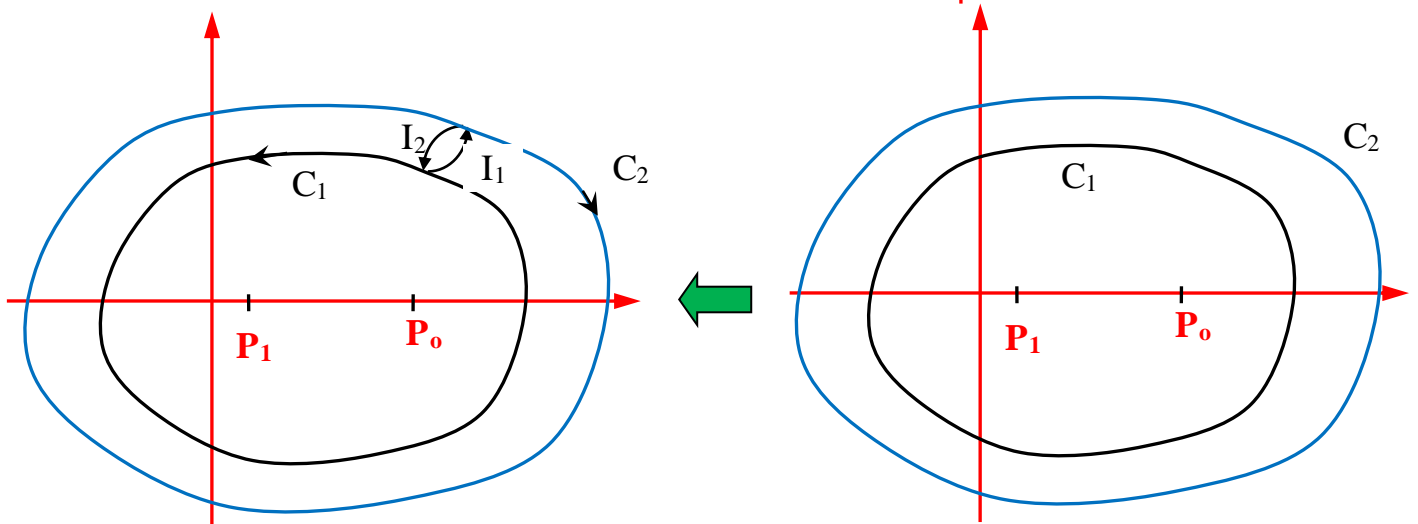
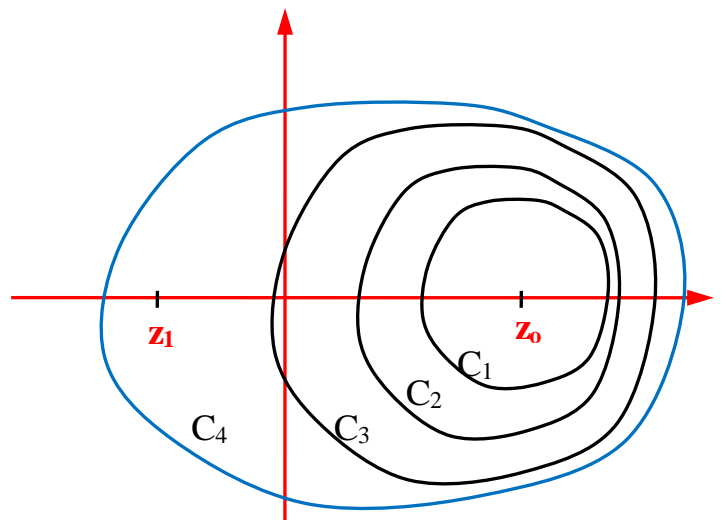
Theorem (3):- The path of integration around z_0 can be deformed freely without affecting the value of integration given that the new path contains the same number of poles.

$$\oint_{c_1} = \oint_{c_2} = \oint_{c_3} \neq \oint_{c_4}$$

Since C_4 contains z_0, z_1

Proof:-

Assume P_1 and P_0 are poles.

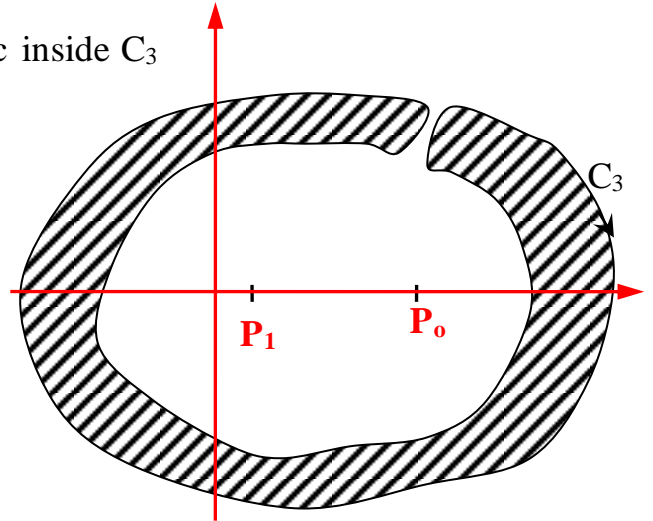


C_3 does not contain any pole, thus $f(z)$ is analytic inside C_3

$$\oint_{c_1} + \int_{I_1} + \int_{I_2} - \oint_{c_2} = 0 = \oint_{c_3}$$

$$\therefore \int_{I_1} = - \int_{I_2}$$

$$\therefore \oint_{c_1} = \oint_{c_2}$$



Theorem (4) Morera's Theorem

If $f(z)$ is analytic inside and on C ;

$$\oint \frac{f(z)}{(z-z_o)^{n+1}} dz = \begin{cases} \frac{i 2\pi}{n!} f^n(z_o) & \text{if } z_o \text{ inside or on } C \\ 0 & \text{if } z_o \text{ outside } C \end{cases}$$

if z_o inside or on C

if z_o outside C

Proof:

Let $n = 0$ ---> **Cauchy's Integral Formula**

$$\oint \frac{f(z)}{(z-z_o)} dz = \oint \frac{f(z) - f(z_o) + f(z_o)}{z-z_o} dz$$

assuming C is very small closed path around z_o .

$$\lim_{z \rightarrow z_o} \oint \frac{f(z) - f(z_o) + f(z_o)}{z-z_o} dz$$

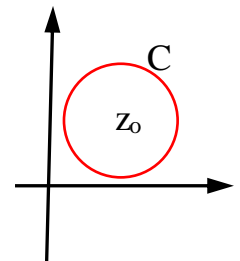
$$= \oint \left[\lim_{z \rightarrow z_o} \frac{f(z) - f(z_o)}{z-z_o} + \lim_{z \rightarrow z_o} \frac{f(z_o)}{z-z_o} \right] dz$$

$$= \oint \left[f'(z_o) + \lim_{z \rightarrow z_o} \frac{f(z_o)}{z-z_o} \right] dz$$

$$= \oint \cancel{f'(z_o)}^{=0} dz + \lim_{z \rightarrow z_o} \oint \frac{f(z_o)}{z-z_o} dz$$

$$= 0 + \lim_{z \rightarrow z_o} f(z_o) \oint \frac{dz}{z-z_o}$$

$$= i 2\pi f(z_o)$$



$$\oint \frac{f(z)}{(z-z_0)} dz = i 2\pi f(z_0) \quad \text{differentiation with respect to } z_0$$

$$\oint (-1) \frac{f(z)}{(z-z_0)^2} (-1) dz = i 2\pi f'(z_0)$$

$$\oint \frac{f(z)}{(z-z_0)^2} dz = \frac{i 2\pi}{1!} f'(z_0) \quad \text{diff. w.r.t. } z_0$$

$$\oint (-2) \frac{f(z)}{(z-z_0)^3} (-1) dz = i 2\pi f''(z_0)$$

$$\oint \frac{f(z)}{(z-z_0)^3} dz = \frac{i 2\pi}{2!} f''(z_0)$$

∴ ∴

$$\therefore \oint \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{i 2\pi}{n!} f^n(z_0)$$

Ex. Evaluate $\oint_C \frac{e^z}{z^2-1} dz$ where C is

- (1) the circle $|z| = 1/2$
- (2) the circle $|z+1| = 1$
- (3) the circle $|z-1| = 1$
- (4) the rectangular from (-2, -2) to (2, 2)
- (5) the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Sol. $\oint_C \frac{e^z}{z^2-1} dz = \oint_C \frac{e^z}{(z-1)(z+1)} dz$

$$\frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$A(z+1) + B(z-1) = 1$$

$$(A+B)z + (A-B) = 1$$

$$\left. \begin{array}{l} A+B=0 \\ A-B=1 \end{array} \right\} \rightarrow \begin{array}{l} A=1/2 \\ B=-1/2 \end{array}$$

Or

$$A = \lim_{z \rightarrow 1} \frac{1}{z+1} = \frac{1}{2}$$

$$B = \lim_{z \rightarrow -1} \frac{1}{z-1} = -\frac{1}{2}$$

$$\oint_c \frac{e^z}{z^2 - 1} dz = \frac{1}{2} \oint_c \frac{e^z}{(z-1)} dz - \frac{1}{2} \oint_c \frac{e^z}{(z+1)} dz$$

Poles $P_1 = 1, P_2 = -1$

(1) C_1 the circle $|z| = 1/2$

$$\oint_{C_1} \frac{e^z}{z^2 - 1} dz = 0 \text{ since } C_1 \text{ does not contain any poles}$$

(2) C_2 the circle $|z + 1| = 1$

Poles P_1 outside C_2, P_2 inside C_2

$$\begin{aligned} \oint_{C_2} \frac{e^z}{z^2 - 1} dz &= \frac{1}{2} \oint_{C_2} \frac{e^z}{(z-1)} dz - \frac{1}{2} \oint_{C_2} \frac{e^z}{(z+1)} dz \\ &= 0 - \frac{1}{2} (i 2\pi e^{-1}) = \frac{-i\pi}{e} \end{aligned}$$

(3) C_3 the circle $|z - 1| = 1$

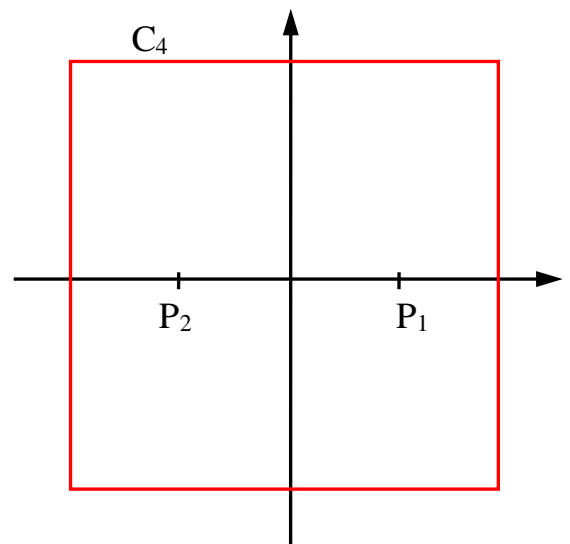
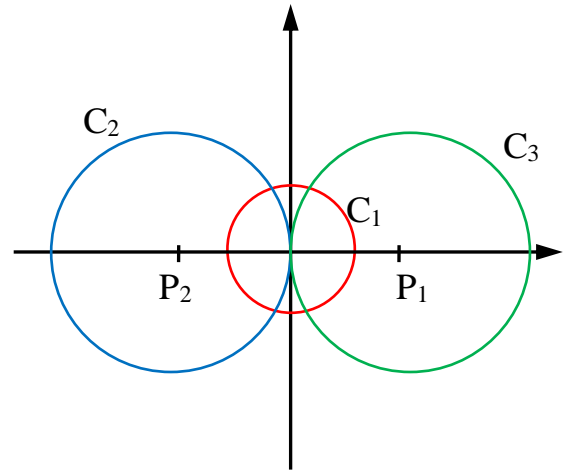
Poles P_1 inside C_3, P_2 Outside C_3

$$\begin{aligned} \oint_{C_3} \frac{e^z}{z^2 - 1} dz &= \frac{1}{2} \oint_{C_3} \frac{e^z}{(z-1)} dz - \frac{1}{2} \oint_{C_3} \frac{e^z}{(z+1)} dz \\ &= \frac{1}{2} (i 2\pi e^1) - 0 = i\pi e \end{aligned}$$

(4) C_4 the rectangular from $(-2, -2)$ to $(2, 2)$

The two poles P_1 and P_2 inside C_4

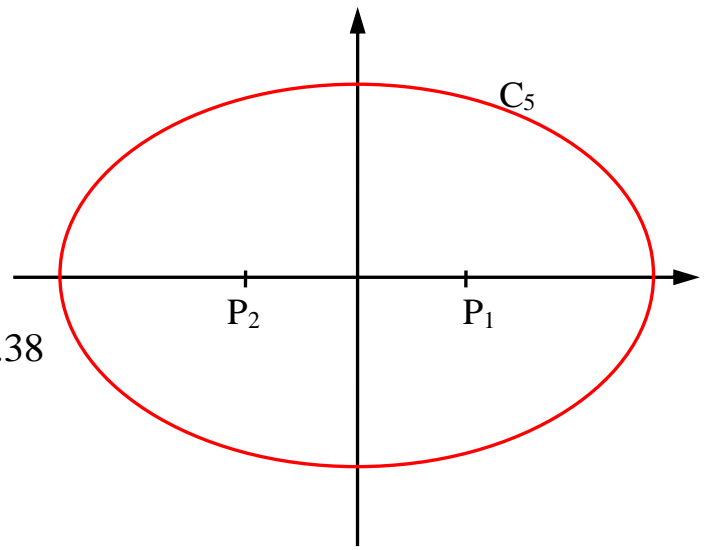
$$\begin{aligned} \oint_{C_4} \frac{e^z}{z^2 - 1} dz &= \frac{1}{2} \oint_{C_4} \frac{e^z}{(z-1)} dz - \frac{1}{2} \oint_{C_4} \frac{e^z}{(z+1)} dz \\ &= \frac{1}{2} (i 2\pi e^1) - \frac{1}{2} (i 2\pi e^{-1}) \\ &= i\pi e^1 - i\pi e^{-1} = i 2\pi \sinh 1 = i 7.38 \end{aligned}$$



(5) the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

The two poles P_1 and P_2 inside C_4

$$\oint_{C_5} \frac{e^z}{z^2 - 1} dz = \oint_{C_4} \frac{e^z}{z^2 - 1} dz = i 2\pi \sinh 1 = i 7.38$$



Ex. Evaluate $\oint_C \frac{z^2}{z^3 + z^2 - z - 1} dz$ where C is

- (1) the circle $|z - 1| = 1$
- (2) the circle $|z + 1 - i| = 2$
- (3) the ellipse $\frac{x^2}{2} + y^2 = 2$

Sol.

$$\begin{aligned} f(z) &= z^2 \\ \therefore z^3 + z^2 - z - 1 &= (z + 1)(z^2 - 1) \\ &= (z + 1)(z + 1)(z - 1) \\ &= (z + 1)^2 (z - 1) \end{aligned}$$

Synthetic division

-1	1	1	-1	-1
	-1	0	0	1
	1	0	-1	0

$(z^2 - 1) = 0$

$$\therefore \frac{1}{z^3 + z^2 - z - 1} = \frac{1}{(z + 1)^2 (z - 1)} = \frac{Az + B}{(z + 1)^2} + \frac{C}{z - 1}$$

$$Az^2 - Az + Bz - B + Cz^2 + 2Cz + C = 1$$

$$\left. \begin{aligned} A + C &= 0 && \rightarrow A = -C \\ B - A + 2C &= 0 && \rightarrow B = -3C \\ C - B &= 1 && \rightarrow C = 1/4 \end{aligned} \right\} \Rightarrow A = \frac{-1}{4}, B = \frac{-3}{4}, C = \frac{1}{4}$$

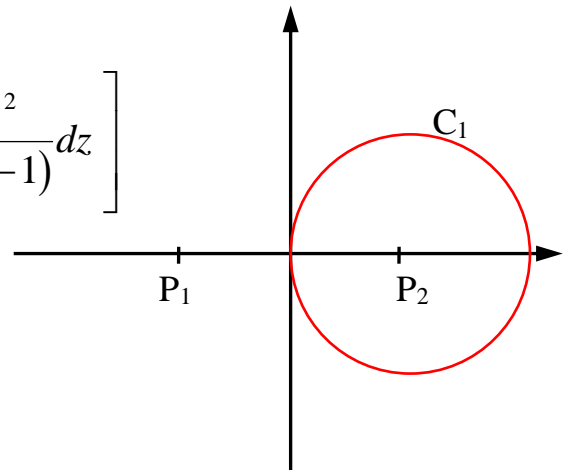
$$\oint_C \frac{z^2}{(z + 1)^2 (z - 1)} dz = \frac{-1}{4} \left[\oint_C \frac{z^2 (z + 3)}{(z + 1)^2} dz - \oint_C \frac{z^2}{z - 1} dz \right]$$

Poles $P_1 = -1, P_2 = 1$

(1) C_1 the circle $|z - 1| = 1$

$$\oint_{C_1} \frac{z^2}{(z+1)^2(z-1)} dz = \frac{-1}{4} \left[\oint_{C_1} \frac{z^2(z+3)}{(z+1)^2} dz - \oint_{C_1} \frac{z^2}{z-1} dz \right]$$

$$= \frac{-1}{4} [0 - i2\pi(1)^2] = \frac{i\pi}{2}$$



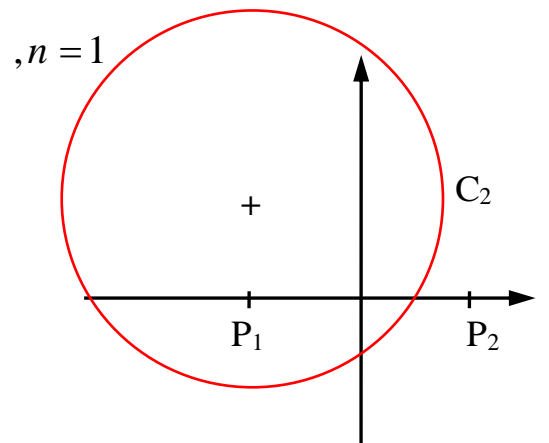
(2) C_2 the circle $|z + 1 - i| = 2$

$$\oint_{C_2} \frac{z^2}{(z+1)^2(z-1)} dz = \frac{-1}{4} \left[\oint_{C_2} \frac{(z^3 + 3z^2)}{(z+1)^2} dz - 0 \right], n=1$$

$$= \frac{-1}{4} \left[\frac{i2\pi}{1!} \frac{d}{dz} (z^3 + 3z^2) \Big|_{z=-1} \right]$$

$$= \frac{-i\pi}{2} [3z^2 + 6z]_{z=-1}$$

$$= \frac{-i\pi}{2} (-3) = \frac{i3\pi}{2}$$

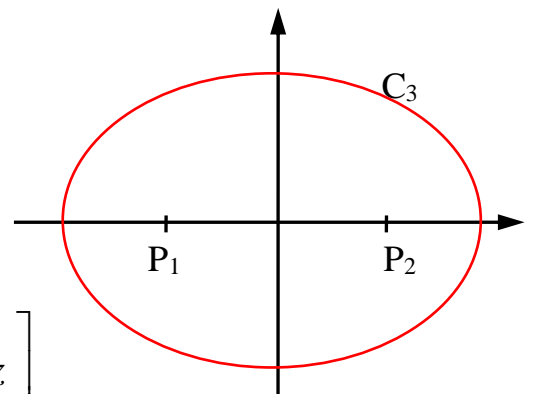


(3) C_3 the ellipse $\frac{x^2}{2} + y^2 = 2$

$$\frac{x^2}{4} + \frac{y^2}{2} = 1 \rightarrow a=2, b=\sqrt{2}$$

$$\oint_{C_3} \frac{z^2}{(z+1)^2(z-1)} dz = \frac{-1}{4} \left[\oint_{C_3} \frac{z^3 + 3z^2}{(z+1)^2} dz - \oint_{C_3} \frac{z^2}{z-1} dz \right]$$

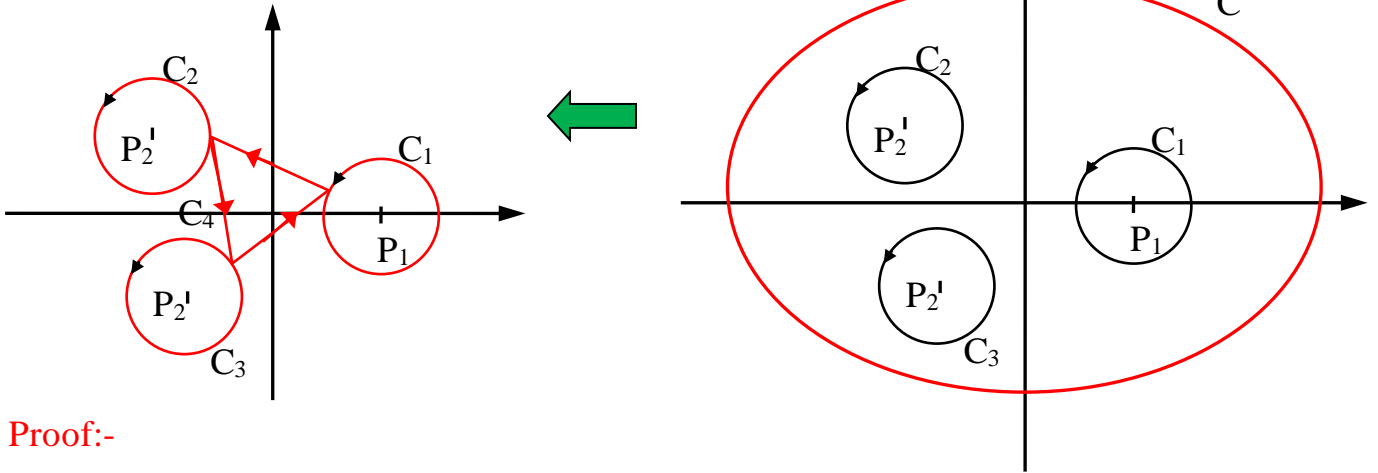
$$= \frac{i3\pi}{2} + \frac{i\pi}{2} = i2\pi$$



Residue Theorem:-

the contour integral around a path containing a number of poles equal to the sum of contour integration around paths of which contains a distinct pole.

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3}$$



Proof:-

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \oint_{C_4} \quad \text{outside poles} = 0$$

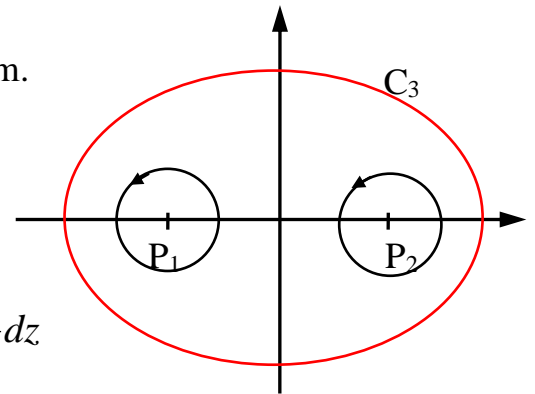
Ex. Resolve the last example by using the residue theorem.

take path (3) the ellipse $\frac{x^2}{2} + y^2 = 2$

$$\oint_{C_3} \frac{z^2}{(z+1)^2(z-1)} dz = \oint_{P_1} \frac{z^2}{(z+1)^2} dz + \oint_{P_2} \frac{z^2}{(z-1)^2} dz$$

$$= \frac{i2\pi}{1!} \left[\frac{d}{dz} \left(\frac{z^2}{(z-1)} \right) \right]_{z=-1} + i2\pi \left[\frac{z^2}{(z+1)^2} \right]_{z=1}$$

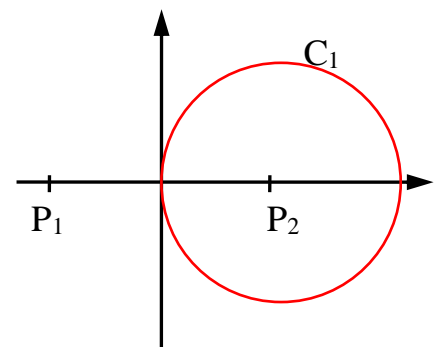
$$= \frac{i2\pi}{1!} \left[\frac{2z(z-1) - z^2}{(z-1)^2} \right]_{z=-1} + \frac{i\pi}{2} = \frac{i3\pi}{2} + \frac{i\pi}{2} = i2\pi$$



(2) (1) C_1 the circle $|z-1|=1$

$$\oint_{C_2} \frac{z^2}{(z+1)^2(z-1)} dz = \oint_{P_2} \frac{z^2}{(z-1)^2} dz$$

$$= \frac{i2\pi}{0!} \left[\frac{z^2}{(z+1)^2} \right]_{z=1} = \frac{i\pi}{2}$$



Ex. Evaluate $\oint_C \frac{\sin z}{z^2 2z + 5} dz$ where C is

(1) the circle $|z - 1 - i| = 2$

(2) the circle $|z| = 3$

$$z = \frac{-2 \mp \sqrt{4 - 20}}{2} = \begin{cases} -1 + i2 & , P_1 \\ -1 - i2 & , P_2 \end{cases}$$

Sol. $\frac{\sin z}{z^2 2z + 5} = \frac{\sin z}{(z + 1 - i2)(z + 1 + i2)}$

(1) C_1 the circle $|z - 1 - i| = 2$

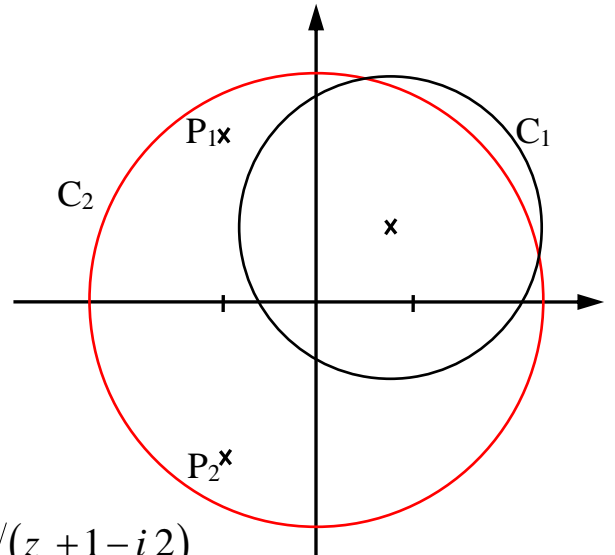
center = $1 + i$

radius = 2

The two poles outside $C_1 \therefore \oint_{C_1} = 0$

(2) C_2 the circle $|z| = 3$

The two poles inside C_2



$$\begin{aligned} \oint_{C_2} \frac{\sin z}{z^2 2z + 5} &= \oint_{P_1} \frac{\sin z / (z + 1 + i2)}{(z + 1 - i2)} dz + \oint_{P_2} \frac{\sin z / (z + 1 - i2)}{(z + 1 + i2)} dz \\ &= i2\pi \left[\frac{\sin z}{z + 1 + i2} \right]_{z=-1+i2} + i2\pi \left[\frac{\sin z}{z + 1 - i2} \right]_{z=-1-i2} \\ &= i2\pi \left[\frac{\sin(-1+i2)}{i4} \right] + i2\pi \left[\frac{\sin(-1-i2)}{-i4} \right] \\ &= \frac{\pi}{2} \sin(-1+i2) - \frac{\pi}{2} \sin(-1-i2) \\ &= \frac{\pi}{2} [2 \cos 1 \sin i2] = i\pi \cos 1 \sinh 2 \end{aligned}$$

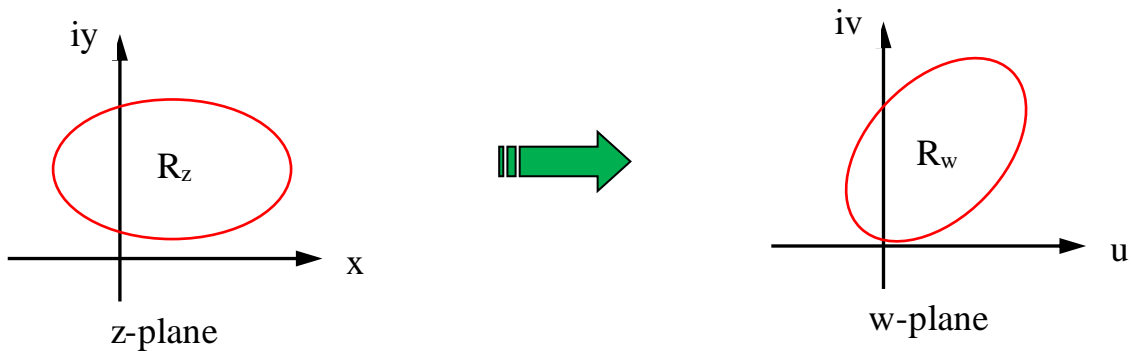
H.W. Solve problems about line and contour integration P.674 and P.703 in "Wylie".

Conformal Mapping

Mapping: is transformation from z -plane into w -plane using a function $w=f(z)$.

If $f(z)$ is analytic, then it is called "**conformal mapping**", that is, **angle-preserving**: the images of any two intersecting curves make the same angle of intersection, in both magnitude and sense, as the curves themselves.

Exceptions are the points at which $f'(z) = 0$. ("**Critical Points**" e.g. $z = 0$ for $w=z^2$)



Ex1. At which points is $w = e^z$ not conformal?

Sol: $f'(z) = e^z$. Since this is never zero the mapping is conformal everywhere

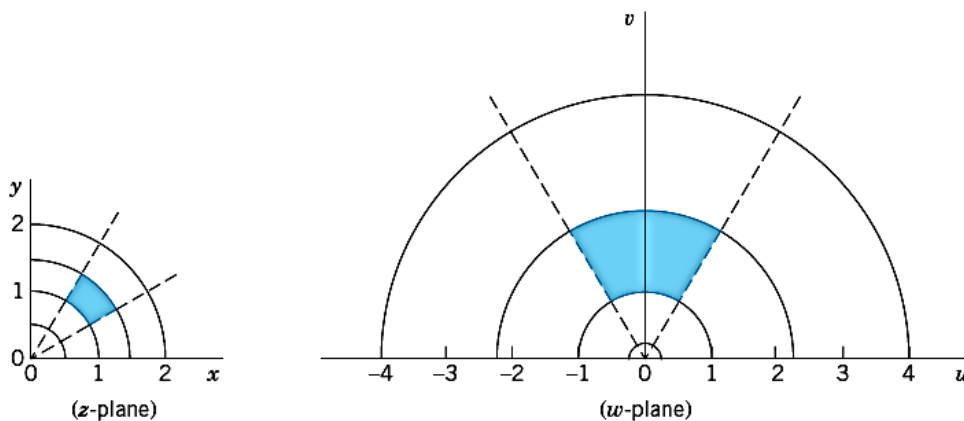
Ex2: Consider the mapping $w = z^2$

Using Polar forms: $z = re^{i\theta}$, and $w = Re^{i\phi}$

We have $w = z^2 = r^2 e^{i2\theta} = R e^{i\phi}$ by Comparing moduli and arguments gives;

$$\therefore R = r^2 \quad \text{and} \quad \phi = 2\theta$$

Ex. The region $1 \leq |z| \leq 3/2, \pi/6 \leq \theta \leq \pi/3$ which is mapped onto the region $1 \leq |w| \leq 9/4, \pi/3 \leq \theta \leq 2\pi/3$



****In Cartesian coordinates** $w = z^2 = (x^2 - y^2) + i2xy = u + iv$

$$u = (x^2 - y^2) \text{ and } v = 2xy$$

The point $z = 2 + i$ maps to $w = (2+i)^2 = 3 + 4i$. The point $z = 2 + i$ lies on the intersection of the two lines $x = 2$ and $y = 1$. To what curves do these map?

(i) Take the line $y = 1$.

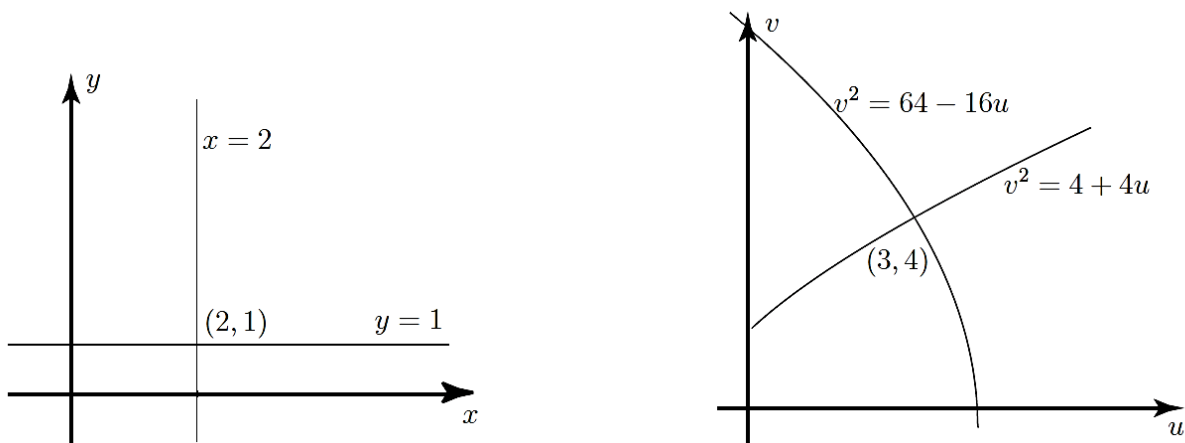
$u = x^2 - 1$ and $v = 2x$ by eliminating x we obtain:

$$v^2 = 4x^2 \text{ \& } x^2 = u + 1 \rightarrow \therefore v^2 = 4(u + 1) \Rightarrow v^2 = 4u + 4 \text{ This parabola eq.}$$

(ii) Take the line $x = 2$.

$u = 4 - y^2$ and $v = 8y$ by eliminating y we obtain:

$$v^2 = 16y^2 \text{ \& } y^2 = 4 - u \rightarrow \therefore v^2 = 16(4 - u) \Rightarrow v^2 = 64 - 16u \text{ This parabola eq.}$$



The angle between the original lines was clearly 90° at z -plane;

What is the angle between the curves at the point of intersection?

The curve $v^2 = 4u + 4$ has a gradient $\frac{dv}{du}$

$$\text{Diff. the eq. implicitly we obtain; } v^2 = 4u + 4 \Rightarrow 2v \frac{dv}{du} = 4 \rightarrow \therefore \frac{dv}{du} = \frac{2}{v}$$

$$\text{At the point } (3,4) \therefore \frac{dv}{du} = \frac{1}{2}$$

$$\text{For the curve } v^2 = 64 - 16u \text{ and evaluate it at the point } (3,4) \therefore \frac{dv}{du} = -2$$

$$\therefore \left. \frac{dv}{du} \right|_{y=1} \times \left. \frac{dv}{du} \right|_{x=2} = -1 \text{ the angle at intersection is also } 90^\circ$$

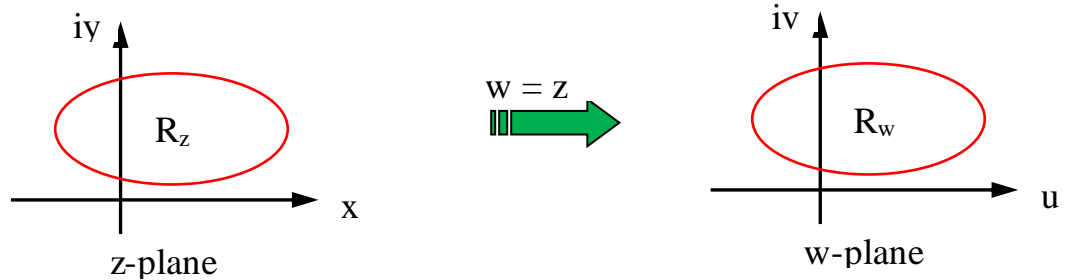
Since the angle between the lines and the angle between the curves is the same we say the angle is preserved.

Types of Mapping:-

1- **Linear mapping** : $w = az + b$ "a and b are complex number"

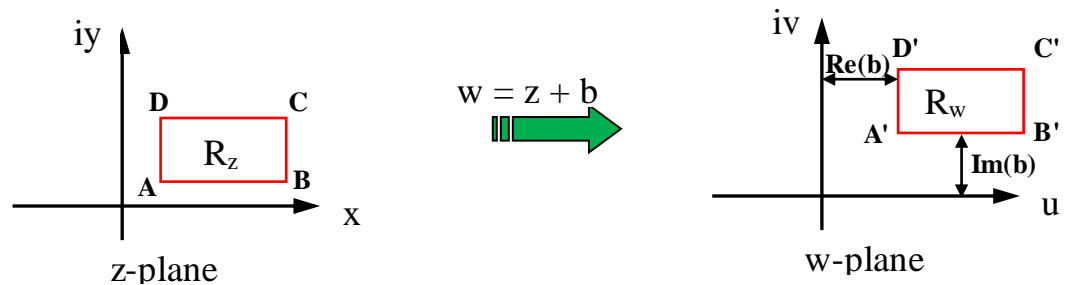
- If $a = 1$, $b = 0$ then:-

$w = z$ "**Identity mapping**"



- If $a = 1$, $b \neq 0$, then:-

$w = z + b$ "**Shifting mapping**"

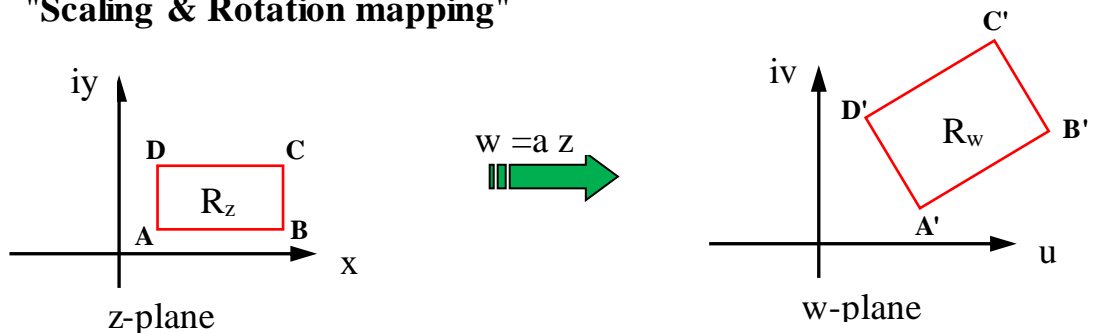


$$\text{Re}(A') = \text{Re}(A) + \text{Re}(b)$$

$$\text{Im}(A') = \text{Im}(A) + \text{Im}(b)$$

- If $b = 0$, $a \neq 0$ then:-

$w = az$ "**Scaling & Rotation mapping**"

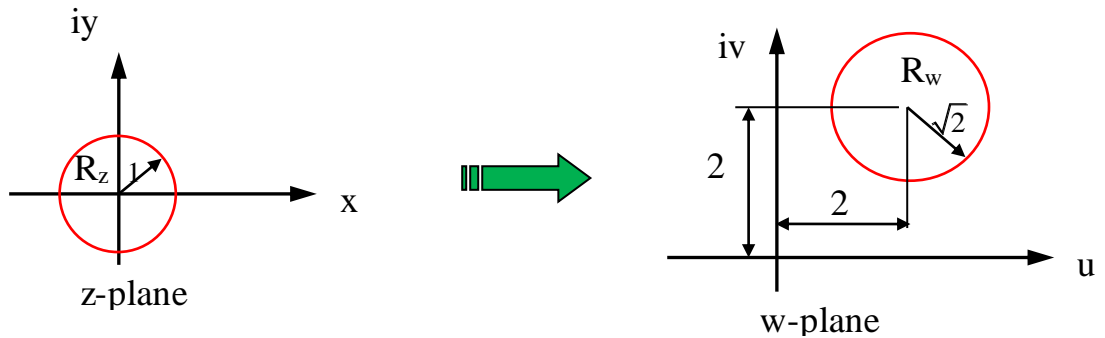


$$|A'| = |A| \cdot |a|$$

$$\text{Arg}(A') = \text{Arg}(A) + \text{Arg}(a)$$

Ex. Find R_w for the following transformation

$$w = (1-i)z + (2+i2) \quad \text{where} \quad R_z : \{z : |z| \leq 1\}$$

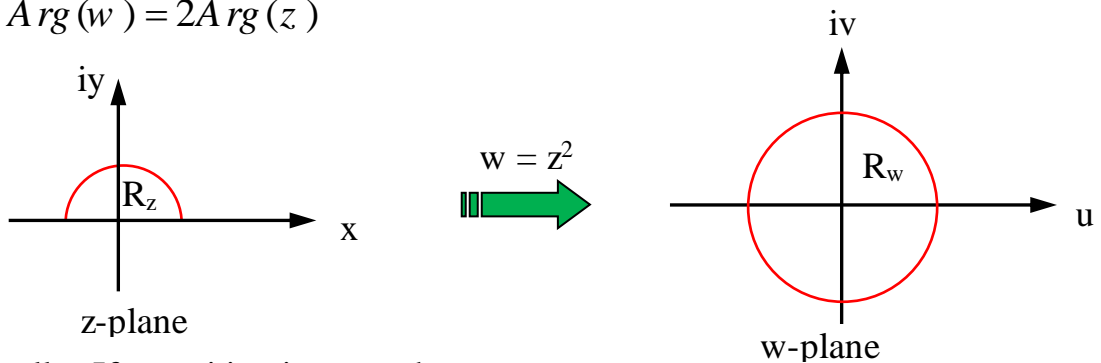


2. Power Mapping: $w = z^n$

- If $n = 1$ "Identity mapping linear"
- If $n = 2$ $\rightarrow w = z^2$ "squaring"

$$w = z^2 = r^2 \underline{2\theta}$$

$$|w| = |z|^2, \quad \text{Arg}(w) = 2\text{Arg}(z)$$



- Generally, If n positive integer, then:

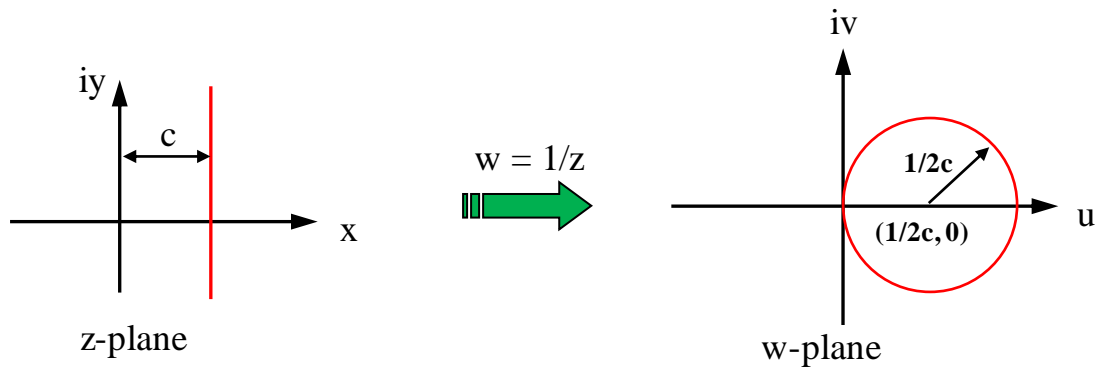
$$w = z^n \quad \Rightarrow \quad |w| = |z|^n, \quad \text{Arg}(w) = n \cdot \text{Arg}(z)$$

(3) Inversion Mapping: $w = \frac{1}{z}$

$$w = \frac{1}{z} \quad \Rightarrow \quad |w| = \frac{1}{|z|}, \quad \text{Arg}(w) = -\text{Arg}(z) \rightarrow \phi = -\theta$$

This mapping translates the straight lines in the z-plane to circles in the w-plane and vice versa.

Ex. The line $x = c$ is mapped into a circle of center $\left(\frac{1}{2c}, 0\right)$ and radius equal to $\frac{1}{2c}$



Sol. $w = \frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}$$

$$u = \frac{c}{c^2 + y^2} \Rightarrow y^2 = \frac{c - c^2 u}{u} \quad \dots(1)$$

$$v = -\frac{y}{c^2 + y^2} = \frac{\mp \sqrt{\frac{c - c^2 u}{u}}}{c^2 + \frac{c - c^2 u}{u}} = \mp \frac{\sqrt{\frac{c - c^2 u}{u}}}{\frac{c}{u}} = \mp \sqrt{\frac{c - c^2 u}{u}} \cdot \frac{u}{c}$$

$$v^2 = \frac{c - c^2 u}{u} \cdot \frac{u^2}{c^2} \Rightarrow v^2 = \frac{u}{c} - u^2$$

$$\therefore v^2 + u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2$$

$$v^2 + \left(u - \frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2 \quad \text{center } (1/2c, 0) \text{ and } r = 1/2c$$

The circle is rewriting as: $\left|w - \frac{1}{2c}\right| = \frac{1}{2c}$

(4) **Bilinear Mapping** : $w = \frac{az + b}{cz + d}$

In condition $\frac{a}{c} \neq \frac{b}{d}$

- If $c = 0, d = 1$ ---> linear mapping

- If $a = 0, b = 1, d = 0$ ----> inversion mapping

Theorem of Bilinear mapping

* The bilinear mapping cannot contain more than two identical points if so, then it is identity mapping.

** If there are three points in z -plane z_1, z_2, z_3 and their images in w -plane w_1, w_2, w_3 then they can be characterized by the bilinear mapping.

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

Ex. Three points in z -plane $i, 1, -1$ and three images in w -plane $2, 1, 0$. Find the bilinear mapping for $f(z)$.

Sol.
$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

$$\frac{w - 2}{w - 0} \cdot \frac{1 - 0}{1 - 0} = \frac{z - i}{z + 1} \cdot \frac{1 + 1}{1 - i}$$

$$\frac{w - 2}{-w} = \frac{2z - i2}{(z + 1)(1 - i)}$$

$$w(2z - i2) = (2 - w)(z + 1)(1 - i)$$

$$w[2z - i2 + (z + 1)(1 - i)] = (z + 1)(2 - i2)$$

$$\therefore w = \frac{(2 - i2)z + (2 - i2)}{(3 - i)z + (1 - i3)}$$

H.W. Find the bilinear mapping maps the x -axis into a semicircle of radius unity as shown.

Ans.
$$w = \frac{z + i}{iz + 1}$$

