Separated variables for PDE

$\boxed{1}$

① wave equation

$$U(x,t) = V(x)\, w(t)$$

$$U_{tt}(x,t) = V(x)\, w''(t)$$

$$V_{xx}(x,t) = w(t)\, V''(x)$$

$$U_{tt} = c^2 U_{xx}$$

$$V(x)\, w''(t) = c^2 V''(x)\, w(t)$$

$$\frac{w''(t)}{c^2 w(t)} = \frac{V''(x)}{V(x)} = m$$

m : separation constant

$$w''(t) = m c^2 w(t) \Rightarrow w''(t) - m c^2 w(t) = 0$$

$$V''(x) = m V(x) \Rightarrow V''(x) - m V(x) = 0$$

Case $m = 0$

$w''(t) = 0$

$V''(x) = 0$

general solution

$w(t) = A + Bt$

$V(x) = C + DX$

$$U(x,t) = (A + Bt)(C + DX)$$

Case 2 $\quad m = \lambda^2 > 0$

$$w''(t) - c^2\lambda^2 w(t) = 0$$

$$V''(x) - \lambda^2 V(x) = 0$$

$$w(t) = e^{c\lambda t},\ e^{-c\lambda t}$$

$$V(x) = e^{\lambda x},\ e^{-\lambda x}$$

$$U(x,t) = e^{c\lambda t} \cdot e^{\lambda x},\ e^{-c\lambda t} e^{\lambda x},\ e^{c\lambda t} e^{-\lambda x},\ e^{-c\lambda t} e^{-\lambda x}$$

Case 3 $\quad m = -\lambda^2 < 0$

$\boxed{2}$

$$w''(t) + c^2\lambda^2 w(t) = 0$$

$$w(t) = \cos c\lambda t,\ \sin c\lambda t$$

$$V''(x) + \lambda V(x) = 0$$

$$V(x) = \cos \lambda x,\ \sin \lambda x$$

$$U(x,t) = \cos c\lambda t \cos \lambda x,\ \cos c\lambda t \sin \lambda x,$$
$$\sin c\lambda t \cos \lambda x,\ \sin c\lambda t \sin \lambda t$$

$\boxed{Ex}$ Solve wave equation for vibration of string stretched between the points $x=0$ and $x=\ell$ and subject the boundary conditions

$\boxed{3}$

ⓐ $u(0,t)=0 \qquad t \geq 0$

ⓑ $u(\ell,t)=0 \qquad t \geq 0$

ⓒ $\dfrac{\partial u(x,0)}{\partial t}=0 \qquad 0 \leq x \leq \ell$ initial velocity

ⓓ $u(x,0)=F(x)$ initial displacement

ⓘ $F(x)=\sin\left(\dfrac{\pi x}{\ell}\right)+\dfrac{1}{4}\sin\left(\dfrac{3\pi x}{\ell}\right)$

ⓘ $F(x)=\begin{cases} x & 0 \leq x \leq \ell/2 \\ \ell-x & \ell/2 \leq x \leq \ell \end{cases}$

صفر ⓘ

From eq ⓓ $\quad m<0$

From ⓐ $u(0,t)=0$

وهنا عندما نعوض عن كل X تكون النتيجة صفر وتتحقق في احدى الحلول العامة

$u(x,t)=\cos c\lambda t \; \sin\lambda x$

or

$u(x,t)=\sin c\lambda t \; \sin\lambda x$

From C $\quad U_t(x,0)=0$

وهنا عندما نشتق $u(x,t)$ ونعوض عن $t=0$ سوف تكون النتيجة صفر

$$\boxed{u(x,t)=\cos c\lambda t \; \sin\lambda x}$$

Prom B $\boxed{4}$

$$\sin \lambda \ell = 0 \Rightarrow \lambda \ell = n\pi$$

$$\cancel{\lambda} = \frac{n\pi}{\ell}$$

$$U(x,t) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{cn\pi}{\ell} t\right) \sin\left(\frac{n\pi}{\ell} x\right)$$

لان يجب ان نجد قيمة b_n من الفقرة (D)

$$U(x,0) = f(x)$$

معناها عندما نعوض عن $t=0$ يكون $U(x,t) = f(x)$ يكون فيه
t

$$\cos\left(\frac{cn\pi}{\ell} \cdot 0\right) = 1$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell} x\right) = \sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{4} \sin\left(\frac{3\pi}{\ell} x\right)$$

$$b_1 \sin\left(\frac{\pi}{\ell} x\right) + b_2\left(\sin \frac{2\pi}{\ell} x\right) + b_3\left(\frac{\sin 3\pi}{\ell} x\right) +$$
$$b_4\left(\frac{\sin 4\pi}{\ell} x\right) + b_5\left(\frac{\sin 5\pi}{\ell} x\right) + \cdots = \sin\left(\frac{\pi}{\ell} x\right) + \frac{1}{4} \sin\left(\frac{3\pi}{\ell} x\right)$$

$$b_1 = 1, \quad b_2 = 0, \quad b_3 = 1/4, \quad b_4, b_5 = \cdots = 0$$

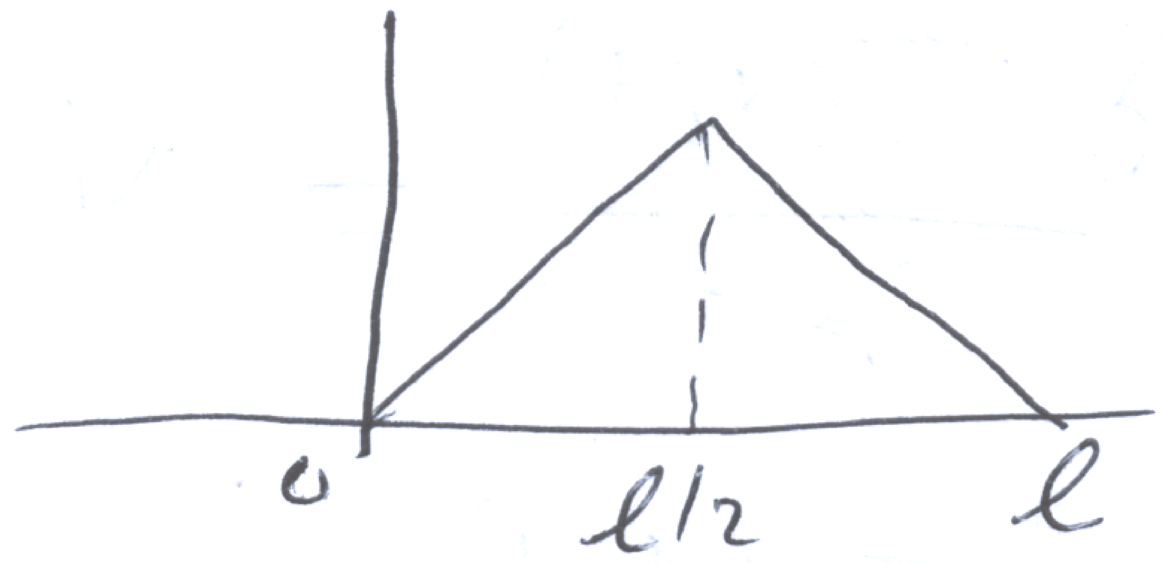
$$U(x,t) = \cos\left(\frac{c\pi}{\ell} t\right) \cdot \sin\left(\frac{\pi}{\ell} x\right) + \frac{1}{4} \cos\left(\frac{3c\pi}{\ell} t\right)$$
$$\sin\left(\frac{3\pi}{\ell} x\right)$$

5

(ii) $f(x) = \begin{cases} x & 0 \le x \le \frac{l}{2} \\ l - x & \frac{l}{2} \le x \le l \end{cases}$

(i) الجزء الاول متشابه مع الفرع

في اختيار المعادلة العامة



$$U(x,t) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{cn\pi}{l} t\right) \sin\left(\frac{n\pi}{l} x\right)$$

عندما نعوض عن $t = 0$ $\underbrace{\cos\left(\frac{cn\pi}{l} \cdot 0\right)}_{=1}$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right) = \begin{cases} x & 0 \le x \le l/2 \\ l - x & l/2 \le x \le l \end{cases}$$

نجد fourier series للموجة المتنقلة

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^{l/2} x \sin\left(\frac{n\pi}{l} x\right) + \frac{2}{l} \int_{l/2}^{l} (l - x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{4l}{\pi^2 n^2} \sin\left(\frac{n\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4l}{\pi^2 n^2} \sin\left(\frac{n\pi}{2}\right) \cdot \cos\left(\frac{cn\pi}{l} t\right) \sin\left(\frac{n\pi}{l} x\right)$$

$$u(x,t) = \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \cos\left(\frac{cn\pi}{l} t\right) \cdot \sin\left(\frac{n\pi}{l} x\right)$$

② heat equation

$T(x,t) = V(x)\,W(t)$

$\boxed{6}$

$$\frac{\partial T(x,t)}{\partial t} = V(x)\,W'(t)$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = W(t)\,V''(x)$$

$$V(x)\,W'(t) = k\,W(t)\,V''(x)$$

$$\frac{W'(t)}{k\,W(t)} = \frac{V''(x)}{V(x)} = m \qquad m : \text{constant}$$

$W'(t) = mk\,W(t)$

$W'(t) - mk\,W(t) = 0$

$V''(x) = m\,V(x)$

$V''(x) - m\,V(x) = 0$

$\boxed{\text{Case } m = 0}$

$W'(t) = 0$

$W(t) = A$

$V''(x) = 0$

$V(x) = B + CX$

$$T(x,t) = A(B + CX) = D + EX$$

A, B, C, D, E Constants

Case $m = \lambda^2 > 0$

$\boxed{7}$

$W'(t) - k\lambda^2 \, w(t) = 0$

$w(t) = e^{k\lambda^2 t}$

$V''(x) - \lambda^2 V(x) = 0$

$V(x) = e^{\lambda x}, \; e^{-\lambda x}$

$T(x,t) = e^{k\lambda^2 t} e^{\lambda x}, \; e^{k\lambda^2 t} e^{-\lambda x}$

Case $m = -\lambda^2 < 0$

$w'(t) + k\lambda^2 w(t) = 0$

$w(t) = e^{-k\lambda^2 t}$

$V''(x) + \lambda^2 V(x) = 0$

$V(x) = \cos\lambda x, \; \sin\lambda x$

$T(x,t) = \cos\lambda x \, e^{-k\lambda^2 t}, \; \sin\lambda x \, e^{-k\lambda^2 t}$

$\boxed{Ex}$ rod $\qquad \ell$

$$\frac{\partial T}{\partial x}(0,t)=0 \qquad\qquad \frac{\partial T}{\partial x}(\ell,T)=0$$

$$T(x,0)=f(x) \qquad 0<x<\ell$$

$$T(x,0)=2-3x$$

$\boxed{8}$

from $\frac{\partial T}{\partial x}(0,t)=0$

$$T(x,t)=\cos\lambda x \, e^{-k\lambda^2 t}$$

$$T(x,t)=\sum_{n=1}^{\infty} b_n \cos\lambda x \, e^{-k\lambda^2 t}+\frac{1}{2}D_0$$

from $\frac{\partial T}{\partial x}(\ell,T)=0$

هنا نشتق $T(x,t)$ ونعوض عن كل x بـ ℓ ونحسب λ

$$\frac{\partial T(x,t)}{\partial x}=-\lambda \sin\lambda x \, e^{-k\lambda^2 t}=0$$

$$\sin\lambda\ell=0 \Longrightarrow \lambda\ell=n\pi \Rightarrow \lambda=\frac{n\pi}{\ell}$$

$$T(x,t)=\frac{D_0}{2}+\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}x\right)\cdot e^{-k\frac{n^2\pi^2}{\ell^2}t}=2-3x$$

$2-3x$) fourrer series نحسب

$$D_0=\frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$D_0=\frac{2}{\ell}\int_0^{\ell}(2-3x)\,dx$$

$$D_0=1$$

$\boxed{9}$

$$b_n = \frac{2}{\ell}\int_0^{\ell} (2-3x)\cos\left(\frac{n\pi}{\ell}x\right) dx$$

$$b_n = \frac{6}{n^2\pi^2}\left[1-(-1)^n\right] = \begin{cases} \frac{12}{n^2\pi^2} & \text{odd no.} \\ 0 & \text{even no.} \end{cases}$$

$$T(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi}{\ell}x\right) \cdot e^{-\frac{kn^2\pi^2}{\ell^2}t}$$

EX/1

$T(0,t) = T(\ell,t) = 0$, $T(x,0) = f(x)$

from $T(0,t)$ the general solution is

$$T(x,t) = \sin(\lambda x)\, e^{-k\lambda^2 t}$$

from $T(\ell,t) = 0$

$$= \sin(\lambda \ell)\, e^{-k\lambda^2 t} = 0$$

$$\lambda \ell = n\pi \Rightarrow \ell = \frac{n\pi}{\ell}$$

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right) \cdot e^{-\frac{kn^2\pi^2}{\ell^2}t}$$

$1 = e^{(\ )}$ $t=0$ being

$$T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right) = f(x)$$

حتى نحصل على قيمة bn نعمل Fourier series للدالة

$$A_n = \int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right) dx$$

$\boxed{3}$ Laplace equation

Let $U(x,y) = V(x)\, w(y)$

$U_{xx}(x,y) = w(y)\, V''(x)$

$U_{yy}(x,y) = V(x)\, w''(y)$

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} = 0$$

$w(y)\, V''(x) + V(x)\, w''(y) = 0$

$w(y)\, V''(x) = -V(x)\, w''(y) =$

$$\frac{V''(x)}{V(x)} = -\frac{w''(y)}{w(y)} = m$$ m Constant

$V''(x) - m V(x) = 0$ $\qquad$ $w''(y) + m w(y) = 0$

$\boxed{\text{Case 1}}$ $\quad m = 0$

$V''(x) = 0$ $\qquad$ $w''(y) = 0$

$V(x) = A + Bx$ $\qquad$ $w(y) = C + Dy$

$U(x,y) = (A + Bx)(C + Dy)$

$\boxed{\text{Case 2)}}$ $m = \lambda^2 > 0$

$V''(x) - \lambda^2 V(x) = 0$

$V(x) = e^{\lambda x}, e^{-\lambda x}$

$W''(y) + \lambda^2 W(y) = 0$

$W(y) = \cos \lambda y, \sin \lambda y$

$U(x,y) = \cos\lambda y\, e^{\lambda x}, \cos\lambda y\, e^{-\lambda x}, \sin\lambda y\, e^{\lambda x}$
$\sin\lambda y\, e^{-\lambda x}$

$\boxed{\text{Case 3)}}$ $m = -\lambda^2 < 0$

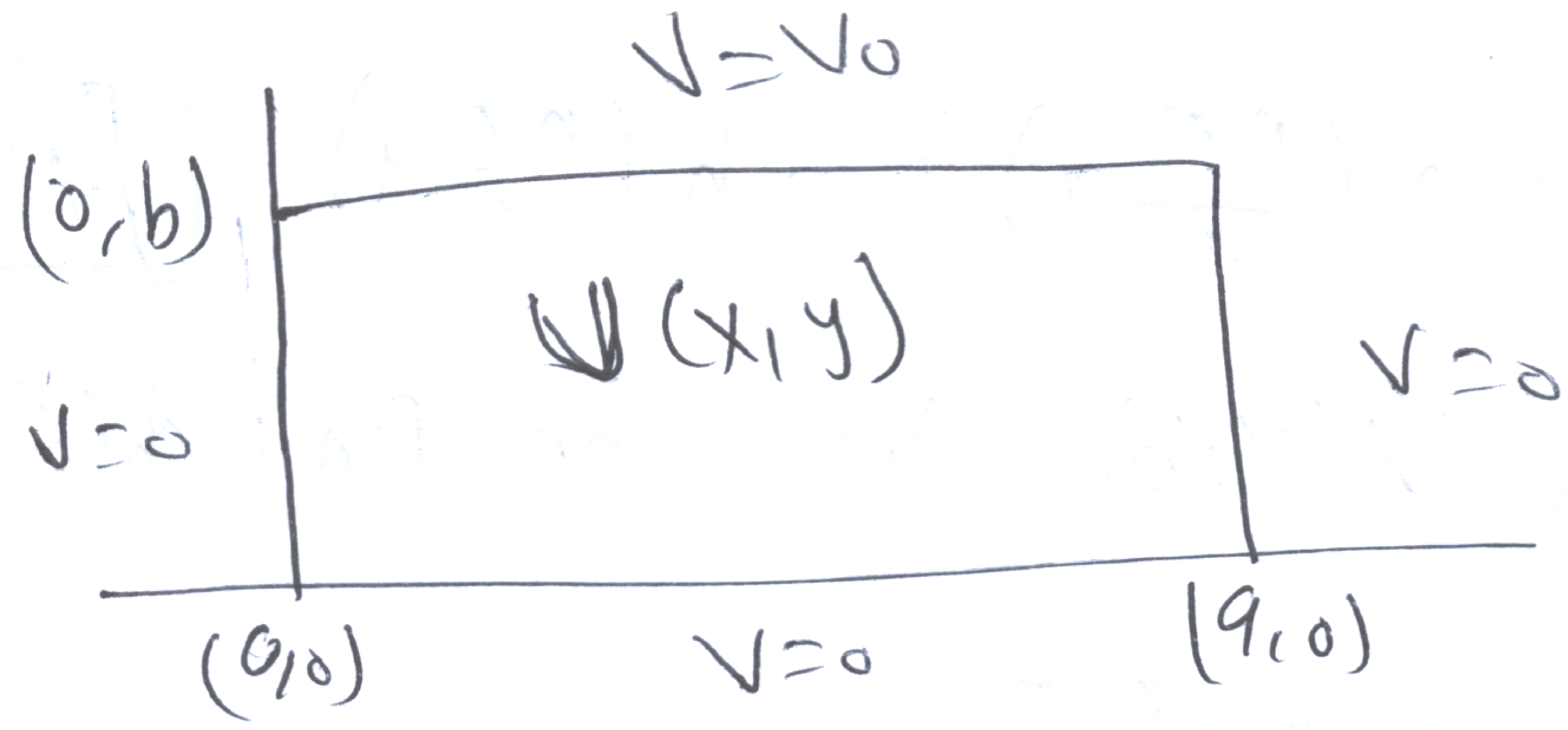
$V''(x) + \lambda^2 V(x) = 0$

$V(x) = \sin\lambda x, \cos\lambda x$

$W''(y) - \lambda^2 W(y) = 0$

$W(y) = e^{\lambda y}, e^{-\lambda y}$

$U(x,y) = \sin\lambda x\, e^{\lambda y}, \sin\lambda x\, e^{-\lambda y},$
$\cos\lambda x\, e^{\lambda y}, \cos\lambda x\, e^{-\lambda y}$



seprated varible

at $X=0$ $\quad U(0,y)=0 \Rightarrow X(0)=0$

$$U(x,y) = \sin\lambda x \, e^{-\lambda y} \text{ or } \sin\lambda x \, e^{\lambda y}$$

at $x=a$ $\quad U(a,y)=0 \Rightarrow X(a)=0$

$$\lambda a = n\pi \Longrightarrow \lambda = \frac{n\pi}{a}$$

$$U(x) = \sin\left(\frac{n\pi}{a}x\right)$$

عند تعويض bondrdy condions of y لا يوجد حل واحد

$$V(x,0) = 0 \qquad e^{-\lambda y} \neq 0 \qquad e^{\lambda y} = 0$$

$\because e^{-\lambda y} \neq 0$ and $e^{\lambda y} \neq 0$

$$\therefore U(x,y) = \sin\left(\frac{n\pi}{a}x\right) \cdot \left[A e^{-\frac{n\pi}{a}y} + B e^{\frac{n\pi}{a}y}\right]$$

~~$U(x,y) = \sum_{n=}$~~ at $y=0$

$$A = -B$$

$$V(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) C_n \left[e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}\right]$$

but $\sinh\left(\frac{n\pi y}{a}\right) = \frac{1}{2}\left[e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}\right]$

$$V(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$ $\boxed{13}$

at $y = b \Rightarrow V(x,b) = V_0$ $b_n = D_n \sinh\left(\frac{n\pi}{a}b\right)$

$$V_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}x\right)$$

forier serries

when $b_n = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi}{a}x\right)dx$

$$b_n = \frac{2V_0}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \frac{4V_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$D_n = \frac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)} = \frac{2V_0\left[1-(-1)^n\right]}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}$$

$$V(x,y) = \sum_{n=1}^{\infty} \frac{2V_0\left[1-(-1)^n\right]}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi}{a}x\right) \cdot \sinh\left(\frac{n\pi}{a}y\right)$$

$$V(x,y) = \frac{2V_0}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi}{a}x\right) \cdot \sinh\left(\frac{n\pi}{a}y\right)$$