IT201

Probability and Statistics for Computer Science

Mathematical Department/ College of Science/ University of Basra

Dr. Alaa Al-Muslimawi

Contents

1.1 Principles of probability theory

Definition: (***Random experiment***) is a process by which we observe something uncertain.

Definition: The set consisting of all possible outcomes of a particular experiment is called the ***sample space*** of that experiment.

Example:

a) Tossing a fair coin a time and observe the sequence of heads/tails. The sample space here may be defined as

C={H,T}

b) Tossing a fair coin two times and observe the sequence of heads/tails. The sample space here may be defined as

C={HH,HT,TH,TT}

c) Tossing a fair coin three times and observe the sequence of heads/tails. The sample space here may be defined as

d)Tossing a dice , the sample space is C={1,2,3,4,5,6}.

Definition: Any subset of the sample space is called an ***event***.

Example:

A coin is tossed twice and the outcome of each is recorded. Then,

C={HH,HT,TH,TT}

The event that the second toss was a Head is the subset

E={HH,TH}

1.2 The algebra of events

DEFINITION: The *union* of E and F, denoted $E \cup F$, consists of those outcomes that are in either E or F (or both). (we write $x \in E \cup F$ if $x \in E$ *or* $x \in F$.)

EXAMPLE: If $E = \{(H, H), (H, T)\}$ (Head on first toss) and $F = \{(T, T), (H, T)\}$ (tail on the second toss). Then,

$$E \cup F = \{(H, H), (H, T), (T, T)\}$$

only missing (T, H).

DEFINITION: The *intersection* of E and F, denoted EF or $E \cap F$, consists of those events that are in both E and F. (we write $x \in EF$ if $x \in E$ *and* $x \in F$.)

EXAMPLE: If $E = \{(H,\ H),\ (H,\ T)\}$ (Head on first toss) and $F = \{(T,\ T),\ (H,\ T)\}$ (tail on the second toss). Then

$$EF = \{(H,\ T)\}$$

DEFINITION: The *empty set* $\{\} = \emptyset$ is the set consisting of nothing.

DEFINITION: Two sets are *mutually exclusive* if $EF = \emptyset$. A set of sets, $\{E_1,\ E_2,\ \ldots\}$ are mutually exclusive if $E_iE_j = \emptyset$ for all $i \neq j$.

EXAMPLE: Let $S = \{(i,\ j) \mid i,\ j \in \{1,\ \ldots,\ 6\}\}$ be the outcome from two rolls of a die. Let E be those such that $i+j = 6$ (i.e. $E = \{(1,5),\ (2,\ 4),\ \ldots\}$) and F be those such that $i+j = 7$ (i.e. $F = \{(1,6),\ (2,\ 5),\ \ldots\}$). Then $EF = \emptyset$ and they are mutually exclusive.

The definition for the union and intersection of a sequence of events $E_1,\ E_2,\ \ldots$ are similar:

1. $\bigcup_{n=1}^{\infty} E_n$ is the event consisting of those outcomes that are in at least one E_n, for $n = 1, 2, \ldots$.

2. $\bigcap_{n=1}^{\infty} E_n$ is the event consisting of those outcomes that are in each E_n for $n = 1, 2, \ldots$.

DEFINITION: The *complement* of E, denoted E^c, consists of those outcomes that are not in E. Note that $S^c = \emptyset$ and that $(E^c)^c = E$.

DEFINITION: E is a *subset* of F if $x \in E$ implies $x \in F$. Notation: $E \subset F$.

DEFINITION: E and F are *equal*, denoted $E = F$, if $E \subset F$ and $F \subset E$.

Important Set Relations

Commutative laws:

$$E \cup F = F \cup E, \quad EF = FE$$

Associative laws:

$$E \cup (F \cup G) = (E \cup F) \cup G, \quad E(FG) = (EF)G$$

Distributive laws:

$$(EF) \cup H = (E \cup H)(F \cup H), \quad (E \cup F)H = (EH) \cup (FH)$$

De Morgan's laws:

$$\left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c$$

Note that if $E_i = \emptyset$ for $i \geq 3$, we have $(E \cup F)^c = E^c \cap F^c$.

$$\left(\bigcap_{i=1}^{\infty} E_i\right)^c = \bigcup_{i=1}^{\infty} E_i^c$$

Note that if $E_i = \emptyset$ for $i \geq 3$, we have $(E \cap F)^c = E^c \cup F^c$.

1.3 Permutations and Combinations

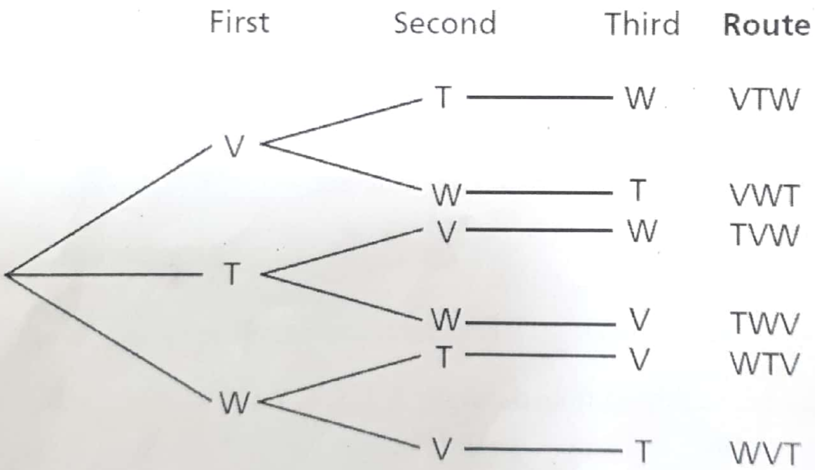
Definition: Each separate arrangement of all or part of a set of items is called **a *permutation***. The number of permutations is the number of different arrangements in which items can be placed. Notice that if the order of the items is changed, the arrangement is different, so we have a different permutation. The number of permutations of n items chosen r at a time is written nPr.

$$_nP_r = \frac{n!}{(n-r)!}$$

Example:

An engineer in technical sales must visit plants in Vancouver, Toronto, and Winnipeg. How many different sequences or orders of visiting these three plants are possible?

Answer: The number of different sequences is equal to $_3P_3 = 3! = 6$ different permutations. This can be verified by the following tree diagram:



Example:

How many 3-letter words with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?

Answer

The word 'LOGARITHMS' has 10 different letters.

Hence, the number of 3-letter words (with or without meaning) formed by using these letters

$= {}^{10}P_3 = 10 \times 9 \times 8 = 720$

Example:

In how many different ways can the letters of the word 'LEADING' be arranged such that the vowels should always come together?

Answer

The word 'LEADING' has 7 letters. It has the vowels 'E','A','I' in it and these 3 vowels should always come together. Hence these 3 vowels can be grouped and considered as a single letter. That is, LDNG(EAI).

Hence we can assume total letters as 5 and all these letters are different.

Number of ways to arrange these letters = 5! = 5 x 4 x 3 x 2 x 1 = 120.

In the 3 vowels (EAI), all the vowels are different.

Number of ways to arrange these vowels among themselves = 3! = 3 x 2 x 1= 6

Hence, required number of ways = 120 x 6= 720.

Example:

How many distinct permutations can be formed from all the letters of each of the following words: (a) them, (b) unusual?

Example:

How many words can be formed by using all letters of the word 'BIHAR'?

Answer

The word 'BIHAR' has 5 letters and all these 5 letters are different.

Total words formed by using all these 5 letters = $^5P_5$ = 5!

= 5 x 4 x 3 x 2 x 1 = 120.

Example:

How many 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9 which are divisible by 5 and none of the digits is repeated?

Answer

A number is divisible by 5 if the its last digit is a 0 or 5

We need to find out how many 3 digit numbers can be formed from the 6 digits (2,3,5,6,7,9) which are divisible by 5.

Since the 3 digit number should be divisible by 5, we should take the digit 5 from the 6 digits(2,3,5,6,7,9) and fix it at the unit place.

There is only 1 way of doing this

		1

Since the number 5 is placed at unit place, we have now five digits(2,3,6,7,9) remaining. Any of these 5 digits can be placed at tens place

	5	1

Since the digits 5 is placed at unit place and another one digits is placed at tens place, we have now four digits remaining. Any of these 4 digits can be placed at hundreds place.

4	5	1

Required Number of three digit numbers = 4 x 5 x 1 = 20

Definition: (Combination) a collection of things, in which the order does not matter.

Combinations are similar to permutations, but with the important difference that combinations take no account of order. Thus, AB and BA are different permutations but the same combination of letters. Then the number of permutations must be larger than the number of combinations.

In general, the number of combinations of n items taken r at a time is

$$_{n}C_{r} = \frac{_{n}P_{r}}{r!} = \frac{n!}{(n-r)!r!}$$

Example:

From a group of 7 persons, three persons are to be selected to form a committee. In how many ways can it be done?

Example:

A box contains 4 red, 3 white and 2 blue balls. Three balls are drawn at random. Find out the number of ways of selecting the balls of different colours?

Answer

1 red ball can be selected in $^{4}C_{1}$ ways

1 white ball can be selected in $^{3}C_{1}$ ways

1 blue ball can be selected in $^{2}C_{1}$ ways

Total number of ways

$= {}^{4}C_{1} \times {}^{3}C_{1} \times {}^{2}C_{1} = 4 \times 3 \times 2 = 24$

Example:

There are 8 men and 10 women and you need to form a committee of 5 men and 6 women. In how many ways can the committee be formed?

Answer

We need to select 5 men from 8 men and 6 women from 10 women

Number of ways to do this

$= {}^{8}C_{5} \times {}^{10}C_{6}$

$= {}^{8}C_{3} \times {}^{10}C_{4}$ [*Applied the formula* ${}^{n}C_{r} = {}^{n}C_{(n-r)}$]

=(8×7×63×2×1)(10×9×8×74×3×2×1)= 56 x 210= 11760

Example:

In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

Answer

In a group of 6 boys and 4 girls, four children are to be selected such that at least one boy should be there.

Hence we have 4 choices as given below

We can select 4 boys ------(Option 1).

Number of ways to this = ${}^{6}C_{4}$

We can select 3 boys and 1 girl ------(Option 2)

Number of ways to this = ${}^{6}C_{3} \times {}^{4}C_{1}$

We can select 2 boys and 2 girls ------(Option 3)

Number of ways to this = ${}^{6}C_{2} \times {}^{4}C_{2}$

We can select 1 boy and 3 girls ------(Option 4)

Number of ways to this = ${}^{6}C_{1} \times {}^{4}C_{3}$

Total number of ways

$= ({}^{6}C_{4}) + ({}^{6}C_{3} \times {}^{4}C_{1}) + ({}^{6}C_{2} \times {}^{4}C_{2}) + ({}^{6}C_{1} \times {}^{4}C_{3})$

$= ({}^{6}C_{2}) + ({}^{6}C_{3} \times {}^{4}C_{1}) + ({}^{6}C_{2} \times {}^{4}C_{2}) + ({}^{6}C_{1} \times {}^{4}C_{1})$ [*Applied the formula* ${}^{n}C_{r} = {}^{n}C_{(n-r)}$]

=[6×52×1]+[(6×5×43×2×1)×4]+[(6×52×1)(4×32×1)]+[6×4]= 15 + 80 + 90 + 24= 209

Example:

From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?

Answer

From a group of 7 men and 6 women, five persons are to be selected with at least 3 men.

Hence we have the following 3 choices

We can select 5 men ------(Option 1)

Number of ways to do this = ${}^7C_5$

We can select 4 men and 1 woman ------(Option 2)

Number of ways to do this = ${}^7C_4 \times {}^6C_1$

We can select 3 men and 2 women ------(Option 3)

Number of ways to do this = ${}^7C_3 \times {}^6C_2$

Total number of ways

$= {}^7C_5 + [{}^7C_4 \times {}^6C_1] + [{}^7C_3 \times {}^6C_2]$

$= {}^7C_2 + [{}^7C_3 \times {}^6C_1] + [{}^7C_3 \times {}^6C_2]$ [*Applied the formula* ${}^nC_r = {}^nC_{(n-r)}$]

$=[7\times62\times1]+[(7\times6\times53\times2\times1)\times6]+[(7\times6\times53\times2\times1)\times(6\times52\times1)]= 21 + 210 + 525 = 756$

1.3 Probability

الاحتمال هو مقياس لحالة حدوث الحدث

Definition: Probability is the measure of the likeliness that an event will occur.

Definition: Let A be event in the sample space C, then the probability of the event A is the ratio between the number of equally probable events of A which define A and the total number of events of C. If C has n elements and A is made up of $m \leq n$ elements, then $P(A)=m/n$

1.3.1 Axiomatic of probability

Let A be event in the sample space C, then

1- $0 \leq p(A) \leq 0$.

2- $p(C) = 1$.

3- $A \cap B = \phi \Rightarrow p(A \cup B) = p(A) + p(B)$

Example:

(a) Tossing a fair coin, in this case the probability measure is given by $p(H) = p(T) = \frac{1}{2}$.

(b) Tossing a fair die. In this case, the probability measure is given by

$$p(1) = p(2) = \cdots = p(6) = \frac{1}{6}.$$

(c) Tossing a fair coin twice, in this case, the probability measure is given by

$$p(HH) = p(HT) = p(TH) = p(TT) = \frac{1}{4}.$$

Theorem For any event E we have

$$P(E^c) = 1 - P(E)$$

Proof: For any set, E and E^c are mutually exclusive. Therefore,

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c) \blacksquare$$

Example: A bag contains 6 red balls, 5 yellow balls and 3 green balls. A ball is drawn at random. What is the probability that the ball is: (a) green, (b) not yellow, (c) red or yellow?

6R

5Y

3G

(a) $p(G)=\dfrac{3}{14}$ or $p(G)=\dfrac{\binom{3}{1}}{\binom{14}{1}}=\dfrac{3}{14}$.

(b) $p(Y^c)=1-p(Y)=1-\dfrac{5}{14}=\dfrac{9}{14}$ or $p(Y^c)=1-p(Y)=1-\dfrac{\binom{5}{1}}{\binom{14}{1}}=1-\dfrac{5}{14}=\dfrac{9}{14}$.

(c) $p(R \text{ or } Y)=\dfrac{6}{14}+\dfrac{5}{14}=\dfrac{11}{14}$ or $p(R \text{ or } Y)=\dfrac{\binom{6}{1}}{\binom{14}{1}}+\dfrac{\binom{5}{1}}{\binom{14}{1}}=\dfrac{11}{14}$

Theorem If $A \subset B$, then $P(A) \leq P(B)$.

Proof: Because $A \subset B$, we can write B as

$$B = A \cup (A^c B)$$

These are mutually exclusive and so

$$P(B) = P(A) + P(A^c B)$$

$P(A^c B) \geq 0$ and so the result is shown. ■

Theorem: $P(E \cup F) = P(E) + P(F) - P(EF)$.

Proof: Note that $E \cup F = E \cup E^c F$, and these are mutually exclusive. Thus,

$$P(E \cup F) = P(E) + P(E^c F)$$

Also, $F = EF \cup E^c F$ (which are mutually exclusive) and so

$$P(F) = P(EF) + P(E^c F)$$

Combining the two equations shows

$$P(E \cup F) = P(E) + P(E^c F) = P(E) + P(F) - P(EF) \ ■$$

1.4 Conditional Probability and Independence

Definition: Suppose that A and B are both events in the sample space C, then the conditional probability of A given B is:

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} \quad ; \quad p(B) \neq 0$$

The conditional probability of B given A is:

$$p(B \mid A) = \frac{p(A \cap B)}{p(A)} \quad ; \quad p(A) \neq 0$$

Example: A box contains 3 white balls and 2 black balls. Two balls are drawn without replacement. What is the probability that the second ball is black given that the first is black? Let A be the event that the first ball is black and let B be the event that the second ball is black. We wish to find $P(B|A)$. First $P(A) = \frac{2}{5}$ given that there are 2 black balls out of a total of 5. Next $A \cap B$ represents that both balls are black. There are $5 * 4 = 20$ possible ordered pairs of balls drawn. Of these, two correspond to $A \cap B$. Thus $P(A \cap B) = \frac{2}{20}$. So:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{5}{20} = \frac{1}{4}$$

We say two events, A and B are independent if the occurrence of one does not affect the occurrence of the other. That is, if we know that A has occurred, then B still occurs with its usual probability. Similarly, if B has occurred, then A occurs with its usual probability. In particular:

$$p(A \mid B) = p(A) \quad and \quad p(B \mid A) = p(B)$$

This leads to a useful formula which is also our definition of independence:

$$p(A \cap B) = p(A)\ p(B)$$

Example: Suppose a box contains three white balls and three black balls. The white balls are labeled with 1,2 and 3 respective, the same for the black balls. Suppose that a ball is drawn and it is noted that it is white, what is the probability that it is labeled with a 1? Are these events independent? Let A be the event that a white ball is drawn and let B be the event that a ball labeled 1 is drawn. Our problem is to find P(B|A) $p(A \mid B)$. . As

$p(A) = \frac{1}{2}$, $p(B) = \frac{1}{3}$ and $p(A \cap B) = \frac{1}{6}$

$$p(B \mid A) = \frac{p(A \cap B)}{p(A)} = \frac{1}{3} = p(B)$$

Note that $p(A \cap B) = \frac{1}{6} = p(A)\,p(B)$, so these events are independent.

Remark: The independence of three events A_1, A_2, A_3 means that we must have:

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

Theorem *If $A_1, A_2, ..., A_n$ are events such that $P(A_1 \cap A_2 \cap ... \cap A_n) \neq 0$ (they can occur simultaneously, i.e. they are compatible), then*

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|(A_1 \cap A_2)) \cdot ... \cdot P(A_n|(A_1 \cap ... \cap A_{n-1})).$$

Proof.

$$P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|(A_1 \cap A_2)) \cdot ... \cdot P(A_n|(A_1 \cap ... \cap A_{n-1})) =$$
$$= P(A_1) \cdot \frac{P(A_1 \cap A_2)}{P(A_1)} \cdot P(A_3|(A_1 \cap A_2)) \cdot ... \cdot P(A_n|(A_1 \cap ... \cap A_{n-1})) =$$
$$= P(A_1 \cap A_2) \cdot \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \cdot ... \cdot P(A_n|(A_1 \cap ... \cap A_{n-1})) =$$
$$\dots\dots\dots$$
$$= P(A_1 \cap ... \cap A_{n-1}) \cdot \frac{P(A_1 \cap ... \cap A_{n-1} \cap A_n)}{P(A_1 \cap ... \cap A_{n-1})} = P(A_1 \cap ... \cap A_n).$$

□

Example 1 *A machine produces parts that are either* good *(90%),* slightly defective *(2%), or* obviously defective *(8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it. What is the quality of the parts that make it through the inspection machine and get shipped?*

Let G (resp., SD, OD) be the event that a randomly chosen shipped part is good (resp., slightly defective, obviously defective). We are told that $P(G) = .90$, $P(SD) = 0.02$, and $P(OD) = 0.08$.

We want to compute the probability that a part is good *given* that it passed the inspection machine (i.e., it is *not* obviously defective), which is

$$P(G|OD^c) = \frac{P(G \cap OD^c)}{P(OD^c)} = \frac{P(G)}{1 - P(OD)} = \frac{.90}{1 - .08} = \frac{90}{92} = .978$$

Example

Let A be the event that a married man watches the show,
B be the event that a married woman watches the show.

Given $P(A) = 0.4$ and $P(B) = 0.5$; also

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = 0.7.$$

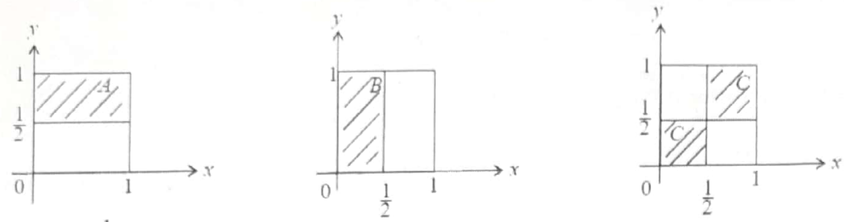
(a) Probability that a married couple watch the show
$= P(A \cap B) = P(B)\, P(A|B) = (0.5)(0.7) = 0.35.$

(b) Probability that a wife watches the show given that her husband does
$= P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.35}{0.4} = 0.875.$

(c) Probability that at least 1 person watches the show
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.35 = 0.55$

Example Two numbers x and y are selected at random between zero and one. Let events A, B and C be defined by

$$A=\left\{y>\frac{1}{2}\right\} \qquad B=\left\{x<\frac{1}{2}\right\} \qquad C=\left\{x<\frac{1}{2},y<\frac{1}{2}\right\}\cup\left\{x>\frac{1}{2},y>\frac{1}{2}\right\}$$



$$P[A\cap B]=\frac{1}{4}=P[A]P[B]$$

$$P[A\cap C]=\frac{1}{4}=P[A]P[C]$$

$$P[B\cap C]=\frac{1}{4}=P[B]P[C]$$

However, $A\cap B\cap C=\phi$, so

$$P[A\cap B\cap C]=P[\phi]=0\neq P[A]P[B]P[C]=\frac{1}{8}.$$

1.5 Total Probability

الحدث الاحتي

Suppose that we want to know the probability that event A happens, but A generally occurs after some possible events, say B_1, B_2 or B_n. Then we can use the average conditional probability to find A.

Theorem *Suppose that $B_1, B_2, \ldots B_n$ is a partition of Ω. Then for any event A,*

$$P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+\cdots+P(A|B_n)P(B_n)$$

Proof: Note that $A=(A\cap B_1)\cup(A\cap B_2)\cup\cdots\cup(A\cap B_n)$ and the sets $A\cap B_i$ are pairwise-disjoint. Thus

$$P(A)=P(A\cap B_1)+(A\cap B_2)+\cdots+P(A\cap B_n)$$

The result follows by replacing $P(A\cap B_i)$ with $P(A|B_i)P(B_i)$. □

Example: Suppose that we have two boxes. The first box (A) has 6 red balls, 5 white balls and 3 black balls and the second box (B) has 2 red balls, 4 white balls and 8 black balls. If a box is chosen at random and a ball drawn, what is the probability that it is a red ball?

$$p(R)=p(R\mid A)\,p(A)+p(R\mid B)\,p(B)=\frac{6}{14}\cdot\frac{1}{2}+\frac{2}{14}\cdot\frac{1}{2}=\frac{4}{14}$$

$$P(A_1)=\frac{1}{2},\quad P(R|A_1)=\frac{6}{14},\quad P(A_2)=\frac{1}{2},\quad P(R|A_2)=\frac{2}{14}$$

1.6 Baye's Theorem

Suppose that $B_1, B_2 \cdots, B_n$ is a partition of the sample space Ω. Then foe any event A

$$p(A_n|B) = \frac{p(A_n \cap B)}{p(B)} = \frac{p(B|A_n)\,p(A_n)}{\sum_{i=1}^{n} p(B|A_i)\,p(A_i)}$$

Example: Suppose that we have three boxes contains balls as

The first box (A1) has 6 red balls, 6 white balls and 8 green balls.

The second box (A2) has 4 red balls, 8 white balls and 8 green balls.

The third box (A3) has 10 red balls, 4 white balls and 6 green balls.

(1) If a box is chosen at random and a ball drawn, if the ball is red, what is the probability that it was from first box?

(2) If a box is chosen at random and a ball drawn, if the ball is white, what is the probability that it was from second box?

(3) If a box is chosen at random and a ball drawn, if the ball is green, what is the probability that it was from third box?

(1)

$$p(A1|R) = \frac{p(A1 \cap R)}{p(R)} = \frac{p(R|A1)\,p(A1)}{p(R\mid A1)\,p(A1) + p(R\mid A2)\,p(A2) + p(R\mid A3)\,p(A3)}$$

$$= \frac{\frac{6}{20}\cdot\frac{1}{3}}{\frac{6}{20}\cdot\frac{1}{3} + \frac{4}{20}\cdot\frac{1}{3} + \frac{10}{20}\cdot\frac{1}{3}} = \frac{3}{10}$$

(2)

$$p(A2|W) = \frac{p(A1 \cap W)}{p(W)} = \frac{p(W\mid A2)\,p(A2)}{p(W\mid A1)\,p(A1) + p(W\mid A2)\,p(A2) + p(W\mid A3)\,p(A3)}$$

$$= \frac{\frac{8}{20}\cdot\frac{1}{3}}{\frac{6}{20}\cdot\frac{1}{3} + \frac{8}{20}\cdot\frac{1}{3} + \frac{4}{20}\cdot\frac{1}{3}} = \frac{4}{9}$$

(3)

$$p(A3|G)=\frac{p(A3\cap G)}{p(G)}=\frac{p(G|A3)\,p(A3)}{p(G\mid A1)\,p(A1)+p(G\mid A2)\,p(A2)+p(G\mid A3)\,p(A3)}$$

$$=\frac{\frac{6}{20}\cdot\frac{1}{3}}{\frac{8}{20}\cdot\frac{1}{3}+\frac{8}{20}\cdot\frac{1}{3}+\frac{6}{20}\cdot\frac{1}{3}}=\frac{3}{11}$$

Example At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio 3:2 in favour of women. If a student is selected at random from among all those over six feet tall, what is the probability that the student is a woman?

Solution

Let M={Student is Male}, F={Student is Female}, (note that M and F partition the sample space of students), T={Student is over 6 feet tall}. We know that $P(M)=2/5$, $P(F)=3/5$. $P(T|M)=4/100$ and $P(T|F)=1/100$. We require $P(F|T)$. Using Bayes' Theorem we have:

$$P(F|T) = \frac{P(T|F)P(F)}{P(T|F)P(F)+P(T|M)P(M)}$$

$$= \frac{\frac{1}{100}\times\frac{3}{5}}{\frac{1}{100}\times\frac{3}{5}+\frac{4}{100}\times\frac{2}{5}}$$

$$=\frac{3}{11}$$

Example A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from

(a) machine A (b) machine B (c) machine C?

Solution

Let D={bolt is defective}, A={bolt is from machine A}, B={bolt is from machine B}, C={bolt is from machine C}. We know that $P(A)=0.25$, $P(B)=0.35$ and $P(C)=0.4$. Also $P(D|A)=0.05$, $P(D|B)=0.04$, $P(D|C)=0.02$. A statement of Bayes' Theorem for three events A, B and C is

$$P(A|D) = \frac{P(D|A)P(A)}{P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C)}$$

$$= \frac{0.05\times 0.25}{0.05\times 0.25+0.04\times 0.35+0.02\times 0.4}$$

$$= 0.362$$

Similarly

$$P(B|D) = \frac{0.04\times 0.35}{0.05\times 0.25+0.04\times 0.35+0.02\times 0.4}$$

$$= 0.406$$

$$P(C|D) = \frac{0.02\times 0.4}{0.05\times 0.25+0.04\times 0.35+0.02\times 0.4}$$

$$= 0.232$$

Exercises

1- A bench can seat 4 people. How many seating arrangements can be made from a group of 10 people?

2- How many distinct permutations can be formed from all the letters of each of the following words: (a) them, (b) unusual?

3- A student is to answer 7 out of 9 questions on a midterm test.

i) How many examination selections has he?

ii) How many if the first 3 questions are compulsory?

iii) How many if he must answer at least 4 of the first 5 questions?

4- A bag contains 6 red balls, 5 yellow balls and 3 green balls. A ball is drawn at random. What is the probability that the ball is: (a) green, (b) not yellow, (c) red or yellow?

5- A fair six-sided die is tossed twice. What is the probability that a five will occur at least once?

6- Three different machines Ml, M2, and M3 are used to produce similar electronic components. Machines Ml, M2, and M3 produce 20%, 30% and 50% of the components respectively. It is known that the probabilities that the machines produce defective components are 1% for M1, 2% for M2, and 3% for M3. If a component is selected randomly from a large batch, and that component is defective, find the probability that it was produced: (a) by M2, and (b) by M3.

7- In a group of 72 students, 14 take neither English nor chemistry, 42 take English and 38 take chemistry. What is the probability that a student chosen at random from this group takes:

a) Both English and chemistry?

b) Chemistry but not English?

8- Two hundred students were sampled in the College of Arts and Science. It was found that: 137 take math, 50 take history, 124 take English, 33 take math and history, 29 take history and English, 92 take math and English, 18 take math, history and English. Find the probability that a student selected at random out of the 200 takes neither math nor history nor English.

9- Three balls are drawn one after the other from a bag containing 6 red balls, 5 yellow balls and 3 green balls. What is the probability that all three balls are yellow if:

a) The ball is replaced after each draw and the contents are well mixed?

b) The ball is not replaced after each draw?

10- A box contains three coins, two of them fair and one two-headed. A coin is selected at random and tossed. If heads appears the coin is tossed again; if tails appears then another coin is selected from the two remaining coins and tossed.

a) Find the probability that heads appears twice.

b) Find the probability that tails appears twice.

---

(3) i- $C^9_7 = \frac{9!}{7!\,2!}$, ii- $C^7_3 \cdot C^5_4$

(4) 6R, 5Y, 3G } $n=14$, $r=1$ $P(G) = \frac{m}{n} = \frac{3}{14}$

$= \frac{\frac{1}{5}}{\frac{31}{80}} = \frac{1}{5} \cdot \frac{80}{31}$

$= \frac{80}{155}$

$= 0.5161$

(9) $r=3$ 6R, 5Y, 3G, $n=14$

H.w:- ~~Two boxes~~ suppouse that we have Two boxes contains balls as, The first box (A_1) has 3 red balls and 5 bule balls, The second box (A_2) has 4 red balls and 6 bule balls, if the chosen at random and ball drawn,

(1) what is the probability that red ball ?

(2) if the ball is red, what is the probability that it was from second box ?

H.w:- $P(R) = P(R|A_1)P(A_1) + P(R|A_2) \cdot P(A_2)$

$\frac{3}{8} \cdot \frac{1}{2} + \frac{4}{10} \cdot \frac{1}{2}$

$= \frac{3}{16} + \frac{1}{5} = \frac{15+16}{80} = \frac{31}{80}$

$P(A_2/R) = \frac{P(R|A_2)\,P(A_2)}{P(R|A_1)\,P(A_1) + P(R|A_2) \cdot P(A_2)} = \frac{\frac{4}{10} \cdot \frac{1}{2}}{\frac{3}{8} \cdot \frac{1}{2} + \frac{4}{10} \cdot \frac{1}{2}}$

المتغير العشوائي: هو دالة منطلقها فضاء العينة S ومستقرها مجموعة الاعداد
الحقيقية (R) ويرمز لها بالرموز الحروف الكبيرة (X,Y,Z,...) ويرمز الى قيمة هذه
المتغيرات بالحروف الصغيرة وسميت بهذا الاسم لان قيمتها تتغير بتكرار التجربة
العشوائية نسبة للتجربة العشوائية.

Chapter 2

Random variable and their functions

2.1 Random variable

Informally, a random variable is a variable that is random, meaning that its value is unknown, uncertain, not observed yet, or something of the sort. The probabilities with which a random variable takes its various possible values are described by a probability model.

In order to distinguish random variables from ordinary, nonrandom variables, we adopt a widely used convention of denoting random variables by capital letters, usually letters near the end of the alphabet, like X, Y, and Z. There is a close connection between random variables and certain ordinary variables. If X is a random variable, we often use the corresponding small letter x as the ordinary variable that takes the same values. Whether a variable corresponding to a real-world phenomenon is considered random may depend on context. In applications, we often say a variable is random before it is observed and nonrandom after it is observed and its actual value is known. Thus the same real-world phenomenon may be symbolized by X before its value is observed and by x after its value is observed.

We begin the discussion with a very simple example. Let the random experiment be the toss of a coin and let the sample space associate with the' experiment be $C = \{c$: where c is Tor c is H$\}$ and T and H represent, respectively, tails and heads. Let X be a function such that $X(c) = 0$ if c is T and let X(t) = I if c is H. Thus X is a real-valued function defined on the sample space C which takes us from the sample space C to a space of real numbers $A = \{0, 1\}$. We call X a random variable and, in this example, the space associated with X is $A = \{0, 1\}$. We now formulate the definition of a random variable and its space.

Example : Consider the experiment of flipping a fair coin three times. The number of tails that appear is noted as a discrete random variable :

X= number of tails that appear in 3 flips of a fair coin.

There are 8 possible outcomes of the experiment : namely the sample space consists of

$$\Omega = \{ HHH, HHT, HTH, HTT, THH, TTT, THT, TTH \} \quad \text{where}$$
$$X = \quad 0 \quad ,1 \quad ,1 \quad ,2 \quad ,1 \quad ,3 \quad ,2 \quad ,2$$

are the corresponding values taken by the random variable X.

Definition: Consider a random experiment with a sample space C. A function X, which assigns to each element $c \in C$ one and only one real number X(c) = x, is called **a random variable**. The space of X is the set of real numbers $A = \{x : x = X(x); c \in C\}$.

Random variable can be **discrete**, that is, taking any of a specified finite or countable list of values, endowed with a probability mass function, characteristic of a probability distribution; or **continuous**, taking any numerical value in an interval or collection of intervals, via a probability density function that is characteristic of a probability distribution; or a mixture of both types.

2.2 Functions of random variables دالة المتغير العشوائي

2.2.1 Probability density function (*p.d.f*) ① دالة كثافة الاحتمال (p.d.f)

Definition: Let *f(x)* be a function of the random variable X, such that

$$(1)\ f(x) > 0 \quad \forall x \in C$$

$$(2)\begin{cases} \sum\limits_{all\ x} f(x) = 1 & \textit{Discrete type} \\ \int\limits_{-\infty}^{\infty} f(x)\,dx = 1 & \textit{Continuous type} \end{cases}$$

متقطع / متصل

then $f(x)$ is called the probability density function (*p.d.f.*) of *X*.

Example: Let

$$p(x) = \begin{cases} \frac{x}{15} & x = 1,2,3,4,5 \\ 0 & elsewhere \end{cases}$$

show that *p(x)* **is** ***p.d.f***

$$(1)\ P(x) > 0 \quad \forall x = 1,2,3,4,5$$

$$(2)\ \sum_{all\ x} p(x) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = \frac{15}{15} = 1$$

Example: Let

$$p(x) = \begin{cases} k\,(x+1) & x = 1,2,3,4 \\ 0 & elsewhere \end{cases},$$

be the p.d.f of the random variable *X*. Find the constant *k*.

Since p(x) is a *p.d.f.* of X, then

$$\sum_{all\ x} p(x) = 1 \Rightarrow k(1+1) + k(2+1) + k(3+1) + k(4+1) = 1 \Rightarrow 14k = 1 \Rightarrow k = \frac{1}{14}$$

* show that $P(X) = \frac{3x+1}{2}$, $X = 0,1,2,3,4$

① $P(X=0) = \frac{1}{2} > 0$

$P(X=1) = 2 > 0$

$P(X=2) = \frac{7}{2} > 0$

$P(X=3) = 5 > 0$

$P(X=4) = \frac{13}{2} > 0$

② $\sum P(X) = \frac{1}{2} + 2 + \frac{7}{2} + 5 + \frac{13}{2} = 7 + 4 + \frac{13}{2} = 11 + \frac{13}{2} \neq 1$

not p.d.f

Ex:- show that $P(x) = \frac{3}{x!(3-x)!}$, $x = 0, 1, 2, 3$ is p.d.f

① $P(X=0) = \frac{3}{0!\,3!} = \frac{1}{2} > 0$

$P(X=1) = \frac{3}{1!\,2!} = \frac{3}{2} > 0$

$P(X=2) = \frac{3}{2!\,1!} = \frac{3}{2} > 0$

$P(X=3) = \frac{3}{2!\,1!} = \frac{1}{2} > 0$

② $\sum p(x) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = 4$

not p.d.f

Example: Let

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & elsewhere \end{cases}$$ **show that** $f(x)$ **is** *p.d.f*

(1) $f(x) > 0 \quad \forall x > 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\infty} = 1$ (where, $e^0 = 1$ and $e^{-\infty} = 0$)

Example: Let

$$f(x) = \begin{cases} ke^{\frac{-x}{30}} & x > 0 \\ 0 & elsewhere \end{cases}$$

be the $p.d.f$. of the random variable X. Find the constant k and $p(10 < x < \infty)$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} k\,e^{\frac{-x}{30}}\,dx = 1 \Rightarrow \left[-30\,e^{\frac{-x}{30}}\right]_0^{\infty} = \frac{1}{k} \Rightarrow k = \frac{1}{30}$$

$$p(10 < x < \infty) = \int_{10}^{\infty} \frac{1}{30} e^{\frac{-x}{30}}\,dx = \left[-e^{\frac{-x}{30}}\right]_{10}^{\infty} = e^{\frac{-1}{3}}$$

2.2.2 Distribution function (*d.f*)

Definition: The distribution function of a random variable X is the function given by

$$F(x) = \begin{cases} \sum_{s=-\infty}^{x} p(s) & \text{discrete type} \\ \int_{-\infty}^{x} f(s)\,ds & \text{continuous type} \end{cases}$$

Theorem 2 (Distribution Function II) *A distribution function $F(x)$ of a random variable X satisfies the following properties.*

1. $P(X > x) = 1 - F(x)$.
2. $P(x < X \le y) = F(y) - F(x)$.
3. $P(X = x) = F(x) - \lim_{y \uparrow x} F(y)$.

Example: Let

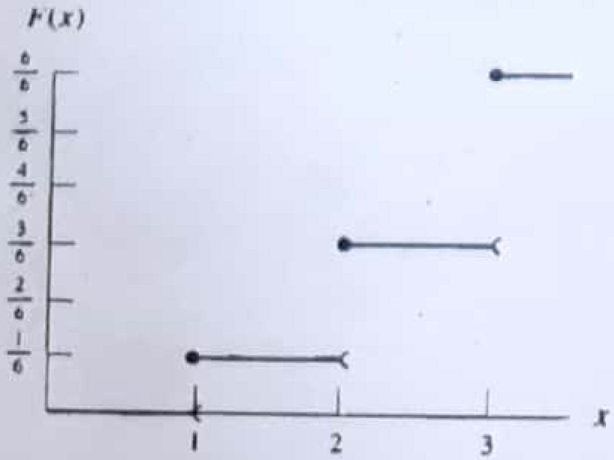
$$p(x) = \begin{cases} \dfrac{x}{15} & x = 1,2,3,4,5 \\ 0 & \textit{elsewhere} \end{cases} \qquad \textbf{find } \boldsymbol{d.f.} \textbf{ of } \boldsymbol{X}.$$

X	1	2	3	4	5
P(x)	1/15	2/15	3/15	4/15	5/15

$$p(x) = \begin{cases} 0 & x < 1 \\ \dfrac{1}{15} & 1 \le x < 2 \\ \dfrac{3}{15} & 2 \le x < 3 \\ \dfrac{6}{15} & 3 \le x < 4 \\ \dfrac{10}{15} & 4 \le x < 5 \\ \dfrac{15}{15} = 1 & x \ge 5 \end{cases}$$

Example . Let the random variable X of the discrete type have the p.d.f. $f(x) = x/6$, $x = 1, 2, 3$, zero elsewhere. The distribution function of X is

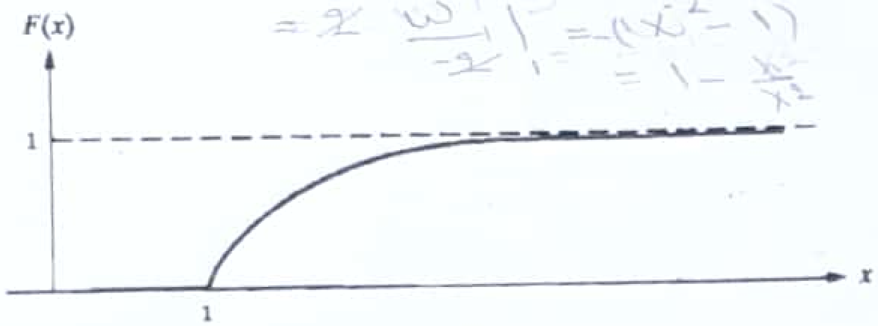
$$\begin{aligned} F(x) &= 0, & x &< 1, \\ &= \tfrac{1}{6}, & 1 \le x &< 2, \\ &= \tfrac{3}{6}, & 2 \le x &< 3, \\ &= 1, & 3 &\le x. \end{aligned}$$



Example . Let the random variable X of the continuous type have the p.d.f. $f(x) = 2/x^3$, $1 < x < \infty$, zero elsewhere. The distribution function of X is

$$F(x) = \int_{-\infty}^{x} 0\, dw = 0, \qquad x < 1,$$

$$= \int_{1}^{x} \frac{2}{w^3}\, dw = 1 - \frac{1}{x^2}, \qquad 1 \le x.$$



Example: Let

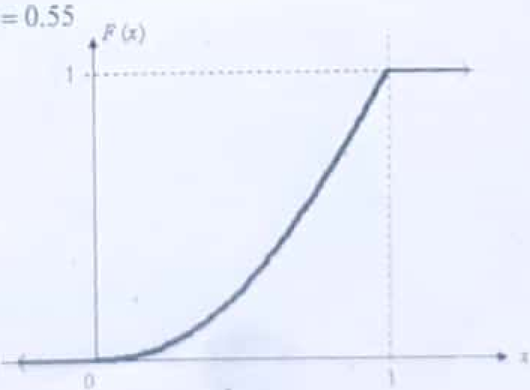
$$p(x) = \begin{cases} k\,x & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

be the p.d.f. of X.

(1) Find the constant k. (2) $p(0.3 < x < 0.8)$ (3) Find the d.f. of X

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 k\,x dx = 1 \Rightarrow \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{k} \Rightarrow k = 2$$

$$p(0.3 < x < 0.8) = \int_{0.3}^{0.8} 2\,x\,dx = \left[x^2\right]_{0.3}^{0.8} = (0.8)^2 - (0.3)^2 = 0.55$$



$$\therefore \quad F(x) = \begin{cases} 0 & (x < 0) \\ x^2 & (0 \le x \le 1) \\ 1 & (x > 1) \end{cases}$$

2.3 Mathematical Expectation

one of the more useful concepts in the problem involving distributions of random variables is that of mathematical expectation.

Definition : Let X be a random variable having p.d.f. $f(x)$ ($p(x)$), and let $u(x)$ be a function of X. The expected value of $u(x)$ is defined as

$$E[u(x)] = \begin{cases} \int_{-\infty}^{\infty} u(x).f(x)dx & \text{if X is continuous} \\ \sum_{all\, x} u(x).f(x) & \text{if X is discrete.} \end{cases}$$

Example1: Let X has the p.d.f.

$$f(x) = \begin{cases} \frac{2}{5}(3x+1) & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then,

$E(X) = \int_0^1 x\, f(x)dx = \int_0^1 \frac{2}{5}x\,(3x+1)dx = \frac{2}{5}\left[x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{3}{5}.$

$E(X^2) = \int_0^1 x^2\, f(x)dx = \int_0^1 \frac{2}{5}x^2\,(3x+1)dx = \frac{2}{5}\left[\frac{3}{4}x^4 + \frac{1}{3}x^3\right]_0^1 = \frac{13}{30}$

$E(2X^2+X) = \int_0^1 (2x^2+x)\, f(x)dx = \int_0^1 \frac{2}{5}(2x^2+x)\,(3x+1)dx = \int_0^1 \frac{2}{5}(6x^3+5x^2+x)dx$

$= \frac{2}{5}\left[\frac{6}{4}x^4 + \frac{5}{3}x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{22}{15}$

Example2: Let X have the p.d.f.

$$f(x) = \begin{cases} \frac{x}{6} & x = 1,2,3 \\ 0 & elsewhere \end{cases}$$

Then,

$$E(X) = \sum_{all\, x} x\, f(x) = \sum_{x=1}^{3} x\,\frac{x}{6} = \sum_{x=1}^{3} \frac{x^2}{6} = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} = \frac{14}{6}$$

$$E(X^2) = \sum_{all\, x} x^2\, f(x) = \sum_{x=1}^{3} x^2\,\frac{x}{6} = \sum_{x=1}^{3} \frac{x^3}{6} = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = \frac{36}{6}$$

$$E(X^3) = \sum_{all\, x} x^3\, f(x) = \sum_{x=1}^{3} x^3\,\frac{x}{6} = \sum_{x=1}^{3} \frac{x^4}{6} = \frac{1}{6} + \frac{16}{6} + \frac{81}{6} = \frac{98}{6}$$

2.3.1 Properties of expected value

1. $E[c] = c$ for a constant c.
2. $E[cg(x)] = cE[g(x)]$ for a constant c.
3. $E[c_1 g(x_1) \pm c_2 g(x_2)] = c_1 E[g(x_1)] \pm c_2 E[g(x_2)]$ for the constants c_1 and c_2.
4. $E[g_1(x)] \leq E[g_2(x)]$ if $g_1(x) \leq g_2(x)$ for all X.

2.4 Variance

Definition : Let X be a random variable, and let μ_X be $E(X)$. The variance of X, denoted by σ_X^2 or $var[X]$ is defined by

$$\sigma_X^2 = \text{var}[X] = E[X^2] - (E[X])^2.$$

2.4.1 Properties of variance

1. $\text{var}[X] \geq 0$
2. $\text{var}[c] = 0$ for a constant c.
3. $\text{var}[cX] = c^2 \text{var}[X]$ for a constant c.
4. $\text{var}[a + bX] = b^2 \text{var}[X]$ for the constants a and b.

Definition : If X is a random variable, the standard deviation of X, denoted by σ_X, is defined as $+\sqrt{\text{var}[X]}$.

Example1: Let X has the p.d.f.

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & elsewhere \end{cases}$$

Then,
$E[X+1] = \int_0^\infty (x+1)e^{-x}dx = -(x+1)e^{-x} + \int_0^\infty e^{-x}dx = [-(x+1)e^{-x} - e^{-x}]_0^\infty = 2$

Similarly, $E[(X+1)^2] = 5$

Thus,
$\text{var}[X+1] = E[(X+1)^2] - (E[X+1])^2 = 5 - (2)^2 = 1.$

2.5 Moment-generating function (m.g.f.)

Definition : let X be a random variable with a p.d.f $f(x)$ The expected value of e^{tx} is defined to be the moment-generating function of X, if the expected value exists for every value of t in some interval $-h < t < h$; $h > 0$. This function, denoted by $M_X(t)$, can be defined as

$$M_X(t) = E[e^{tX}] = \begin{cases} \int\limits_{-\infty}^{\infty} e^{tx} f(x) & if\, X \text{ is continuous} \\ \\ \sum\limits_{-\infty}^{\infty} e^{tx} f(x) & if\, X \text{ is discrete.} \end{cases} \qquad (1.3)$$

Where,
$-h < t < h$; $h > 0$.

2.5.1 Properties of moment-generating function

1. $M_X(0) = 1$.
2. $M'_X(0) = E[X]$.
3. $M''_X(0) = E[X^2]$.
4. $\text{var}[X] = M''_X(0) - (M'_X(0))^2$.

Example1: Let X has the p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}x} & 0 < x < \infty \\ \\ 0 & elsewhere \end{cases}$$

Then,
$M_X(t) = E[e^{tX}] = \int_0^\infty \frac{1}{2}e^{tx}e^{-\frac{1}{2}x}dx = \frac{1}{2}\int_0^\infty e^{-x(\frac{1}{2}-t)}dx = \frac{1}{2}\left(\frac{-1}{\frac{1}{2}-t}\right)\left[e^{-x(\frac{1}{2}-t)}\right]_0^\infty = \frac{1}{1-2t}$.
Where, $t \neq \frac{1}{2}$.
To find the expectated value and variance using the properties of m.g.f.

$M'_X(t) = \frac{2}{(1-2t)^2} \Rightarrow M'_X(0) = 2.$
$M''_X(t) = \frac{8}{(1-2t)^3} \Rightarrow M''_X(0) = 8.$

Therefore, we have

$\text{var}[X] = M''_X(0) - (M'_X(0))^2 = 8 - (2)^2 = 4.$

$$\frac{3}{2}\int_0^{x_2} \frac{x_1^3}{3} + z^2 x_1 \, dz$$

Example2: Let X has the p.d.f.

$$f(x) = \begin{cases} \frac{1}{3} & -1 < x < 2 \\ 0 & elsewhere \end{cases}$$

Here, we need to prove that, the moment-generating function of X is

$$M_X(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t} & t \neq 0, \\ 1 & t = 0. \end{cases}$$

For $t \neq 0$

$$M_X(t) = E[e^{tX}] = \int_{-1}^{2} \frac{1}{3} e^{tx} dx = \frac{1}{3t} \left[e^{tx}\right]_{-1}^{2} = \frac{1}{3t} \left[e^{2t} - e^{-t}\right]$$

For $t = 0$

$$M_X(0) = E[e^{0.X}] = E[1] = 1.$$

EXERCISES:

1. Find the mean and variance, if they exist, of each of the following distributions.

   a) $f(x) = \frac{3!}{x!(3-x)!} \left(\frac{1}{2}\right)^3$, $x = 0, 1, 2, 3$, *zero elsewhere*.

   b) $f(x) = 6x(1-x)$, $0 < x < 1$, *zero elsewhere*.

   c) $f(x) = \frac{2}{x^3}$, $1 < x < \infty$, *zero elsewhere*.

2. Let $f(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \cdots$, zero elsewhere, be the p.d.f. of the random variable X. Find the moment-generating function, the mean, and the variance of X.

3. Let the random variable X has mean μ, standard deviation σ, and moment-generating function $M_X(t)$, $-h < t < h$. Show that,

   a) $E\left[\frac{X-\mu}{\sigma}\right] = 0$.

   b) $E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = 1$.

   c) $E\left\{\exp\left[t\left(\frac{X-\mu}{\sigma}\right)\right]\right\} = e^{\frac{-\mu t}{\sigma}} M_X\left(\frac{t}{\sigma}\right)$, $-h\sigma < x < h\sigma$.

$M_x(t) = \frac{1}{3t} e^{2t} - \frac{1}{3t} e^{-t}$ $\quad M'_x(t) = \frac{2}{3t} e^{2t} + e^{2t} \cdot \frac{-3}{(3t)^2}$

2.6 Joint, marginal and conditional distributions

Definition : Let X_1 and X_2 denote radom variables with a joint function $f(x_1, x_2)$ $(p(x_1, x_2))$. This joint function is said to be joint p.d.f. if it satisfies:

a) $f(x_1, x_2) > 0$ $(p(x_1, x_2) > 0)$; $\forall x_1, x_2$

b) $$\begin{cases} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 dx_2 = 1 & \textit{continuous case} \\ \sum\limits_{-\infty}^{\infty}\sum\limits_{-\infty}^{\infty} p(x_1, x_2) = 1 & \textit{discrete case.} \end{cases}$$

Definition : Let X_1 and X_2 be jointly random variables with the joint p.d.f. $f(x_1, x_2)(p(x_1, x_2))$. The marginal p.d.f. of X_1 is given by

$$\begin{cases} f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 & \textit{Continuous case} \\ p_1(x_1) = \sum\limits_{all\ x_2} p(x_1, x_2) & \textit{Discrete case} \end{cases}$$

Similarly,the marginal p.d.f. of X_2 is given by

$$\begin{cases} f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_1 & \textit{Continuous case} \\ p_2(x_2) = \sum\limits_{all\ x_1} p(x_1, x_2) & \textit{Discrete case} \end{cases}$$

Definition : Let X_1 and X_2 be jointly random variables with the joint p.d.f. $f(x_1, x_2)(p(x_1, x_2))$ and the marginal p.d.f. $f_1(x_1)$ $(p_1(x_1))$ and $f_2(x_2)$ $(p_2(x_2))$, respectively.The conditional p.d.f. of X_1, given X_2, is

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}; \quad f_2(x_2) > 0$$

The conditional p.d.f. of X_2, givenX_1, is

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}; \quad f_1(x_1) > 0$$

Definition : Let X_1 and X_2 denote random variables that have the joint p.d.f. $f(x_1, x_2)(p(x_1, x_2))$. The joint distribution function of X_1 and X_2 can be defined as

$$F(x_1, x_2) = \Pr(X_1 \le x_1, X_2 \le x_2) = \begin{cases} \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f(s, z)dsdz & \textit{Continuous case} \\ \sum\limits_{z=-\infty}^{x_2}\sum\limits_{s=-\infty}^{x_1} p(s, z) & \textit{Discrete case.} \end{cases}$$

Example1: Let X_1 and X_2 have the joint p.d.f.

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_1 is

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 = \int_0^1 (x_1 + x_2)dx_2 = \begin{cases} x_1 + \frac{1}{2} & 0 < x_1 < 1 \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_2 is

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_1 = \int_0^1 (x_1 + x_2)dx_1 = \begin{cases} x_2 + \frac{1}{2} & 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The conditional p.d.f. of X_1, given X_2, is

$$f(x_1|x_2) = \frac{f(x_1,x_2)}{f_2(x_2)} = \frac{x_1+x_2}{x_2+\frac{1}{2}} \ ; \quad 0 < x_1 < 1,\ 0 < x_2 < 1$$

The conditional p.d.f. of X_2, given X_1, is

$$f(x_2|x_1) = \frac{f(x_1,x_2)}{f_1(x_1)} = \frac{x_1+x_2}{x_1+\frac{1}{2}} \ ; \quad 0 < x_1 < 1,\ 0 < x_2 < 1$$

2.7 Independence

Definition : Let the random variables X_1 and X_2 have the joint p.d.f. $f(x_1, x_2)(p(x_1, x_2))$ and the marginal p.d.f. $f_1(x_1)$ $(p_1(x_1))$ and $f_2(x_2)$ $(p_2(x_2))$, respectively. The random variables X_1 and X_2 are said to be stochastically independent if and only if

$$f(x_1, x_2) = f_1(x_1).f_2(x_2)$$

Random variables that are not stochastically independent are said to be stochastically dependent.

Example1: I the above example, we had

$f_1(x_1) = x_1 + \frac{1}{2}$ and $f_2(x_2) = x_2 + \frac{1}{2}$

Since $f_1(x_1).f_2(x_2) = (x_1 + \frac{1}{2})(x_2 + \frac{1}{2}) = x_1x_2 + \frac{1}{2}(x_1 + x_2) + \frac{1}{4} \neq f(x_1, x_2)$, the random variable X_1 and X_2 are stochastically dependent.

Example2: Let X_1 and X_2 have the joint p.d.f.

$$f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_1 is

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 = \int_{x_1}^{1} 2dx_2 = \begin{cases} 2(1-x_1) & 0 < x_1 < 1 \\ 0 & elsewhere \end{cases}$$

$$F(x_1,x_2) = \int_{x_1}^{x_2}\int_0^{x_1} 2\, ds\, dz$$
$$= \int_{x_1}^{x_2} 2s\Big|_0^{x_1} dz$$
$$= \int_{x_1}^{x_2} 2x_1\, dz$$
$$= 2x_1 z\Big|_{x_1}^{x_2}$$
$$= 2x_1(x_2 - x_1)$$
$$= 2x_1x_2 - 2x_1^2$$

The marginal p.d.f. of X_2 is

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1,x_2)dx_1 = \int_0^{x_2} 2dx_1 = \begin{cases} 2x_2 & 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The conditional p.d.f. of X_1, givenX_2, is

$$f(x_1|x_2) = \frac{f(x_1,x_2)}{f_2(x_2)} = \begin{cases} \frac{1}{x_2} & 0 < x_1 < x_2,\ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The conditional p.d.f. of X_2, givenX_1, is

$$f(x_2|x_1) = \frac{f(x_1,x_2)}{f_1(x_1)} = \begin{cases} \frac{1}{1-x_1} & 0 < x_1 < x_2,\ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

Since $f_1(x_1).f_2(x_2) = 2(1-x_1)(2x_2) = 4x_2(1-x_1) \neq f(x_1,x_2)$, the random variable X_1 and X_2 are stochastically dependent.

Example2: Let X_1 and X_2 have the joint p.d.f.

$$p(x_1,x_2) = \begin{cases} \frac{1}{3} & (x_1,x_2) = (0,0),\ (0,1),\ (1,1) \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_1 is

For $x_1 = 0$

$$p_1(x_1=0) = \sum_{all\ x_2} p(x_1=0,x_2) = p(x_1=0,x_2=0) + p(x_1=0,x_2=1) = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

For $x_1 = 1$

$$p_1(x_1=1) = \sum_{all\ x_2} p(x_1=1,x_2) = p(x_1=1,x_2=0) + p(x_1=1,x_2=1) = 0+\frac{1}{3} = \frac{1}{3}$$

Thus,

$$p_1(x_1) = \begin{cases} \frac{2}{3} & x_1 = 0 \\ \frac{1}{3} & x_1 = 1 \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_2 is

For $x_2 = 0$

$$p_2(x_2=0) = \sum_{all\ x1} p(x_1,x_2=0) = p(x_1=0,x_2=0) + p(x_1=1,x_2=0) = \frac{1}{3}+0 = \frac{1}{3}$$

For $x_2 = 1$

$$p_2(x_2=1) = \sum_{all\ x_1} p(x_1,x_2=1) = p(x_1=0,x_2=1) + p(x_1=1,x_2=1) = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

$$F(x_1, x_2) = \int_0^{x_2}\int_0^{x_1} \frac{3}{2} s^2 + z^2 \, ds\, dz$$

$$= \frac{3}{2}\int_0^{x_2} \frac{s^3}{3} + z^2 s \Big|_0 dz = \frac{3}{2}\int_0^{x_2} \frac{x_1^3}{3} + x_1 z^2$$

$$= \frac{3}{2}\left[\frac{x_1^3}{3} z + x_1 \frac{z^3}{3}\right]_0^{x_2}$$

$$= \frac{3}{2}\left[x_2 \frac{x_1^3}{3} + x_1 \frac{x_2^3}{3}\right] = df$$

Thus,

$$p_2(x_2) = \begin{cases} \frac{1}{3} & x_2 = 0 \\ \frac{2}{3} & x_2 = 1 \\ 0 & elsewhere \end{cases}$$

2.8 Expectation

Definition : Let X_1 and X_2 be random variables, with the joint p.d.f. $f(x_1, x_2)(p(x_1, x_2))$, and $g(x_1, x_2)$ function of X_1 and X_2. The expected valuse of $g(x_1, x_2)$ is defined as:

$$E[g(x_1, x_2)] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2) f(x_1, x_2) dx_1 dx_2 & continuous\ case \\ \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} g(x_1, x_2) p(x_1, x_2) & discrete\ case \end{cases}$$

Example1: Let X_1 and X_2 have the joint p.d.f.

$$f(x_1, x_2) = \begin{cases} k(x_1^2 + x_2^2) & 0 < x_1 < 1, \ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

Since $f(x_1, x_2)$ is a joint p.d.f., then

$$\int_0^1\int_0^1 k(x_1^2 + x_2^2) dx_1 dx_2 = 1 \Rightarrow \int_0^1 \left[\frac{1}{3}x_1^3 + x_1 x_2^2\right]_0^1 dx_2 = \frac{1}{k} \Rightarrow \int_0^1 \left(\frac{1}{3} + x_2^2\right) dx_2 = \frac{1}{k}$$

$$\Rightarrow \left[\frac{1}{3}x_2 + \frac{1}{3}x_2^3\right]_0^1 = \frac{1}{k} \Rightarrow k = \frac{3}{2}$$

Thus,

$$f(x_1, x_2) = \begin{cases} \frac{3}{2}(x_1^2 + x_2^2) & 0 < x_1 < 1, \ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_1 is

$$f_1(x_1) = \int_0^1 \frac{3}{2}(x_1^2 + x_2^2)\, dx_2 = \frac{3}{2}\left[x_1^2 x_2 + \frac{1}{3}x_2^3\right]_0^1 = f(x_1, x_2) = \begin{cases} \frac{3}{2}(x_1^2 + \frac{1}{3}) & 0 < x_1 < 1 \\ 0 & elsewhere \end{cases}$$

The marginal p.d.f. of X_2 is

$$f_2(x_2) = \int_0^1 \frac{3}{2}(x_1^2 + x_2^2)\, dx_1 = \frac{3}{2}\left[\frac{1}{3}x_1^3 + x_1 x_2^2\right]_0^1 = f(x_1, x_2) = \begin{cases} \frac{3}{2}(x_2^2 + \frac{1}{3}) & 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

The conditional p.d.f. of X_1, given X_2 is

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = \frac{\frac{3}{2}(x_1^2 + x_2^2)}{\frac{3}{2}(x_2^2 + \frac{1}{3})} = \frac{(x_1^2 + x_2^2)}{(x_2^2 + \frac{1}{3})} \ ; \ 0 < x_1 < 1, \ 0 < x_2 < 1$$

$\frac{3}{2}(\frac{1}{4}+\frac{1}{6}) = \frac{3}{2}(\frac{6+4}{24}) = \frac{3}{2}(\frac{10}{24})$

$= \frac{15}{24} = \frac{5}{8}$

The conditional p.d.f. of X_2, given X_1 is

$$f(x_2|x_1) = \frac{f(x_1,x_2)}{f_1(x_1)} = \frac{\frac{3}{2}(x_1^2+x_2^2)}{\frac{3}{2}(x_1^2+\frac{1}{3})} = \frac{(x_1^2+x_2^2)}{(x_1^2+\frac{1}{3})}\ ;\ 0 < x_1 < 1,\quad 0 < x_2 < 1$$

Moreover,

$$E[x_1x_2] = \frac{3}{2}\int_0^1\int_0^1 x_1x_2(x_1^2+x_2^2)dx_1dx_2 = \frac{3}{2}\int_0^1\left[\frac{1}{4}x_1^4x_2+\frac{1}{2}x_1^2x_2^3\right]_0^1 dx_2 = \frac{3}{2}\int_0^1(\frac{1}{4}x_2+\frac{1}{2}x_2^3)\,dx_2$$
$$= \frac{3}{2}\left[\frac{1}{8}x_2^2+\frac{1}{8}x_2^4\right]_0^1 = \frac{3}{8}$$

2.9 Covariance and correlation coeficient

Definition : Let X_1 and X_2 be random variables. The covariance of X_1 and X_2, denoted by $cov[X_1, X_2]$ is defined as:

$$\text{cov}[X_1, X_2] = E[X_1X_2] - E[X_1].E[X_2]$$

2.9.1 Properties of covariance

1. $cov[X_1, X_2] = \text{cov}[X_2, X_1]$.

2. $\text{cov}[X_1, a] = 0$, for a constant a.

3. $\text{cov}[X_1, -X_2] = \text{cov}[-X_1, X_2] = -\text{cov}[X_1, X_2]$..

4. $\text{cov}[X_1, X_1] = \text{var}[X_1]$.. item $\text{cov}[aX_1+b, cX_2+d] = ac\ \text{cov}[X_1, X_2]$, for the constants a, b, c and d

Definition : Let X_1 and X_2 be random variables with variances σ_1 and σ_2, respectively, and covariance $cov[X_1, X_2]$. The correlation coeficient of X_1 and X_2, denoted by $\rho[X_1, X_2]$ or ρ_{12} is defined as: معامل الارتباط

$$\rho[X_1, X_2] = \frac{cov[X_1, X_2]}{\sigma_1\sigma_2}$$

Remark: $|\rho| \leq 1 \quad \Leftrightarrow \quad -1 \leq \rho \leq 1$.

Example 1: In the preceding example

$$E[X_1] = \int_0^1 x_1 f_1(x_1)\,dx_1 = \frac{3}{2}\int_0^1 x_1(x_1^2 + \frac{1}{3})\,dx_1 = \frac{3}{2}\left[\frac{1}{4}x_1^4 + \frac{1}{6}x_1^2\right]_0^1 = \frac{5}{8}$$

and

$$E[X_1^2] = \int_0^1 x_1^2 f_1(x_1)\,dx_1 = \frac{3}{2}\int_0^1 x_1^2(x_1^2 + \frac{1}{3})\,dx_1 = \frac{3}{2}\left[\frac{1}{5}x_1^5 + \frac{1}{9}x_1^3\right]_0^1 = \frac{7}{15}$$

Then,

$$\text{var}[X_1] = E[X_1^2] - (E[X_1])^2 = \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{73}{960}$$

Similarly, we can find that

$$\text{var}[X_2] = \text{var}[X_1] = \frac{73}{960}$$

Also we have,

$$\text{cov}[X_1, X_2] = E[X_1X_2] - E[X_1]\,E[X_2] = \frac{3}{8} - \left(\frac{5}{8}\right)\left(\frac{5}{8}\right) = -\frac{1}{64}$$

Therefore, the correlation coeficient equal to

$$\rho[X_1, X_2] = \frac{cov[X_1,X_2]}{\sigma_1\,\sigma_2} = \frac{-\frac{1}{64}}{\frac{73}{960}} \cong -0.21$$