1- Fourier Series and Fourier Transform:

Fourier Series, trigonometric FS, Compact form, Complex FS, Symmetry, Half wave Symmetry, Parseval's theorem, Fourier Transform, Properties, convolution theorem, power spectral, density and correlations, signals and linear systems, applications.

Do not worry about your difficulties in mathematics, I assure you that mine are greater.

—Albert Einstein

1.1 Introduction:

In this chapter, we consider representing a signal as a weighted superposition of orthogonal sinusoids, if such a signal is applied to a linear system, then the system output is a weighted superposition of the system response to each complex sinusoid.

Representation of signals as superposition of orthogonal sinusoid not only lead to a useful expression for the system output but also provides a very insightful characterization of signals and systems. The focus of this chapter is representation of signals using orthogonal sinusoids and the properties of such representations. Applications of these representations to system and signal analysis are emphasized in the communication system theory and circuit analysis. The Fourier series is named after Jean Baptiste Joseph Fourier (1768–1830). In 1822, Fourier's genius came up with the insight that any practical periodic function can be represented as a sum of sinusoids. Such a representation, along with the superposition theorem, allows us to find the response of circuits to arbitrary periodic inputs using phasor techniques.

We begin with the trigonometric Fourier series. Later we consider the exponential Fourier series. We then apply Fourier series in circuit analysis. Finally, practical applications of Fourier series in spectrum analyzers and filters are demonstrated.

1.2 TRIGONOMETRIC FOURIER SERIES

According to the Fourier theorem, any practical periodic function of frequency ω_0 can be expressed as an infinite sum of sine or cosine functions that are integral multiples of ω_0. Thus, f(t) can be expressed as

$$f(t) = a_0 + a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

or

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

where $\omega_0 = 2\pi/T$ is called the fundamental frequency in radians per second. The sinusoid $\sin n\omega_0 t$ or $\cos n\omega_0 t$ is called the nth harmonic of f(t); it is an odd harmonic if n is odd and an even harmonic if n is even. The constants a_n and b_n are the Fourier coefficients. The coefficient a_0 is the dc component or the average value of f(t).

The Fourier series of a periodic function $f(t)$ is a representation that resolves $f(t)$ into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

A function that can be represented by a Fourier series as in Eq. (16.3) must meet certain requirements, because the infinite series in Eq. (16.3) may or may not converge. These conditions on f(t) to yield a convergent Fourier series are as follows:

1. $f(t)$ is single-valued everywhere.
2. $f(t)$ has a finite number of finite discontinuities in any one period.
3. $f(t)$ has a finite number of maxima and minima in any one period.
4. The integral $\int_{t_0}^{t_0+T} |f(t)|\, dt < \infty$ for any t_0.

These conditions are called Dirichlet conditions. Although they are not necessary conditions, they are sufficient conditions for a Fourier series to exist.

A major task in Fourier series is the determination of the Fourier coefficients a0,an, and bn. The process of determining the coefficients is called Fourier analysis. The following trigonometric integrals are very helpful in Fourier analysis. For any integers m and n,

$$\int_0^T \sin n\omega_0 t \, dt = 0$$

$$\int_0^T \cos n\omega_0 t \, dt = 0$$

$$\int_0^T \sin n\omega_0 t \cos m\omega_0 t \, dt = 0$$

$$\int_0^T \sin n\omega_0 t \sin m\omega_0 t \, dt = 0, \qquad (m \neq n)$$

$$\int_0^T \cos n\omega_0 t \cos m\omega_0 t \, dt = 0, \qquad (m \neq n)$$

$$\int_0^T \sin^2 n\omega_0 t \, dt = \frac{T}{2}$$

$$\int_0^T \cos^2 n\omega_0 t \, dt = \frac{T}{2}$$

By using the above identities we can

$$a_0 = \frac{1}{T} \int_0^T f(t) \, dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\, dt$$

An alternative form of Eq. (16.3) is the amplitude-phase form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

Where

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

The plot of the amplitude A_n of the harmonics versus $n\omega_0$ is called the amplitude spectrum of f(t); the plot of the phase φ_n versus $n\omega_0$ is the phase spectrum of f(t). Both the amplitude and phase spectra form the frequency spectrum of f(t).

The frequency spectrum of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency.

Thus, the Fourier analysis is also a mathematical tool for finding the spectrum of a periodic signal.

One-Sided or Positive-frequency line spectra

Consider the arbitrary sinusoid

$$f(t) = A\cos(w_0 t + \theta) \text{ where } w_0 = 2 * \pi * f_0 = 2\pi/T_0$$

Which can be written as:

$$A\cos(w_0 t + \theta) = Re\left[Ae^{j(w_0 t + \theta)}\right] = Re[Ae^{j(w_0 t)}e^{j(\theta)}]$$

This is called a Phasor representation because the term inside the brackets may be viewed as a rotating vector in a complex plane whose axes are the real and imaginary parts as shown below.

The phasor has length A , rotates counterclockwise ate a rate frequency f_0 revolutions per second, and at time =0 makes an angle θ with respect to the positive real axis. At any time t the projection of the phasor on the real axis i.e its real part equals the sinusoid f(t).

Note carefully that only three parameters are needed to specify a phasor: amplitude, relative phase and rotational frequency. To describe the same phasor in the frequency domain , we see that it is defined only for particular frequency f_0. With this frequency we must associate the corresponding amplitude and phase. Hence, a suitable frequency domain description would be the line spectrum of Figure below, which consists of two plots, amplitude versus frequency and phase versus frequency. Four standard conventions used in constructing line spectra listed .

1- In all our spectra drawings the independent variable will be cyclical frequency f in hertz, rather than radian frequency w.
2- Phase angles will be measured with respect to Cosine waves or equivalently, with respect to the positive real axis of the phasor diagram. $\sin(wt)=\cos(wt-90^0)$.
3- We regard amplitude as always being a positive quantity; when negative appear, they must be absorbed in the phase $-A\cos(wt)=A\cos(wt\pm180^0)$.

There is another spectra representation which is only slightly more complicated and turns out to be much more useful. It is based on writing a sinusoid as a sum of two exponentials,

$$A cos(w_0 t + \theta) = \left[\frac{A}{2} e^{j(w_0 t+\theta)} + \frac{A}{2} e^{-j(w_0 t+\theta)}\right]$$

$$= [\frac{A}{2} e^{j(w_0 t)} e^{j(\theta)} + \frac{A}{2} e^{-j(w_0 t)} e^{-j(\theta)}]$$

Which will call the conjugate-phasor representation since the two terms are complex conjugates of each other.

- The spectrum of periodic signals is not continuous but exist only at discrete frequencies.
- The idea of negative frequency is not hard to be understand. Both signals $e^{j(w_0 t)}$, $e^{-j(w_0 t)}$ oscillate with the same frequency but they may looked upon two phasor rotating in opposite directions and when added yield a real time domain function.

Example 1.1: Draw the line spectra for the signal consist of sum of sinusoids, such as f(t)= 2+6cos(2π10t+30^0)+ 3sin(2π30t+30^0)- 4cos(2π35t).

Solution :

f(t)= 2+6cos(2π10t+30^0)+ 3cos(2π30t+30^0-90)+ 4cos(2π35t-180).

Some Useful Trigonometric integration formulas

To evaluate the Fourier coefficients a_0, a_n, and b_n, we often need to apply the following integrals:

$$\int \cos at\, dt = \frac{1}{a} \sin at$$

$$\int \sin at\, dt = -\frac{1}{a} \cos at$$

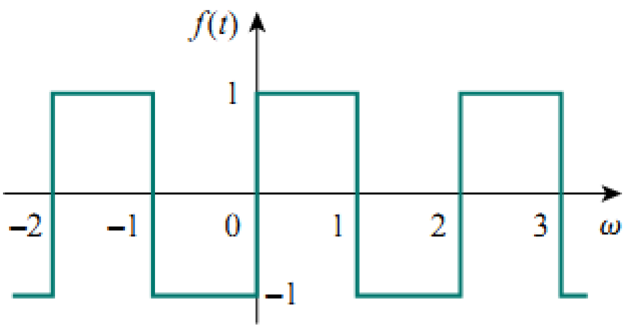
$$\int t \cos at\, dt = \frac{1}{a^2} \cos at + \frac{1}{a} t \sin at$$

$$\int t \sin at\, dt = \frac{1}{a^2} \sin at - \frac{1}{a} t \cos at$$

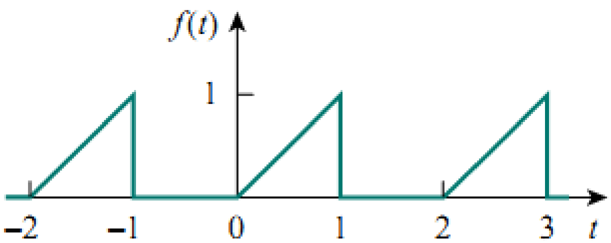
It is also useful to know the values of the cosine, sine, and exponential functions for integral multiples of π

Function	Value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos \frac{n\pi}{2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$
$\sin \frac{n\pi}{2}$	$\begin{cases} (-1)^{(n-1)/2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ j(-1)^{(n-1)/2}, & n = \text{odd} \end{cases}$

Example 1.2 : Find the Fourier series of the square wave in Fig. 16.5. Plot the amplitude and phase spectra.



Example 1.3 : Obtain the Fourier series for the periodic function in Fig. 16.7 and plot the amplitude and phase spectra.



1.3 SYMMETRY CONSIDERATIONS

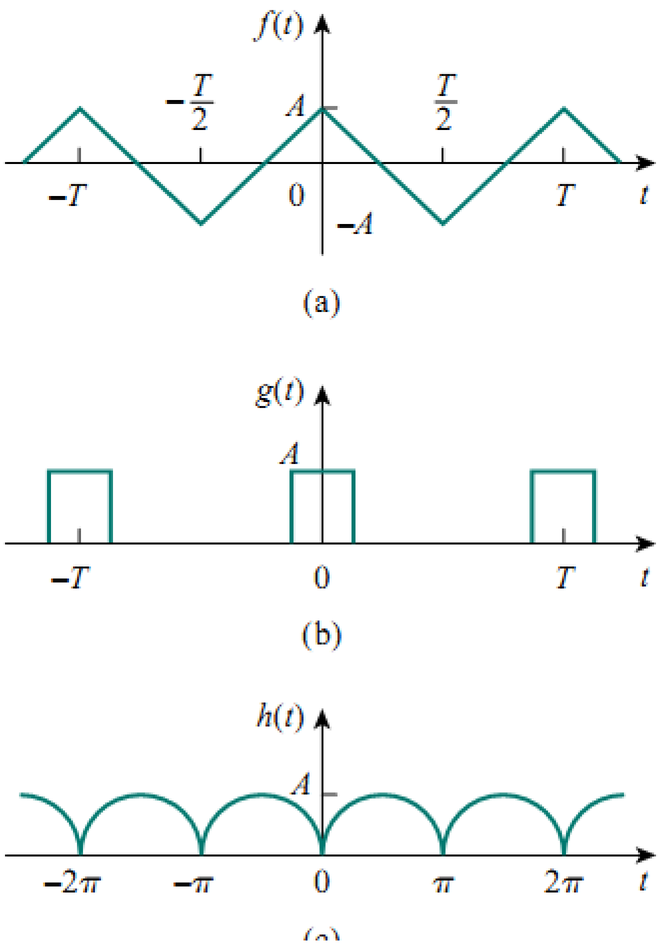
Some Fourier coefficients would be zero and avoid the unnecessary work involved in the tedious process of calculating them. Such a method does exist; it is based on recognizing the existence of symmetry. Here we discuss three types of symmetry: (1) even symmetry, (2) odd symmetry, (3) half-wave symmetry.

1.3.1 Even Symmetry

A function f(t) is even if its plot is symmetrical about the vertical axis;

that is,

$$\boxed{f(t) = f(-t)}$$



the Fourier coefficients for an even function become

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

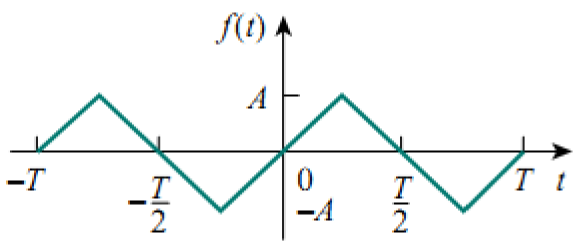
$$b_n = 0$$

Since $b_n = 0$, above equation becomes a Fourier cosine series.

1.3.2 Odd Symmetry

A function f(t) is said to be odd if its plot is anti-symmetrical about the vertical axis:

$$f(-t) = -f(t)$$



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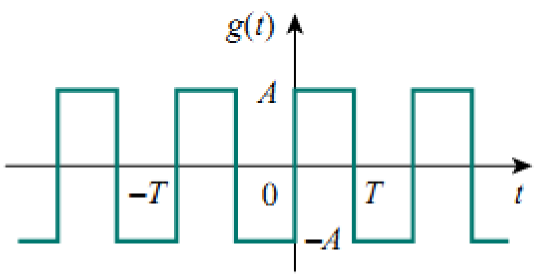

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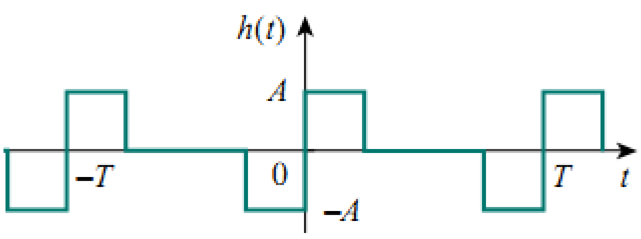


the Fourier coefficients for an odd function become

$$a_0 = 0, \qquad a_n = 0$$
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$

which give us a Fourier sine series.

It is interesting to note that any periodic function f(t) with neither even nor odd symmetry may be decomposed into even and odd parts.

$$f(t) = \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{\text{even}} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{\text{odd}} = f_e(t) + f_o(t)$$

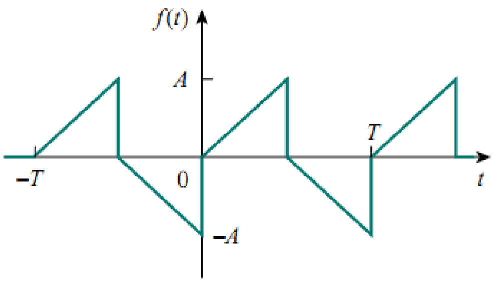
$$f(t) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin n\omega_0 t}_{\text{odd}} = f_e(t) + f_o(t)$$

1.3.3 Half-Wave Symmetry

A function is half-wave (odd) symmetric if

$$f\left(t - \frac{T}{2}\right) = -f(t)$$

which means that each half-cycle is the mirror image of the next half-cycle.



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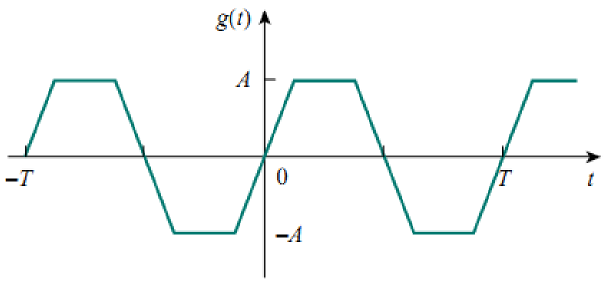

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The Fourier coefficients become

$$a_0 = 0$$

$$a_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

$$b_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

Table below summarizes the effects of these symmetries on the Fourier coefficients.

Symmetry	a_0	a_n	b_n	Remarks
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Half-wave	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0$ $b_{2n+1} \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.

1.4 CIRCUIT and SYSTEM APPLICATIONS

To find, the steady-state response of a circuit to a non-sinusoidal periodic excitation requires the application of a Fourier series, ac phasor analysis, and the superposition principle. The procedure usually involves three steps.

Steps for Applying Fourier Series:

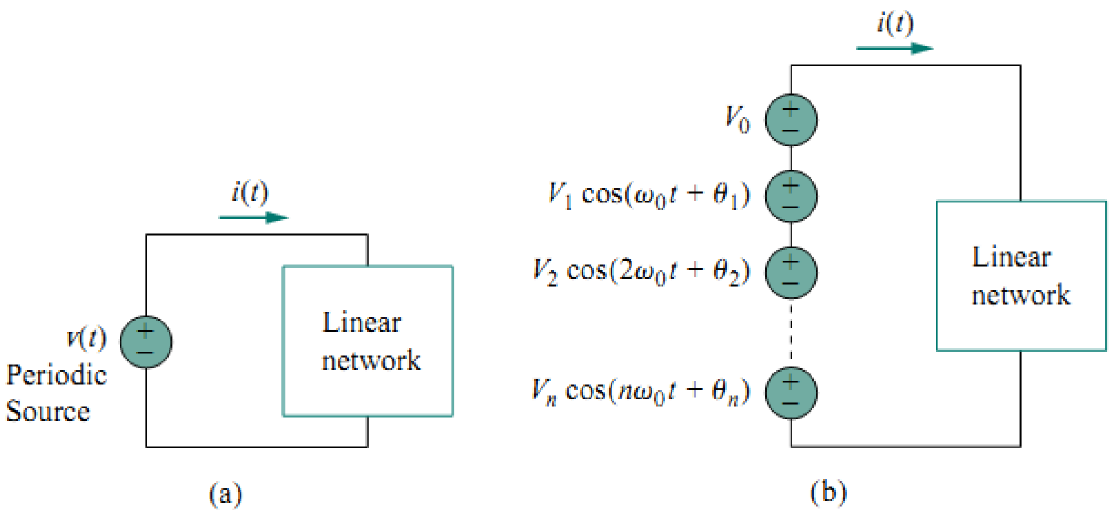
1. Express the excitation as a Fourier series.
2. Find the response of each term in the Fourier series.
3. Add the individual responses using the superposition principle.

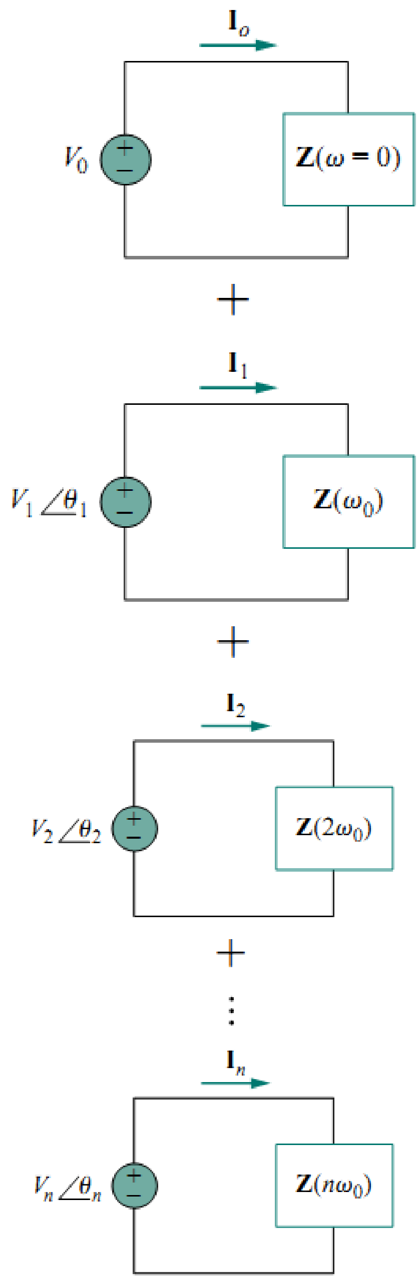
The Fourier series is expressed as

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n)$$

Finally, following the principle of superposition, we add all the individual responses.

$$\begin{aligned} i(t) &= i_0(t) + i_1(t) + i_2(t) + \cdots \\ &= \mathbf{I}_0 + \sum_{n=1}^{\infty} |\mathbf{I}_n| \cos(n\omega_0 t + \psi_n) \end{aligned}$$





1.5 EXPONENTIAL FOURIER SERIES

The other representation of Fourier series is the complex exponential form which , can be derived from the Euler identity:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

This is the complex or exponential Fourier series representation of f(t).

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt$$

where $\omega_0 = 2\pi/T$, as usual. The plots of the magnitude and phase of c_n versus $n\omega_0$ are called the complex amplitude spectrum and complex phase spectrum of f(t), respectively. The two spectra form the complex frequency spectrum of f(t).

The exponential Fourier series of a periodic function $f(t)$ describes the spectrum of $f(t)$ in terms of the amplitude and phase angle of ac components at positive and negative harmonic frequencies.

The coefficients of the three forms of Fourier series (sine-cosine form, amplitude-phase form, and exponential form) are related by

$$A_n \angle \phi_n = a_n - jb_n = 2c_n$$

$$c_n = |c_n| \angle \theta_n = \frac{\sqrt{a_n^2 + b_n^2}}{2} \angle -\tan^{-1} b_n/a_n$$

Example 1.6:

Find the exponential Fourier series expansion of the periodic function $f(t) = e^t, 0 < t < 2\pi$ with $f(t + 2\pi) = f(t)$.

1.6 System Applications:

The Fourier series has many other practical applications, particularly in communications and signal processing. Typical applications include spectrum analysis, filtering, rectification, and harmonic distortion. We will consider two of these: spectrum analyzers and filters.

1.6.1 Spectrum Analyzers

A spectrum analyzer is an instrument that displays the amplitude of the components of a signal versus frequency. In other words, it shows the various frequency components (spectral lines) that indicate the amount of energy at each frequency. It is unlike an oscilloscope, which displays the entire signal (all components) versus time. An oscilloscope shows the signal in the time domain, while the spectrum analyzer shows the signal in the frequency domain. There is perhaps no instrument more useful to a circuit analyst than the spectrum analyzer. An analyzer can conduct noise and spurious signal analysis, phase checks, electromagnetic interference and filter examinations, vibration measurements, radar measurements, and more. Spectrum analyzers are commercially available in various sizes and shapes.

The Fourier series provides the spectrum of a signal. As we have seen, the spectrum consists of the amplitudes and phases of the harmonics versus frequency. By providing the spectrum of a signal f(t), the Fourier series helps us identify the pertinent features of the signal. It demonstrates which frequencies are playing an important role in the shape of the output and which ones are not.

A periodic function is said to be band-limited if its amplitude spectrum contains only

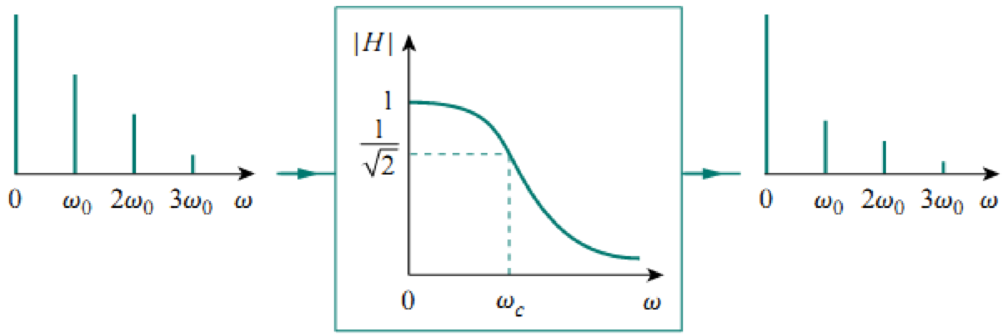
a finite number of coefficients An or cn. In this case, the Fourier series becomes

$$f(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} = a_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t + \phi_n)$$

1.6.2 Filters

Filters are an important component of electronics and communications systems. In communication, course a full discussion on passive and active filters. Here, we investigate how to design filters to select the fundamental component (or any desired harmonic) of the input signal and reject other harmonics. This filtering process cannot be accomplished without the Fourier series expansion of the input signal. For the purpose of illustration, we will consider two cases, a low pass filter and a band pass filter.

The output of a low pass filter depends on the input signal, the transfer function $H(\omega)$ of the filter, and the corner or half-power frequency ω_c. We recall that $\omega_c = 1/RC$ for an RC passive filter. As shown in Figure below, the low pass filter passes the dc and low-frequency components, while blocking the high-frequency components. By making ω_c sufficiently large ($\omega_c >> \omega_0$, e.g., making C small), a large number of the harmonics can be passed. On the other hand, by making ω_c sufficiently small ($\omega_c << \omega_0$), we can block out all the ac components and pass only dc, series expansion of the square wave.)



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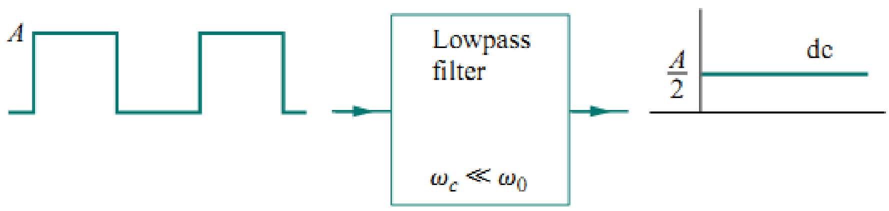

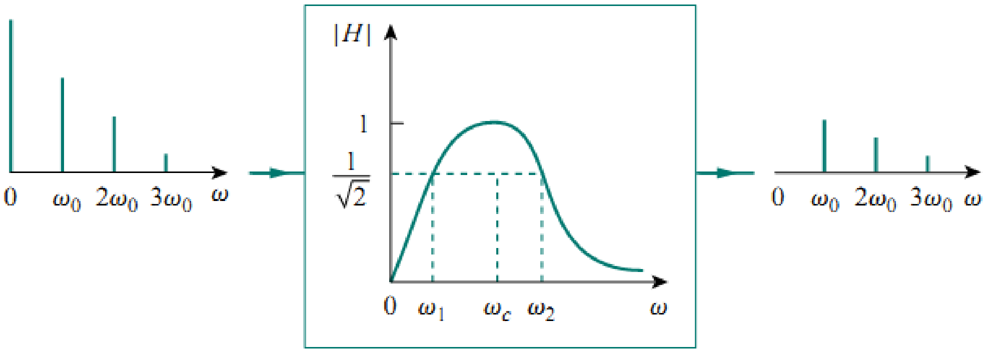

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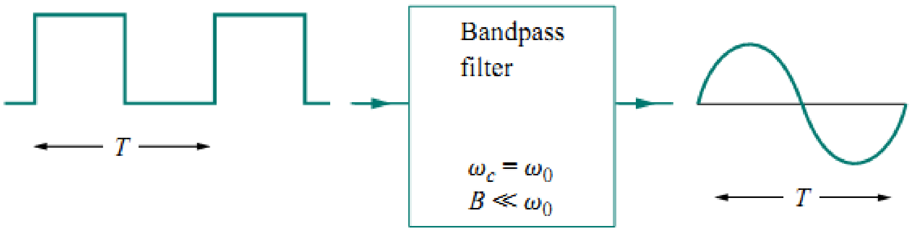

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If the input x(t)= $\sum_{n=-\infty}^{\infty} c_x e^{j2\pi n t f_0}$ and the transfer function H(f), then y(t)= $\sum_{n=-\infty}^{\infty} c_x e^{j2\pi n t f_0}$, where $c_y=H(f)c_x$

1.7 AVERAGE POWER AND RMS VALUES

$$P = V_{dc} I_{dc} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

Given a periodic function f(t), its rms value (or the effective value) is given by

$$F_{rms} = \sqrt{\frac{1}{T} \int_0^T f^2(t)\, dt}$$

$$F_{\text{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2$$

$$F_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

$$F_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)}$$

If f(t) is the current through a resistor R, then the power dissipated in the resistor is

$$P = RF_{\text{rms}}^2$$

One can avoid specifying the nature of the signal by choosing a 1Ohm resistance. The power dissipated by the 1 Ohm resistance is

$$P_{1\Omega} = F_{\text{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

This result is known as Parseval's theorem. Notice that a^2_0 is the power in the dc component, while $1/2(a^2_n + b^2_n)$ is the ac power in the n^{th} harmonic. Thus, Parseval's theorem states that the average power in aperiodic signal is the sum of the average power in its dc component and the average powers in its harmonics.

Example 1.7:

1.8 Fourier Transform

No human investigation can claim to be scientific if it doesn't pass the test of mathematical proof.

—Leonardo da Vinci

Fourier series enable us to represent a periodic function as a sum of sinusoids and to obtain the frequency spectrum from the series. The Fourier transform allows us to extend the concept of a frequency spectrum to non-periodic functions. The transform assumes that a non-periodic function is a periodic function with an infinite period. Thus, the Fourier transform is an integral representation of a non-periodic function that is analogous to a Fourier series representation of a periodic function.

The Fourier transform is an integral transform like the Laplace transform. It transforms a function in the time domain into the frequency domain. The Fourier transform is very useful in communications systems and digital signal processing, in situations where the Laplace transform does not apply. While the Laplace transform can only handle circuits with inputs for t> 0 with initial conditions, the Fourier transform can handle circuits with inputs for t< 0 as well as those for t> 0.

The Fourier transform is given by:

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

where $\mathcal{F}$ is the Fourier transform operator.

The Fourier transform is an integral transformation of $f(t)$ from the time domain to the frequency domain.

In general, F(ω)is a complex function; its magnitude is called the amplitude spectrum, while its phase is called the phase spectrum. Thus F(ω)is the spectrum. The inverse Fourier transform as

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

The function f(t) and its transform F(ω) form the Fourier transform pairs :

$$f(t) \quad \Longleftrightarrow \quad F(\omega)$$

The Fourier transform F(ω)exists when the Fourier integral converges. A sufficient but not necessary condition that f(t) has a Fourier transform is that it be completely integrable in the sense that

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

For example, the Fourier transform of the unit ramp function tu(t) does not exist, because the function does not satisfy the condition above.

Table below illustrate some important and useful functions with their Fourier transform.

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$2\frac{\sin\omega\tau}{\omega}$
$\lvert t\rvert$	$\frac{-2}{\omega^2}$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$e^{at}u(-t)$	$\frac{1}{a-j\omega}$
$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$

1.8.1 PROPERTIE S O F TH E FOURIE R TRANSFORM

We now develop some properties of the Fourier transform that are useful in finding the transforms of complicated functions from the transforms of simple functions. For each property, we will first state and derive it, and then illustrate it with some examples.

1- Linearity

If $F_1(\omega)$ and $F_2(\omega)$ are the Fourier transforms of $f_1(t)$ and $f_2(t)$, respectively, then

$$\boxed{\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(\omega) + a_2 F_2(\omega)} \tag{17.12}$$

where a_1 and a_2 are constants. This property simply states that the Fourier transform of a linear combination of functions is the same as the linear combination of the transforms of the individual functions. The proof of the linearity property in Eq. (17.12) is straightforward. By definition,

$$\begin{aligned}\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] &= \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)] e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} a_1 f_1(t) e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} a_2 f_2(t) e^{-j\omega t}\, dt \\ &= a_1 F_1(\omega) + a_2 F_2(\omega)\end{aligned} \tag{17.13}$$

For example, $\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$. Using the linearity property,

$$\begin{aligned}F[\sin \omega_0 t] &= \frac{1}{2j}[\mathcal{F}(e^{j\omega_0 t}) - \mathcal{F}(e^{-j\omega_0 t})] \\ &= \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]\end{aligned} \tag{17.14}$$

2- Time Scaling

If $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \tag{17.15}$$

where a is a constant. Equation (17.15) shows that time expansion ($|a| > 1$) corresponds to frequency compression, or conversely, time compression ($|a| < 1$) implies frequency expansion. The proof of the time-scaling property proceeds as follows.

For example, for the rectangular pulse $p(t)$ in Example 17.2,

$$\mathcal{F}[p(t)] = A\tau \operatorname{sinc} \frac{\omega\tau}{2} \tag{17.18a}$$

Using Eq. (17.15),

$$\mathcal{F}[p(2t)] = \frac{A\tau}{2} \operatorname{sinc} \frac{\omega\tau}{4} \tag{17.18b}$$

3- Time Shifting

If $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}[f(t - t_0)] = e^{-j\omega t_0} F(\omega) \tag{17.20}$$

that is, a delay in the time domain corresponds to a phase shift in the frequency domain. To derive the time shifting property, we note that

For example, from Example 17.3,

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a + j\omega}$$

The transform of $f(t) = e^{-(t-2)}u(t-2)$ is

$$F(\omega) = \mathcal{F}[e^{-(t-2)}u(t-2)] = \frac{e^{-j2\omega}}{1 + j\omega}$$

4- Frequency Shifting (or Amplitude Modulation)

This property states that if $F(\omega) = \mathcal{F}[f(t)]$, then

$$\boxed{\mathcal{F}[f(t)e^{j\omega_0 t}] = F(\omega - \omega_0)} \tag{17.25}$$

meaning, a frequency shift in the frequency domain adds a phase shift to the time function. By definition,

For example, $\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$. Using the property in Eq. (17.25),

$$\begin{aligned} \mathcal{F}[f(t)\cos\omega_0 t] &= \frac{1}{2}\mathcal{F}\left[f(t)e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[f(t)e^{-j\omega_0 t}\right] \\ &= \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0) \end{aligned} \tag{17.27}$$

5- Time Differentiation

Given that $F(\omega) = \mathcal{F}[f(t)]$, then

$$\boxed{\mathcal{F}[f'(t)] = j\omega F(\omega)} \tag{17.28}$$

In other words, the transform of the derivative of $f(t)$ is obtained by multiplying the transform of $f(t)$ by $j\omega$. By definition,

In general

$$\boxed{\mathcal{F}[f^{(n)}(t)] = (j\omega)^n F(\omega)}$$

For example, if $f(t) = e^{-at}$, then

$$f'(t) = -ae^{-at} = -af(t) \tag{17.32}$$

Taking the Fourier transforms of the first and last terms, we obtain

$$j\omega F(\omega) = -aF(\omega) \qquad \Longrightarrow \qquad F(\omega) = \frac{1}{a + j\omega} \tag{17.33}$$

which agrees with the result in Example 17.3.

6- Time Integration

Given that $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}\left[\int_{-\infty}^{t} f(t)\,dt\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) \tag{17.34}$$

that is, the transform of the integral of $f(t)$ is obtained by dividing the transform of $f(t)$ by $j\omega$ and adding the result to the impulse term that reflects the dc component $F(0)$. Someone might ask, "How do we know that when we take the Fourier transform for time integration, we should integrate over the interval $[-\infty, t]$ and not $[-\infty, \infty]$?" When we integrate over $[-\infty, \infty]$, the result does not depend on time anymore, and the Fourier transform of a constant is what we will eventually get. But when we integrate over $[-\infty, t]$, we get the integral of the function from the past to time t, so that the result depends on t and we can take the Fourier transform of that.

For example, we know that $\mathcal{F}[\delta(t)] = 1$ and that integrating the impulse function gives the unit step function [see Eq. (7.39a)]. By applying the property in Eq. (17.34), we obtain the Fourier transform of the unit step function as

$$\mathcal{F}[u(t)] = \mathcal{F}\left[\int_{-\infty}^{t} \delta(t)\,dt\right] = \frac{1}{j\omega} + \pi\delta(\omega) \tag{17.36}$$

7- Reversal

If $F(\omega) = \mathcal{F}[f(t)]$, then

$$\mathcal{F}[f(-t)] = F(-\omega) = F^*(\omega) \tag{17.37}$$

where the asterisk denotes the complex conjugate. This property states that reversing $f(t)$ about the time axis reverses $F(\omega)$ about the frequency axis. This may be regarded as a special case of time scaling for which $a = -1$ in Eq. (17.15).

8- Duality

This property states that if $F(\omega)$ is the Fourier transform of $f(t)$, then the Fourier transform of $F(t)$ is $2\pi f(-\omega)$; we write

$$\boxed{\mathcal{F}[f(t)] = F(\omega) \qquad \Longrightarrow \qquad \mathcal{F}[F(t)] = 2\pi f(-\omega)} \tag{17.38}$$

This expresses the symmetry property of the Fourier transform. To derive

For example, if $f(t) = e^{-|t|}$, then

$$F(\omega) = \frac{2}{\omega^2 + 1} \tag{17.41}$$

By the duality property, the Fourier transform of $F(t) = 2/(t^2 + 1)$ is

$$2\pi f(\omega) = 2\pi e^{-|\omega|} \tag{17.42}$$

9- Convolution

If $X(\omega)$, $H(\omega)$, and $Y(\omega)$ are the Fourier transforms of $x(t)$, $h(t)$, and $y(t)$, respectively, then

$$\boxed{Y(\omega) = \mathcal{F}[h(t) * x(t)] = H(\omega)X(\omega)} \tag{17.44}$$

which indicates that convolution in the time domain corresponds with multiplication in the frequency domain.

Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-j\omega a} F(\omega)$
Frequency shift	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Modulation	$\cos(\omega_0 t)\ f(t)$	$\frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$

Property	$f(t)$	$F(\omega)$
Time differentiation	$\frac{df}{dt}$	$j\omega F(\omega)$
	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t)\, dt$	$\frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$
Reversal	$f(-t)$	$F(-\omega)$ or $F^*(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in t	$f_1(t) * f_1(t)$	$F_1(\omega) F_2(\omega)$
Convolution in ω	$f_1(t) f_1(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

1.9 PARSEVAL' S THEOREM

Parseval's theorem demonstrates one practical use of the Fourier trans-form. It relates the energy carried by a signal to the Fourier transform of the signal. If p(t) is the power associated with the signal, the energy carried by the signal is

$$W = \int_{-\infty}^{\infty} p(t)\, dt$$

Parseval's theorem states that this same energy can be calculated in the frequency domain as

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

Parseval's theorem states that the total energy delivered to a 1-Ω resistor equals the total area under the square of $f(t)$ or $1/2\pi$ times the total area under the square of the magnitude of the Fourier transform of $f(t)$.

Parseval's theorem relates energy associated with a signal to its Fourier transform. It provides the physical significance of $F(\omega)$, namely, that $|F(\omega)|^2$ is a measure of the energy density (in joules per hertz) corresponding to $f(t)$.

We may also calculate the energy in any frequency band $\omega_1 < \omega < \omega_2$ As

$$W_{1\Omega} = \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 \, d\omega$$

Example 1.:

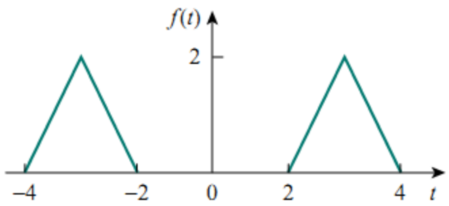


Figure 17.16 For Practice Prob. 17.5.

Determine the Fourier transform of the function in Fig. 17.16.

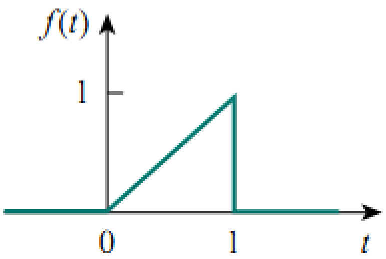
Answer: $(8\cos 3\omega - 4\cos 4\omega - 4\cos 2\omega)/\omega^2$.

Determine the Fourier transforms of these functions: (a) gate function $g(t) = u(t) - u(t-1)$, (b) $f(t) = te^{-2t}u(t)$, and (c) sawtooth pulse $f(t) = 10t[u(t) - u(t-2)]$.

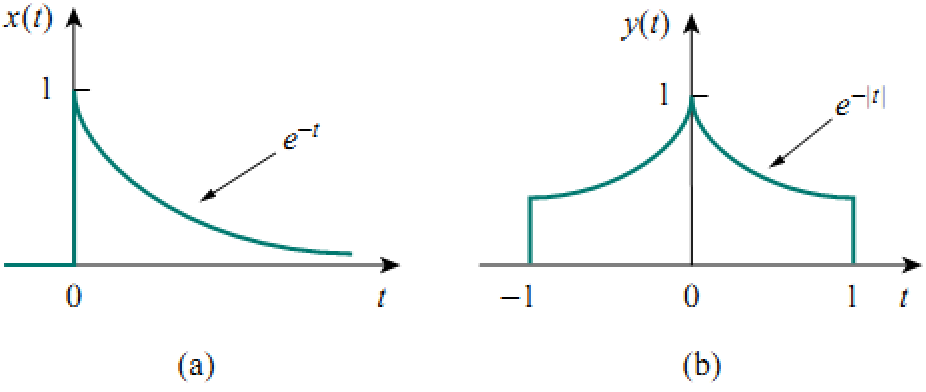
Answer: (a) $(1 - e^{-j\omega})\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$, (b) $\frac{1}{(2+j\omega)^2}$,

(c) $\frac{10(e^{-j2\omega} - 1)}{\omega^2} + \frac{20j}{\omega}e^{-j2\omega}$.

17.2 What is the Fourier transform of the triangular pulse in Fig. 17.27?



17.6 Obtain the Fourier transforms of the signals shown in Fig. 17.31.



17.8 Determine the Fourier transforms of these functions:

(a) $f(t) = e^t[u(t) - u(t-1)]$

(b) $g(t) = te^{-t}u(t)$

(c) $h(t) = u(t+1) - 2u(t) + u(t-1)$

17.9 Find the Fourier transforms of these functions:

(a) $f(t) = e^{-t}\cos(3t + \pi)u(t)$

(b) $g(t) = \sin \pi t[u(t+1) - u(t-1)]$

(c) $h(t) = e^{-2t}\cos \pi t u(t-1)$

(d) $p(t) = e^{-2t}\sin 4t u(-t)$

(e) $q(t) = 4\,\mathrm{sgn}(t-2) + 3\delta(t) - 2u(t-2)$

17.14 Find the Fourier transform of
$f(t) = \cos 2\pi t[u(t) - u(t-1)]$.

17.15 (a) Show that a periodic signal with exponential Fourier series

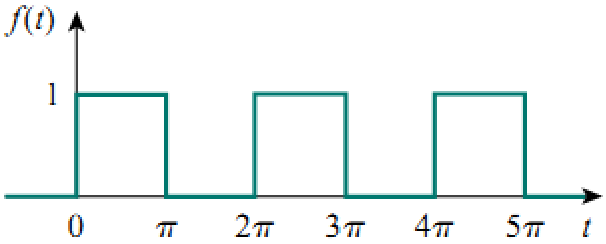
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

has the Fourier transform

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$$

where $\omega_0 = 2\pi/T$.

(b) Find the Fourier transform of the signal in Fig. 17.33.



17.16 Prove that if $F(\omega)$ is the Fourier transform of $f(t)$,

$$\mathcal{F}[f(t)\sin\omega_0 t] = \frac{j}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]$$

17.17 If the Fourier transform of $f(t)$ is

$$F(\omega) = \frac{10}{(2+j\omega)(5+j\omega)}$$

determine the transforms of the following:

(a) $f(-3t)$ (b) $f(2t-1)$ (c) $f(t)\cos 2t$

(d) $\frac{d}{dt}f(t)$ (e) $\int_{-\infty}^{t} f(t)\,dt$

17.18 Given that $\mathcal{F}[f(t)] = (j/\omega)(e^{-j\omega} - 1)$, find the Fourier transforms of:

(a) $x(t) = f(t) + 3$ (b) $y(t) = f(t-2)$

(c) $h(t) = f'(t)$

(d) $g(t) = 4f(\frac{2}{3}t) + 10f(\frac{5}{3}t)$

17.26 A linear system has a transfer function

$$H(\omega) = \frac{10}{2 + j\omega}$$

Determine the output $v_o(t)$ at $t = 2$ s if the input $v_i(t)$ equals:

(a) $4\delta(t)$ V (b) $6e^{-t}u(t)$ V (c) $3\cos 2t$ V

17.54 A signal with Fourier transform

$$F(\omega) = \frac{20}{4 + j\omega}$$

is passed through a filter whose cutoff frequency is 2 rad/s (i.e., $0 < \omega < 2$). What fraction of the energy in the input signal is contained in the output signal?

17.53 The voltage signal at the input of a filter is $v(t) = 50e^{-2|t|}$ V. What percentage of the total 1-Ω energy content lies in the frequency range of $1 < \omega < 5$ rad/s?

Inverse Fourier Transform

Obtain the inverse Fourier transform of:

(a) $F(\omega) = \dfrac{10j\omega + 4}{(j\omega)^2 + 6j\omega + 8}$ (b) $G(\omega) = \dfrac{\omega^2 + 21}{\omega^2 + 9}$

Solution:

(a) To avoid complex algebra, we can replace $j\omega$ with s for the moment. Using partial fraction expansion,

$$F(s) = \frac{10s + 4}{s^2 + 6s + 8} = \frac{10s + 4}{(s + 4)(s + 2)} = \frac{A}{s + 4} + \frac{B}{s + 2}$$

where

$$A = (s + 4)F(s)\Big|_{s=-4} = \frac{10s + 4}{(s + 2)}\Big|_{s=-4} = \frac{-36}{-2} = 18$$

$$B = (s + 2)F(s)\Big|_{s=-2} = \frac{10s + 4}{(s + 4)}\Big|_{s=-2} = \frac{-16}{2} = -8$$

Substituting $A = 18$ and $B = -8$ in $F(s)$ and s with $j\omega$ gives

$$F(j\omega) = \frac{18}{j\omega + 4} + \frac{-8}{j\omega + 2}$$

$f(t) = (18e^{-4t} - 8e^{-2t})u(t)$

(b) We simplify $G(\omega)$ as

$$G(\omega) = \frac{\omega^2 + 21}{\omega^2 + 9} = 1 + \frac{12}{\omega^2 + 9}$$

$g(t) = \delta(t) + 2e^{-3|t|}$

Find the inverse Fourier transform of:

(a) $H(\omega) = \dfrac{6(3 + j2\omega)}{(1 + j\omega)(4 + j\omega)(2 + j\omega)}$

(b) $Y(\omega) = \pi\delta(\omega) + \dfrac{1}{j\omega} + \dfrac{2(1 + j\omega)}{(1 + j\omega)^2 + 16}$

Answer: (a) $h(t) = (2e^{-t} + 3e^{-2t} - 5e^{-4t})u(t)$,
(b) $y(t) = (1 + 2e^{-t}\cos 4t)u(t)$.

***17.22** Determine the inverse Fourier transforms of:

(a) $F(\omega) = 4\delta(\omega + 3) + \delta(\omega) + 4\delta(\omega - 3)$

(b) $G(\omega) = 4u(\omega + 2) - 4u(\omega - 2)$

(c) $H(\omega) = 6\cos 2\omega$

***17.23** Determine the functions corresponding to the following Fourier transforms:

(a) $F_1(\omega) = \dfrac{e^{j\omega}}{-j\omega + 1}$ (b) $F_2(\omega) = 2e^{|\omega|}$

(c) $F_3(\omega) = \dfrac{1}{(1+\omega^2)^2}$ (d) $F_4(\omega) = \dfrac{\delta(\omega)}{1 + j2\omega}$

***17.24** Find $f(t)$ if:

(a) $F(\omega) = 2\sin\pi\omega[u(\omega + 1) - u(\omega - 1)]$

(b) $F(\omega) = \dfrac{1}{\omega}(\sin 2\omega - \sin\omega) + \dfrac{j}{\omega}(\cos 2\omega - \cos\omega)$

17.25 Determine the signal $f(t)$ whose Fourier transform is shown in Fig. 17.34.
(*Hint:* Use the duality property.)