9- Partial Differential Equations: (10 hrs)

Wave equation, Laplace equation, solution of boundary condition problems, general solution, solution by separation of variables.

Introduction

In second year, we considered the role of ordinary differential equations in engineering. However, many physical processes fundamental to science and engineering are governed by partial differential equations, which is equations involving partial derivatives. The most familiar of these processes are heat conduction and wave propagation. Unless the situation is very simple, there will be many independent variables, for example a time variable t and a space variable x, and the differential equations must involve partial derivatives.

The application of partial differential equations is much wider than the simple situations already mentioned. Maxwell's equations comprise a set of partial differential equations that form the basis of electromagnetic theory, and are fundamental to electrical engineers and physicists. The equations of fluid flow are partial differential equations, and are widely used in aeronautical engineering, acoustics.

One of the major difficulties with partial differential equations is that it is extremely

difficult to illustrate their solutions geometrically, in contrast to single-variable problems, where a simple curve can be used.

The solution of partial differential equations has been greatly eased by the use of computers, which have allowed the rapid numerical solution of problems that would otherwise have been intractable.

There are three basic types of equation that appear in most areas of science and engineering, and it is essential to understand their solutions before any progress can be made on more complicated sets of equations, nonlinear equations or equations with variable coefficients.

General discussion

The three basic types of equation are referred to as the wave equation, the heat-conduction or diffusion equation and the Laplace equation.

1- **Wave equation**

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla^2 u$$

Thus the displacement of the string satisfies the one-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(9.4)}$$

and the propagation of the disturbance in the string is given by a solution of this equation where $c^2 = T/\rho$.

Example 9.1 Show that

$$u = u_0 \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c t}{L}\right)$$

satisfies the one-dimensional wave equation and the conditions

(a) a given initial displacement $u(x, 0) = u_0 \sin(\pi x/L)$, and

(b) zero initial velocity, $\partial u(x, 0)/\partial t = 0$.

Solution Clearly the condition (a) is satisfied by inspection. If we now partially differentiate u with respect to t,

$$\frac{\partial u}{\partial t} = -\frac{u_0 \pi c}{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi c t}{L}\right)$$

so that at $t = 0$ we have $\partial u/\partial t = 0$ and (b) is satisfied.

It remains to show that (9.4) is also satisfied. Using the standard subscript notation for partial derivatives,

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = -\frac{u_0 \pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c t}{L}\right)$$

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = -\frac{u_0 \pi^2 c^2}{L^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c t}{L}\right)$$

so that the equation is indeed satisfied.

Example 9.2 Verify that the function

$$u = a\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right]$$

satisfies the wave equation (9.4). Sketch the graphs of the solution u against x at $t = 0$, $t = 2h/c$ and $t = 4h/c$.

Solution Evaluate the partial derivatives as

$$u_x = \frac{-2a(x - ct)}{h^2}\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right]$$

$$u_t = \frac{2ac(x - ct)}{h^2}\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right]$$

and

$$u_{xx} = \frac{-2a}{h^2}\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right] + \frac{4a(x - ct)^2}{h^4}\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right]$$

$$u_{tt} = \frac{-2ac^2}{h^2}\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right] + \frac{4a(x - ct)^2c^2}{h^4}\exp\left[-\left(\frac{x}{h} - \frac{ct}{h}\right)^2\right]$$

Clearly (9.4) is satisfied by these second derivatives.

2- Heat-conduction or diffusion equation

$$\frac{1}{\kappa}\frac{\partial u}{\partial t} = \nabla^2 u$$

where k is the thermal conductivity and the minus sign takes into account the fact that heat flows from hot to cold. Substitution for Q in (9.6) gives the **one-dimensional heat equation**

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2} \tag{9.7}$$

where $\kappa = k/c\rho$ is called the **thermal diffusivity**.

Example 9.3 Show that

$$T = T_\infty + (T_m - T_\infty)\,e^{-U(x-Ut)/\kappa} \quad (x \geqslant Ut)$$

satisfies the one-dimensional heat-conduction equation (9.7), together with the boundary conditions $T \to T_\infty$ as $x \to \infty$ and $T = T_m$ at $x = Ut$.

Solution The second term vanishes as $x \to \infty$, for any fixed t, and hence $T \to T_\infty$. When $x = Ut$, the exponential term is unity, so the T_∞s cancel and $T = T_m$. Hence the two boundary conditions are satisfied. Checking both sides of the heat-conduction equation (9.7),

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{1}{\kappa}(T_m - T_\infty)\frac{U^2}{\kappa}e^{-U(x-Ut)/\kappa}$$

$$\frac{\partial^2 T}{\partial x^2} = (T_m - T_\infty)\frac{U^2}{\kappa^2}e^{-U(x-Ut)/\kappa}$$

which are obviously equal, so that the equation is satisfied.

The example models a block of material being melted at a temperature T_m, with the melting boundary having constant speed U, and with a steady temperature T_∞ at great distances. An application of this model would be a heat shield on a re-entry capsule ablated by frictional heating.

Example 9.4 Show that the function

$$T = \frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4\kappa t}\right)$$

satisfies the one-dimensional heat-conduction equation (9.7). Plot T against x for various times t, and comment.

Solution We first calculate the partial derivatives

$$\frac{\partial T}{\partial t} = -\frac{1}{2}\frac{1}{t^{3/2}}\exp\left(\frac{-x^2}{4\kappa t}\right) + \frac{1}{\sqrt{t}}\frac{-x^2}{4\kappa}\frac{-1}{t^2}\exp\left(\frac{-x^2}{4\kappa t}\right)$$

$$\frac{\partial T}{\partial x} = \frac{1}{\sqrt{t}}\frac{-2x}{4\kappa t}\exp\left(\frac{-x^2}{4\kappa t}\right)$$

and

$$\frac{\partial^2 T}{\partial x^2} = \frac{-1}{2\kappa t^{3/2}}\exp\left(\frac{-x^2}{4\kappa t}\right) + \frac{-x}{2\kappa t^{3/2}}\frac{-2x}{4\kappa t}\exp\left(\frac{-x^2}{4\kappa t}\right)$$

3- Laplace equation

$$\nabla^2 u = 0 \tag{9.8}$$

The simplest physical interpretation of this equation has already been mentioned, namely as the steady-state heat equation. So, for example, the two-dimensional Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

could represent the steady-state distribution of temperature over a thin rectangular plate in the (x, y) plane.

Example 9.5 Show that

$$u = x^4 - 2x^3y - 6x^2y^2 + 2xy^3 + y^4$$

satisfies the Laplace equation.

Solution Differentiating

$$u_x = 4x^3 - 6x^2y - 12xy^2 + 2y^3, \quad u_y = -2x^3 - 12x^2y + 6xy^2 + 4y^3$$

$$u_{xx} = 12x^2 - 12xy - 12y^2, \quad u_{yy} = -12x^2 + 12xy + 12y^2$$

so clearly

$$u_{xx} + u_{yy} = 0$$

and the two-dimensional Laplace equation is satisfied.

Example 9.6 Show that the function

$$\psi = Uy\left(1 - \frac{a^2}{x^2 + y^2}\right)$$

satisfies the Laplace equation, and sketch the curves $\psi =$ constant.

Solution First calculate the partial derivatives:

$$\psi_x = \frac{2xyUa^2}{(x^2 + y^2)^2}$$

$$\psi_y = U - \frac{Ua^2}{x^2 + y^2} + \frac{2y^2Ua^2}{(x^2 + y^2)^2}$$

$$\psi_{xx} = \frac{2yUa^2}{(x^2 + y^2)^2} - \frac{8x^2yUa^2}{(x^2 + y^2)^3}$$

$$\psi_{yy} = \frac{2yUa^2}{(x^2 + y^2)^2} + \frac{4yUa^2}{(x^2 + y^2)^2} - \frac{8y^3Ua^2}{(x^2 + y^2)^3}$$

Substituting into (9.8) gives

$$\nabla^2\psi = \frac{8yUa^2}{(x^2+y^2)^2} - \frac{8y(x^2+y^2)Ua^2}{(x^2+y^2)^3} = 0$$

and hence the Laplace equation is satisfied.

Arbitrary functions and first-order equations

In each of the examples in this section, a solution has been given; it has been checked that the solution satisfies the appropriate partial differential equation. In no case has the boundary condition been part of the specification of the problem, although in several cases boundary conditions were checked. In the next sections the boundary conditions are given as part of the set-up of the example. This is the natural way that a physical problem is specified and it proves to be a much tougher proposition.

The most significant difference between ordinary and partial differential equations is the treatment of the 'arbitrary constants'. Consider the examples:

ODE

Solve the ordinary differential equation

$$\frac{dy(t)}{dt} = 3t^2$$

Integrating gives

$$y(t) = t^3 + K$$

where K is an **arbitrary constant**, since differentiating $y(t)$ with respect to t produces $3t^2$ whatever the value of the constant K.

PDE

Solve the partial differential equation

$$\frac{\partial z(x,t)}{\partial t} = 3t^2$$

Integrating gives

$$z(x,t) = t^3 + f(x)$$

where $f(x)$ is an **arbitrary function**. Differentiating with respect to t produces $3t^2$ for any function $f(x)$ because x is kept constant in the partial differentiation.

Extending this idea it can be seen that each partial integration introduces an arbitrary function into the solution. Sufficient conditions must be given to determine these arbitrary functions. It is not always easy to decide exactly what conditions are required, but in subsequent sections an idea will be given for the three classic equations, the wave equation, the heat-conduction equation and the Laplace equation. An extended discussion can be found in Section 9.8.

Consider for the moment a first-order equation. Such equations are of less interest in applications to engineering and science, but there is a comprehensive theory for their solution which will illustrate the use of arbitrary functions.

Example 9.7 Find the general solution, $u(x, t)$, of the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

and find the particular solution when $u(x, 0) = x^2$.

Solution Change the variables $z = x - t$ and $T = t$ and use the chain rule to evaluate the terms in the equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial T}\frac{\partial T}{\partial t} = -\frac{\partial u}{\partial z} + \frac{\partial u}{\partial T}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial T}\frac{\partial T}{\partial x} = \frac{\partial u}{\partial z}$$

Putting these differentials into the equation

$$0 = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial u}{\partial T}$$

Thus $u(z, T)$ can be deduced as

$u(z, T) = f(z)$, where f is an arbitrary function

Reverting to the original variables

$$u(x, t) = f(x - t)$$

and a general solution of the partial differential equation has been obtained.

For the particular solution with initial conditions written in parametric form, $x = s$, $t = 0$, $u = s^2$, it is easily deduced that $s^2 = f(s)$ and hence

$$u(x, t) = (x - t)^2$$

Solution of the wave equation

In this section, we will try to solve the three types of PDE's by three different methods:

1- D'Alembert method.

2- Separated Variables.

3- Laplace method.

D'Alembert solution and characteristics

A classical solution of the one-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(9.4)}$$

is obtained by changing the axes to reduce the equation to a particularly simple form. Let

$$r = x + ct, \qquad s = x - ct$$

Then, using the chain rule procedure for transformation of coordinates (see Section 3.1.1),

$$u_{xx} = u_{rr} + 2u_{rs} + u_{ss}$$

$$u_{tt} = c^2(u_{rr} - 2u_{rs} + u_{ss})$$

so that the wave equation (9.4) becomes

$$4c^2 u_{rs} = 0$$

This equation can now be integrated once with respect to s to give

$$u_r = \frac{\partial u}{\partial r} = \theta(r)$$

where θ is an arbitrary function of r. Now, integrating with respect to r, we obtain

$$u = f(r) + g(s)$$

which, on substituting for r and s, gives the solution of the wave equation (9.4) as

$$u = f(x + ct) + g(x - ct) \qquad \textbf{(9.17)}$$

where f and g are *arbitrary functions* and f is just the integral of the arbitrary function θ.

The solution (9.17) is one of the few cases where the general solution of a partial differential equation can be found. However, finding the precise form of the arbitrary functions f and g that satisfy given initial data is not always easy. The initial conditions must give just enough information to evaluate f and g, which are functions of the *single* variables $r = x + ct$ and $s = x - ct$ respectively.

Example 9.10 Check that $u = 1/[1 + (x + ct)^2]$ satisfies the wave equation (9.4) and show that it represents a travelling wave in the $-x$ direction.

Solution Differentiating partially with respect to x and t

$$u_x = \frac{-2(x+ct)}{[1+(x+ct)^2]^2}, \qquad u_{xx} = \frac{2[-1+3(x+ct)^2]}{[1+(x+ct)^2]^3}$$

$$u_t = \frac{-2c(x+ct)}{[1+(x+ct)^2]^2}, \qquad u_{tt} = \frac{2c^2[-1+3(x+ct)^2]}{[1+(x+ct)^2]^3}$$

and the wave equation is satisfied. Plots of the function u against x for various values

Example 9.11 Solve the wave equation (9.4) subject to the conditions

(a) zero initial velocity, $\partial u(x, 0)/\partial t = 0$ for all x, and

(b) an initial displacement given by

$$u(x, 0) = F(x) = \begin{cases} 1 - x & (0 \leqslant x \leqslant 1) \\ 1 + x & (-1 \leqslant x \leqslant 0) \\ 0 & \text{otherwise} \end{cases}$$

Solution This example corresponds physically to an infinite string initially at rest, and displaced as in Figure 9.9, which is then released.

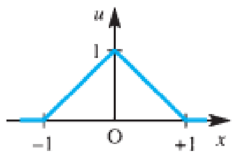

Figure 9.9 Initial displacement in Example 9.11.

From (9.17) we have a solution of the wave equation as

$$u = f(x + ct) + g(x - ct)$$

We now fit the given boundary data. Condition (a) gives

$$0 = cf'(x) - cg'(x) \quad \text{for all } x$$

so that

$$f(x) - g(x) = K = \text{an arbitrary constant}$$

and thus

$$u = f(x + ct) + f(x - ct) - K$$

Similarly, condition (b) gives

$$F(x) = 2f(x) - K$$

so that

$$u = \tfrac{1}{2}F(x + ct) + \tfrac{1}{2}F(x - ct) \tag{9.18}$$

We now have the solution to the equation in terms of the function F defined in condition (b). (Note that the same is true for any function F.)

The solution is plotted in Figure 9.10 as u against x for given times. It may be observed from this example that we have two **travelling waves**, one propagating to the right and one to the left. The initial shape is propagated exactly, except for a factor of two, and the shape discontinuities are not smoothed out, as noted in Section 9.2.1.

Separated solutions

A method of considerable importance is the **method of separation of variables**. The basis of the method is to attempt to look for solutions $u(x, y)$ of a partial differential equation as a product of functions of single variables

$$u(x, y) = X(x)Y(y)$$

The advantage of this approach is that it is sometimes possible to find X and Y as solutions of *ordinary* differential equations. These are very much easier to solve than partial differential equations, and it may be possible to build up solutions of the full equation in terms of the solutions for X and Y. A simple example illustrates the general strategy. Suppose that we wish to solve

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

Then we should write $u = X(x)Y(y)$ and substitute

$$Y\frac{dX}{dx} + X\frac{dY}{dy} = 0, \quad \text{or} \quad \frac{1}{X}\frac{dX}{dx} = -\frac{1}{Y}\frac{dY}{dy}$$

Note that the partial differentials become ordinary differentials, since the functions are just functions of a single variable. Now

$$\text{LHS} = \frac{1}{X}\frac{dX}{dx} = \text{a function of } x \text{ only}$$

$$\text{RHS} = -\frac{1}{Y}\frac{dY}{dy} = \text{a function of } y \text{ only}$$

Since LHS = RHS for *all* x and y, the only way that this can be achieved is for each side to be a *constant*. We thus have two ordinary differential equations

$$\frac{1}{X}\frac{dX}{dx} = \lambda, \quad -\frac{1}{Y}\frac{dY}{dy} = \lambda$$

These equations can be solved easily as

$$X = B\,e^{\lambda x}, \quad Y = C\,e^{-\lambda y}$$

and thus the solution of the original partial differential equation is

$$u(x, y) = X(x)Y(y) = A\,e^{\lambda(x-y)}$$

where $A = BC$. The constants A and λ are arbitrary. The crucial question is whether the boundary conditions imposed by the problem can be satisfied by a sum of solutions of this type.

The method of separation of variables can be a very powerful technique, and we shall see it used on all three of the basic partial differential equations. It should be noted, however, that all equations do not have separated solutions, see Example 9.2, and even when they can be obtained it is not always possible to satisfy the boundary conditions with such solutions.

In the case of the heat-conduction equation and the wave equation, the form of one of the functions in the separated solution is dictated by the physics of the problem. We shall see that the separation technique becomes a little simpler when such physical arguments are used. However, for the Laplace equation there is no help from the physics, so the method just described needs to be applied.

In most wave equation problems we are looking for either a travelling-wave solution as in Section 9.3.1 or for periodic solutions, as a result of plucking a violin string for instance. It therefore seems natural to look for specific solutions that have periodicity built into them. These will not be general solutions, but they will be seen to be useful for a whole class of problems. The essential mathematical simplicity of the method comes from only having to solve ordinary differential equations.

The above argument suggests that we seek solutions of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(9.4)}$$

of the form either

$$u = \sin(c\lambda t)v(x) \qquad \textbf{(9.24a)}$$

or

$$u = \cos(c\lambda t)v(x) \qquad \textbf{(9.24b)}$$

both of which when substituted into (9.4) give the ordinary differential equation

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\lambda^2 v$$

This is a simple harmonic equation with solutions $v = \sin \lambda x$ or $v = \cos \lambda x$. We can thus build up a general solution of (9.4) from linear multiples of the four basic solutions

$$u_1 = \cos \lambda ct \sin \lambda x \qquad \textbf{(9.25a)}$$

$$u_2 = \cos \lambda ct \cos \lambda x \qquad \textbf{(9.25b)}$$

$$u_3 = \sin \lambda ct \sin \lambda x \qquad \textbf{(9.25c)}$$

$$u_4 = \sin \lambda ct \cos \lambda x \qquad \textbf{(9.25d)}$$

and try to satisfy the boundary conditions using appropriate linear combinations of solutions of this type. We saw an example of such a solution in Example 9.1.

Example 9.14 Solve the wave equation (9.4) for the vibration of a string stretched between the points $x = 0$ and $x = l$ and subject to the boundary conditions

(a) $u(0, t) = 0 \quad (t \geqslant 0)$ (fixed at the end $x = 0$);

(b) $u(l, t) = 0 \quad (t \geqslant 0)$ (fixed at the end $x = l$);

(c) $\partial u(x, 0)/\partial t = 0 \quad (0 \leqslant x \leqslant l)$ (with zero initial velocity);

(d) $u(x, 0) = F(x)$ (given initial displacement).

Consider the two cases

(i) $F(x) = \sin(\pi x/l) + \frac{1}{4}\sin(3\pi x/l)$

(ii) $F(x) = \begin{cases} x & (0 \le x \le \frac{1}{2}l) \\ l - x & (\frac{1}{2}l \le x \le l) \end{cases}$

Solution Clearly, we are solving the problem of a stretched string, held at its ends $x = 0$ and $x = l$ and released from rest.

By inspection, we see that the solutions (9.25b, d) cannot satisfy condition (a). We see that condition (b) is satisfied by the solutions (9.25a, c), provided that

$$\sin \lambda l = 0, \quad \text{or} \quad \lambda l = n\pi \quad (n = 1, 2, 3, \dots)$$

It may be noted that only specific values of λ in (9.25) give permissible solutions. Thus the string can only vibrate with given frequencies, $nc/2l$. The solution (9.25) appropriate to this problem takes the form either

$$u = \cos\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \tag{9.26a}$$

or

$$u = \sin\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \tag{9.26b}$$

($n = 1, 2, 3, \dots$). To satisfy condition (c) for all x, we must choose the solution (9.26a) and omit (9.26b). Clearly, it is not possible to satisfy the initial condition (d) with (9.26a). However, because the wave equation is linear, any *sum* of such solutions is also a solution. Thus we build up a solution

$$u = \sum_{n=1}^{\infty} b_n \cos\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

Case (i)

The initial condition (d) for $u(x, 0)$ gives

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \sin\left(\frac{\pi x}{l}\right) + \frac{1}{4}\sin\left(\frac{3\pi x}{l}\right)$$

and the values of b_n can be evaluated by inspection as

$$b_1 = 1,\ b_2 = 0,\ b_3 = \tfrac{1}{4},\ b_4 = b_5 = \dots = 0$$

The full solution is therefore

$$u = \cos\left(\frac{\pi ct}{l}\right)\sin\left(\frac{\pi x}{l}\right) + \frac{1}{4}\cos\left(\frac{3\pi ct}{l}\right)\sin\left(\frac{3\pi x}{l}\right)$$

The solution is illustrated in Figure 9.17.

Case (ii)

The condition (d) for $u(x, 0)$ simply gives

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x) = \begin{cases} x & (0 \leqslant x \leqslant \frac{1}{2}l) \\ l - x & (\frac{1}{2}l \leqslant x \leqslant l) \end{cases}$$

and thus to determine b_n we must find the Fourier sine series expansion of the function $f(x)$ over the finite interval $0 \leqslant x \leqslant l$. We have from (7.33) that

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x \\ &= \frac{2}{l}\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x + \frac{2}{l}\int_{l/2}^{l} (l-x)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x \\ &= \frac{4l}{\pi^2 n^2}\sin(\tfrac{1}{2}n\pi) \quad (n = 1, 2, 3, \dots) \end{aligned}$$

The complete solution of the wave equation in this case is therefore

$$u(x, t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\tfrac{1}{2}n\pi)\cos\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \tag{9.28}$$

or

$$u(x, t) = \frac{4l}{\pi^2}\left[\cos\left(\frac{c\pi t}{l}\right)\sin\left(\frac{\pi x}{l}\right) - \tfrac{1}{9}\cos\left(\frac{3c\pi t}{l}\right)\sin\left(\frac{3\pi x}{l}\right) + \tfrac{1}{25}\cos\left(\frac{5c\pi t}{l}\right)\sin\left(\frac{5\pi x}{l}\right) + \dots\right]$$

Example 9.15 Solve the wave equation (9.4) for vibrations in an organ pipe subject to the boundary conditions

(a) $u(0, t) = 0 \quad (t \geqslant 0)$ (the end $x = 0$ is closed);

(b) $\partial u(l, t)/\partial x = 0 \quad (t \geqslant 0)$ (the end $x = l$ is open);

(c) $u(x, 0) = 0 \quad (0 \leqslant x \leqslant l)$ (the pipe is initially undisturbed);

(d) $\partial u(x, 0)/\partial t = v = \text{constant} \quad (0 \leqslant x \leqslant l)$ (the pipe is given an initial uniform blow).

Solution From the solution (9.25), we deduce from condition (a) that solutions (9.25b, d) must be omitted, and similarly from condition (c) that solution (9.25a) is not useful. We are left with the solution (9.25c) to satisfy the boundary condition (b). This can only be satisfied if

$$\cos\lambda l = 0, \quad \text{or} \quad \lambda l = (n + \tfrac{1}{2})\pi \quad (n = 0, 1, 2, \dots)$$

Thus we obtain solutions of the form

$$u = b_n \sin\left[\frac{(n+\frac{1}{2})\pi ct}{l}\right] \sin\left[\frac{(n+\frac{1}{2})\pi x}{l}\right] \quad (n = 0, 1, 2, \dots)$$

giving a general solution

$$u = \sum_{n=0}^{\infty} b_n \sin\left[\frac{(n+\frac{1}{2})\pi ct}{l}\right] \sin\left[\frac{(n+\frac{1}{2})\pi x}{l}\right]$$

The condition (d) gives

$$v = \sum_{n=0}^{\infty} b_n \frac{(n+\frac{1}{2})\pi c}{l} \sin\left[\frac{(n+\frac{1}{2})\pi x}{l}\right]$$

which, on using (7.33) to obtain the coefficients of the Fourier sine series expansion of the constant v over the finite interval $0 \leqslant x \leqslant l$, gives

$$b_n = \frac{2v}{(n+\frac{1}{2})\pi} \frac{l}{(n+\frac{1}{2})\pi c} = \frac{8lv}{\pi^2 c} \frac{1}{(2n+1)^2}$$

Our complete solution of the wave equation is therefore

$$u = \frac{8lv}{\pi^2 c} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sin\left[(n+\tfrac{1}{2})\pi \frac{ct}{l}\right] \sin\left[(n+\tfrac{1}{2})\pi \frac{x}{l}\right]$$

or,

$$u = \frac{8lv}{\pi^2 c}\left[\sin\left(\frac{\pi ct}{2l}\right)\sin\left(\frac{\pi x}{2l}\right) + \tfrac{1}{9}\sin\left(\frac{3\pi ct}{2l}\right)\sin\left(\frac{3\pi x}{2l}\right) + \tfrac{1}{25}\sin\left(\frac{5\pi ct}{2l}\right)\sin\left(\frac{5\pi x}{2l}\right) + \dots\right]$$

Laplace transform solution

For linear problems that are time varying from 0 to ∞, as in the case of the wave equation, Laplace transforms provide a formal method of solution. The only difficulty is whether the final inversion can be performed.

First we obtain the Laplace transforms of the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial t^2}$$

of the function $u(x, t)$, $t \geqslant 0$. Using the same procedure as that used to obtain the Laplace transform of standard derivatives in Section 5.3.1, we have the following:

(a) $$\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\,dt = \frac{d}{dx}\int_0^\infty e^{-st}u(x, t)\,dt$$

using Leibniz's rule (see *Modern Engineering Mathematics*) for differentiation under an integral sign. Noting that

$$\mathscr{L}\{u(x, t)\} = U(x, s) = \int_0^\infty e^{-st}u(x, t)\,dt$$

we have

$$\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d}{dx}U(x, s) \qquad \textbf{(9.29)}$$

(b) Writing $y(x, t) = \partial u/\partial x$, repeated application of the result (9.29) gives

$$\mathscr{L}\left\{\frac{\partial y}{\partial x}\right\} = \frac{d}{dx}\mathscr{L}\{y(x, t)\} = \frac{d}{dx}\left(\frac{d}{dx}U(x, s)\right)$$

so that

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U(x, s)}{dx^2} \qquad \textbf{(9.30)}$$

(c) $$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\,dt$$

$$= [e^{-st}u(x, t)]_0^\infty + s\int_0^\infty e^{-st}u(x, t)\,dt = [0 - u(x, 0)] + sU(x, s)$$

so that

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0)$$

where we have assumed that $u(x, t)$ is of exponential order.

(d) Writing $v(x, t) = \partial u/\partial t$, repeated application of (9.31) gives

$$\mathscr{L}\left\{\frac{\partial v}{\partial t}\right\} = sV(x, s) - v(x, 0)$$

$$= s[sU(x, s) - u(x, 0)] - v(x, 0)$$

so that

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x, s) - su(x, 0) - u_t(x, 0)$$

where $u_t(x, 0)$ denotes the value of $\partial u/\partial t$ at $t = 0$.

Let us now return to consider the wave equation (9.4)

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

subject to the boundary conditions $u(x, 0) = f(x)$ and $\partial u(x, 0)/\partial t = g(x)$. Taking Laplace transforms on both sides of (9.4) and using the results (9.30) and (9.32) gives

$$c^2\frac{d^2U(x, s)}{dx^2} = s^2U(x, s) - g(x) - sf(x) \qquad \textbf{(9.33)}$$

The problem has thus been reduced to an ordinary differential equation in $U(x, s)$ of a straightforward type. It can be solved for given conditions at the ends of the x range, and the solution can then be inverted to give $u(x, t)$.

Example 9.16 Solve the wave equation (9.4) for a semi-infinite string by Laplace transforms, given that

(a) $u(x, 0) = 0 \quad (x \geq 0)$ (string initially undisturbed);

(b) $\partial u(x, 0)/\partial t = x\,e^{-x/a} \quad (x \geq 0)$ (string given an initial velocity);

(c) $u(0, t) = 0 \quad (t \geq 0)$ (string held at $x = 0$);

(d) $u(x, t) \to 0$ as $x \to \infty$ for $t \geq 0$ (string held at infinity).

Solution Using conditions (a) and (b) and substituting for $f(x)$ and $g(x)$ in the result (9.33), the transformed equation in this case is

$$c^2 \frac{d^2}{dx^2} U(x, s) = s^2 U(x, s) - x\,e^{-x/a}$$

By seeking a particular integral of the form

$$U = \alpha x\,e^{-x/a} + \beta\,e^{-x/a}$$

we obtain a solution of the differential equation as

$$U(x, s) = A\,e^{sx/c} + B\,e^{-sx/c} - \frac{e^{-x/a}}{c^2/a^2 - s^2}\left[x + \frac{2c^2/a}{c^2/a^2 - s^2}\right]$$

where A and B are arbitrary constants.

Transforming the given boundary conditions (c) and (d), we have $U(0, s) = 0$ and $U(x, s) \to 0$ as $x \to \infty$, which can be used to determine A and B. From the second condition $A = 0$, and the first condition then gives

$$B = \frac{2c^2/a}{(c^2/a^2 - s^2)^2}$$

so that the solution becomes

$$U(x, s) = \frac{2c^2/a}{(c^2/a^2 - s^2)^2}\,e^{-sx/c} - \frac{e^{-x/a}}{(c^2/a^2 - s^2)}\left[x + \frac{2c^2/a}{(c^2/a^2 - s^2)}\right]$$

Fortunately in this case these transforms can be inverted from tables of Laplace transforms.

Using the second shift theorem (5.45) together with the Laplace transform pairs

$$\mathscr{L}\{\sinh \omega t\} = \frac{\omega}{s^2 - \omega^2}, \quad \mathscr{L}\{\cosh \omega t\} = \frac{s}{s^2 - \omega^2}$$

$$\mathscr{L}\left\{\frac{\omega t \cosh \omega t - \sinh \omega t}{2\omega^3}\right\} = \frac{1}{(s^2 - \omega^2)^2}$$

we obtain the solution as

$$u = \frac{a}{c}\left[(ct - x)\cosh\left(\frac{ct - x}{a}\right)H(ct - x) - ct\,e^{-x/a}\cosh\left(\frac{ct}{a}\right)\right]$$
$$+ \frac{a}{c}\left[e^{-x/a}(x + a)\sinh\left(\frac{ct}{a}\right) - a\sinh\left(\frac{ct - x}{a}\right)H(ct - x)\right]$$

where $H(t)$ is the Heaviside step function defined in Section 5.5.1.