University of Basrah
College of Engineering
Electrical Engineering Department

Electric Circuits Analysis

Asst. Lect: Hamzah Abdulkareem ***Asst. Lect: Mohamed Jasim***

Chapter Three

Sinusoidal Steady State Analysis

- **Sinusoidal analysis and phasor**
- **Phasor Relationships for Circuit Elements**
- **Nodal Analysis for AC circuits**
- **Mesh Analysis for AC circuits**
- **Superposition AC Analysis**
- **Thevenin and Norton AC analysis**
- **AC Power Analysis**

Chapter Three

Sinusoidal Steady State Analysis

3.1 Sinusoidal analysis and phasor

Consider a sinusoidally varying voltage

$$\boldsymbol{v(t) = V_m\, sin\omega t} \qquad (3.1)$$

Where

V_m = the amplitude of the sinusoid

ω = the angular frequency in radians/s

$\omega t =$ the argument of the sinusoid

The sinusoid is shown in Fig. 3.1(a) as a function of its argument and in Fig. 3.1(b) as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the period of the sinusoid. From the two plots in Fig. 3.1, we observe that $\omega\, T = 2\pi$

$$\boxed{T = \frac{2\pi}{\omega}} \qquad \textbf{(3.2)}$$

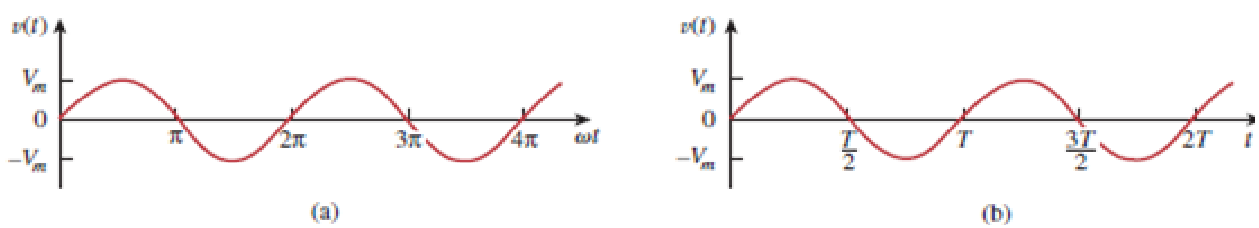


Figure 3.1

A sketch of Vm sin ωt: (a) as a function of ωt, (b) as a function of t.

A more general form of the sinusoid,

$$\boldsymbol{v(t) = Vm\, sin(\omega t + \theta)} \qquad (3.3)$$

includes a phase angle θ in its argument. Equation [1] is plotted in Fig. 3.2 as a function of ωt. Since corresponding points on the sinusoid $V_m \sin(\omega t + \theta)$ occur θ rad, or θ/ω seconds, earlier, we say that

- $V_m \sin(\omega t + \theta)$ ***leads*** $V_m \sin\omega t$ by θ rad.
- $V_m \sin \omega t$ ***lags*** $V_m \sin(\omega t + \theta)$ by θ rad.

In either case, ***leading or lagging***, we say that the sinusoids are ***out of phase***. If the phase angles are ***equal***, the sinusoids are said to be ***in phase***.

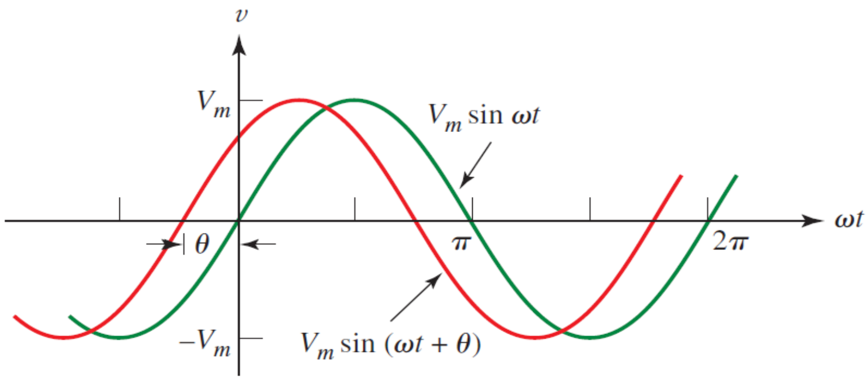


Fig. 3.2: The sine wave $V_m \sin(\omega t + \theta)$ leads $V_m \sin \omega t$ by θ rad.

The complex quantities are usually written in polar form rather than exponential form in order to achieve a slight additional saving of time and effort. For example, a source voltage

$$v(t) = Vm \cos\omega t = Vm \cos(\omega t + 0^\circ) \tag{3.4}$$

we now represent in complex form as

$$v(t) = Vm \angle 0^\circ \tag{3.5}$$

and its current response

$$\mathbf{i(t) = I_m \cos(\omega t + \phi)} \tag{3.6}$$

becomes

$$\mathbf{i(t) = I_m \angle\phi} \tag{3.7}$$

This abbreviated complex representation is called a phasor.

A real sinusoidal current

$$i(t) = I_m \cos(\omega t + \phi) = Re\{I_m e^{j(\omega t+\phi)}\} \tag{3.8}$$

We then represent the current as a complex quantity by dropping the instruction Re{}, thus adding an imaginary component to the current without affecting the real component; further simplification is achieved by suppressing the factor $e^{j\omega t}$:

$$I = I_m e^{j\phi} \tag{3.9}$$

and writing the result in polar form:

$$I = I_m \angle\phi \tag{3.10}$$

The process of returning to the time domain from the frequency domain is exactly the reverse of the previous sequence. Thus, given the phasor voltage

$$V = 115\angle{-45^\circ} \quad \textbf{volts}$$

and the knowledge that ω = 500 rad/s, we can write the time-domain equivalent directly:

$$v(t) = 115\cos(500t - 45^\circ) \quad \text{volts}$$

If desired as a sine wave, v(t) could also be written

$$v(t) = 115\sin(500t + 45^\circ) \quad \text{volts}$$

We can proceed to our simplification of sinusoidal steady-state analysis by establishing the relationship between the phasor voltage and phasor current for each of the three passive elements.

Example 3.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\,cos(50\,t + 10^0)$$

Example 3.2

Calculate the phase angle between $v_1 = -10\,Sin\,(\omega t + 50^0)$ and $v_1 = 12\,Sin\,(\omega t - 10^0)$

State which sinusoid is leading

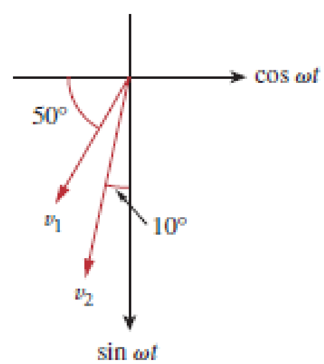


Figure 3.3
For Example 3.2.

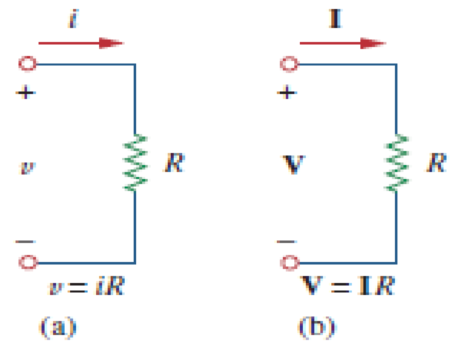
3.2 Phasor Relationships for Circuit Elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

3.2.1 The Resistor

We begin with the resistor. If the current through a resistor R is $\boldsymbol{i} = \boldsymbol{I_m Cos(\omega t} + \emptyset)$ the voltage across it is given by Ohm's law as

$$\boldsymbol{v(t)} = \boldsymbol{R\, i(t)} = \boldsymbol{R\, I_m Cos(\omega t} + \emptyset) \qquad (3.11)$$



The phasor form of this voltage is $\boldsymbol{V} = \boldsymbol{R\, I_m} \angle\emptyset$

But the phasor representation of the current is $\boldsymbol{I} = \boldsymbol{I_m} \angle\emptyset$

Here $\boldsymbol{V} = \boldsymbol{R\, I}$ $\qquad$ (3.12)

Figure 3.4

Voltage-current relations for a resistor in the:

(a) time domain, (b) frequency domain.

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 3.4 illustrates the voltage-current relations of a resistor. We should note from Eq. (3.12) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 3.5.

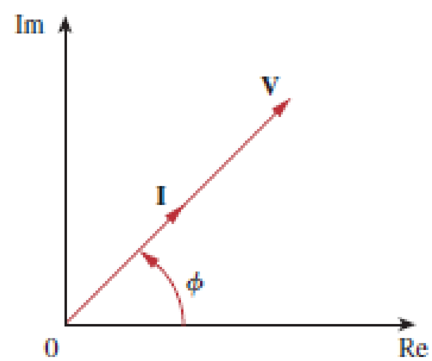


Figure 3.5

Phasor diagram for the resistor.

3.2.2 The Inductor

For the inductor L, assume the current through it is $\boldsymbol{i} = \boldsymbol{I_m Cos(\omega t} + \emptyset)$ The voltage across the inductor is

$$\boldsymbol{v} = \boldsymbol{L}\frac{\boldsymbol{di}}{\boldsymbol{dt}} = -\boldsymbol{\omega L I_m Sin(\omega t} + \emptyset) \qquad (3.13)$$

Where $-\sin A = \cos(A + 90)$

$$\boldsymbol{v = \omega L I_m Cos(\omega t + \emptyset + 90)} \quad (3.14)$$

Where

$$V = \boldsymbol{\omega L I_m \angle\emptyset + 90 \quad and\ I = I_m \angle\emptyset \quad and\ \angle 90 = j}$$

$$V = \boldsymbol{j\omega L I} \quad (3.15)$$

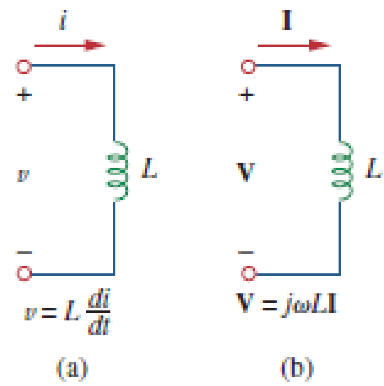


Figure 3.6

Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

showing that the voltage has a magnitude of and a phase of The voltage and current are out of phase. Specifically, the current lags the voltage by Figure 3.6 shows the voltage-current relations for the inductor. Figure 3.7 shows the phasor diagram.

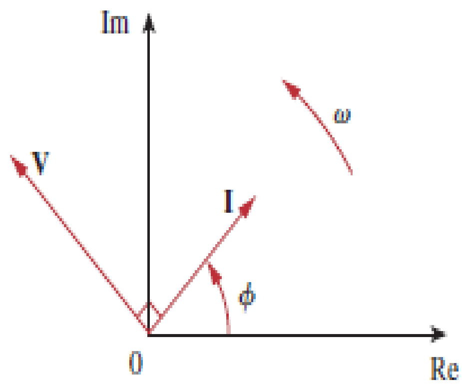


Figure 3.7

Phasor diagram for the inductor; I lags V.

3.2.3 The Capacitor

For the capacitor C, assume the voltage across it is $\boldsymbol{v = V_m Cos(\omega t + \emptyset)}$ The current through the capacitor is

$$\boldsymbol{i = C\frac{dv}{dt}} \quad (3.16)$$

By following the same steps as we took for the inductor or by applying Eq. (2.13) on Eq. (3.15), we obtain

$$\boldsymbol{I = j\omega C V} \quad \Longrightarrow \boldsymbol{V = \frac{I}{j\omega C}} \quad (3.17)$$

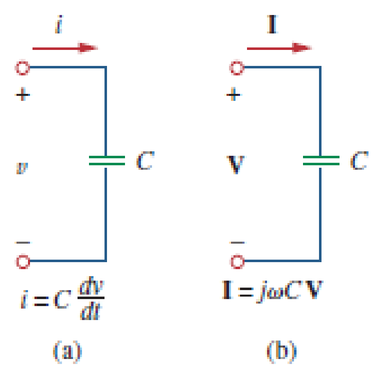


Figure 3.8

Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

showing that the current and voltage are out of phase. To be specific, the current leads the voltage by Figure 3.8 shows the voltage current relations for the capacitor; Fig. 3.9 gives the phasor diagram.

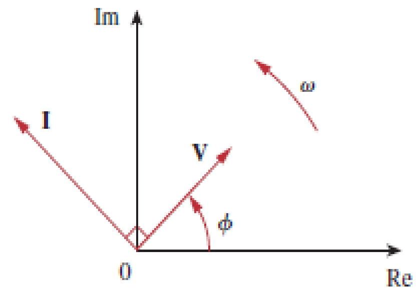
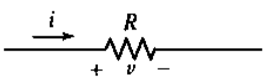
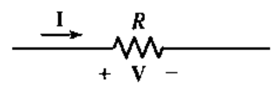
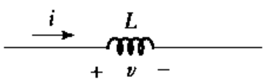
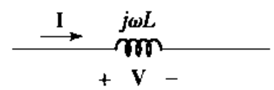
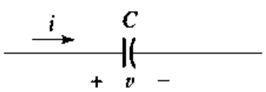
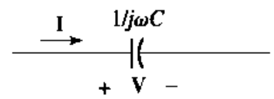


Figure 3.9

Phasor diagram for the capacitor; I leads V.

Table 3.1 summarizes the time domain and phasor domain representations of the circuit elements.

Time Domain		Frequency Domain	
	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	
	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$	
	$v = \dfrac{1}{C}\int i\,dt$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$	

3.3 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{3.18}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{3.19}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I}} \tag{3.20}$$

where Z is a frequency-dependent quantity known as **impedance**, measured in ohms.

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω).

we may choose to express impedance in either **rectangular** ($\boldsymbol{Z = R + jX}$) or **polar** ($\boldsymbol{Z = |Z|\angle\theta}$) form.

In rectangular form, we can see clearly

- The real part, which arises only from real resistances (R)
- The imaginary component, termed the reactance, which arises from the energy storage elements (XL and XC).

Both resistance and reactance have units of ohms, but reactance will always depend upon frequency. An ideal resistor has zero reactance; an ideal inductor or capacitor is purely reactive (i.e., characterized by zero resistance).

The **admittance** Y is the reciprocal of impedance, measured in siemens (S).

The admittance **Y** of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

- The real part of the admittance is the conductance G.
- The imaginary part is the susceptance B.

All three quantities (Y, G, and B) are measured in siemens.

$$\boldsymbol{Y = G + jB = \frac{1}{Z} = \frac{1}{R + jX}} \qquad (3.21)$$

3.4 Nodal Analysis for A.C circuits

The basic of Nodal analysis is KCL as shown in chapter one

Example 3.3

Find $\mathbf{i_x}$ for circuit shown in figure 3.10.1

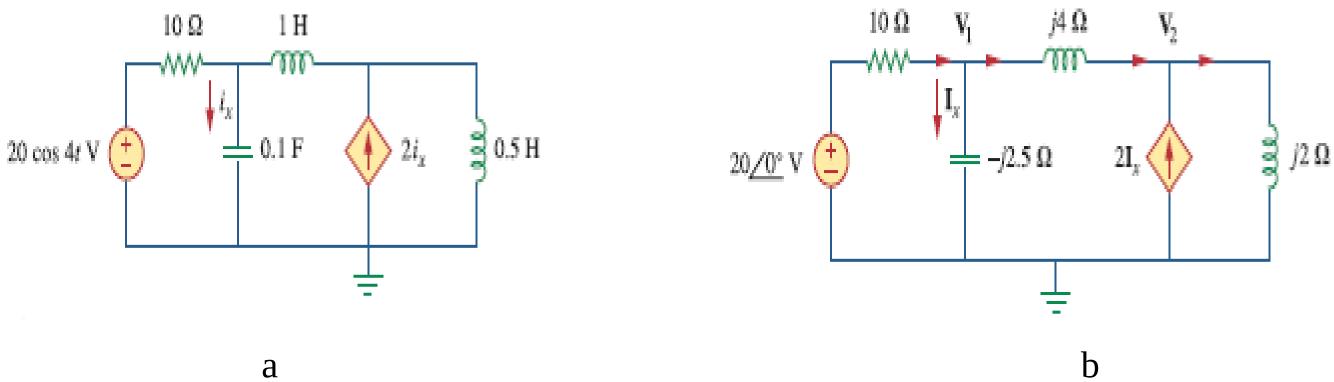


Figure 3.10.

Example 3.4

Determine $\mathbf{V_1}$ and $\mathbf{V_2}$ in the circuit shown in figure 3.11

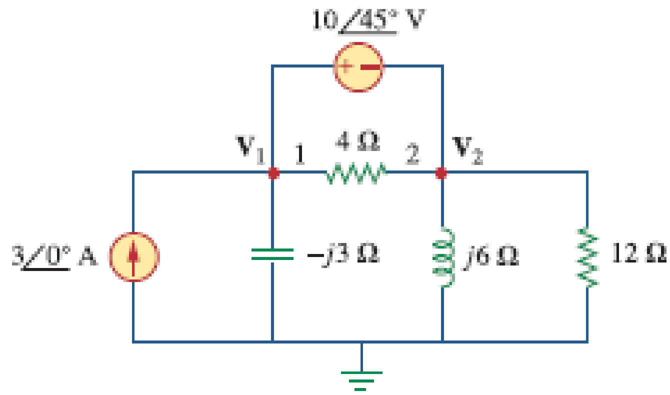
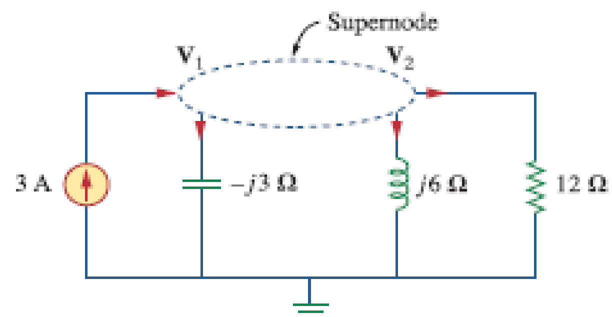


Figure 3.11



H.W

1. by using Nodal analysis find v_1 and v_2 in the circuit shown in figure 3.12

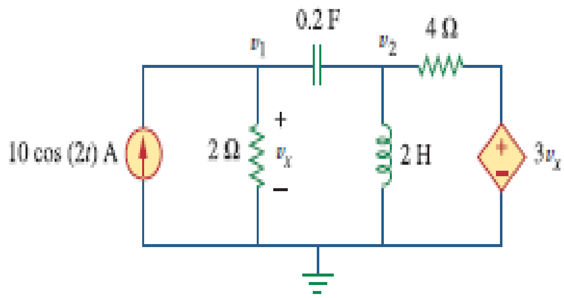


Figure 3.12

2. Determine $\mathbf{V_1}$ and $\mathbf{V_2}$ in the circuit shown in figure 3.13

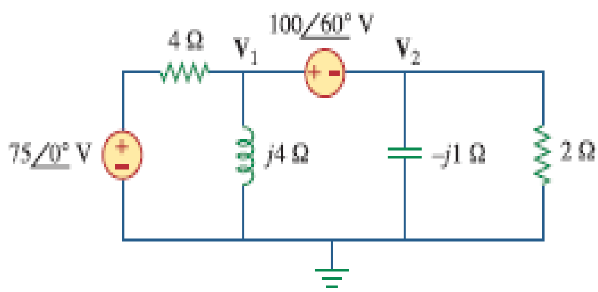


Figure 3.13

3. Find the time-domain node voltages $V_1(t)$ and $V_2(t)$ in the circuit shown in Fig. 3.14.

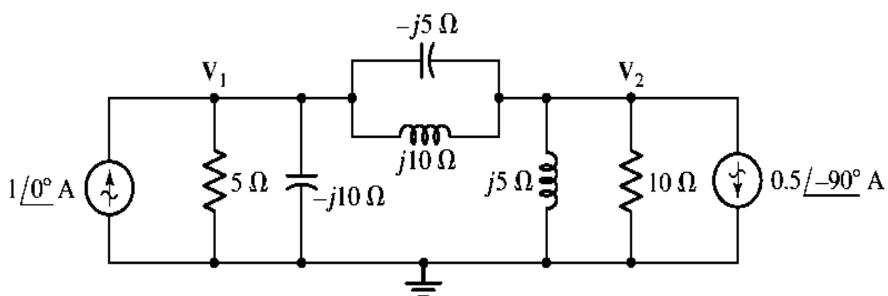


Fig. 3.14.

3.5 Mesh Analysis for A.C circuits

In mesh analysis, KVL is used for each loop as we studied in chapter one

Example 3.5

Determine current $\mathbf{I_0}$ in the circuit shown in figure 3.15

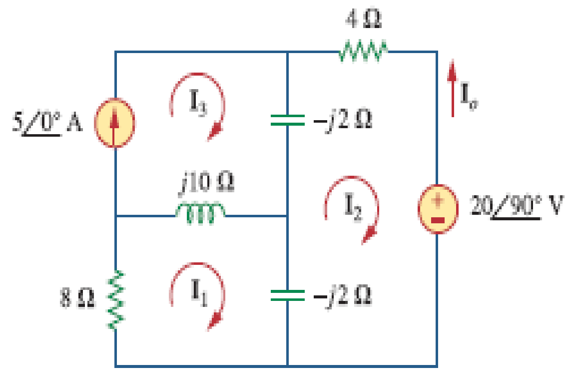


Figure 3.15

Example 3.6

Find $\mathbf{V_o}$ in the circuit shown in figure 3.16

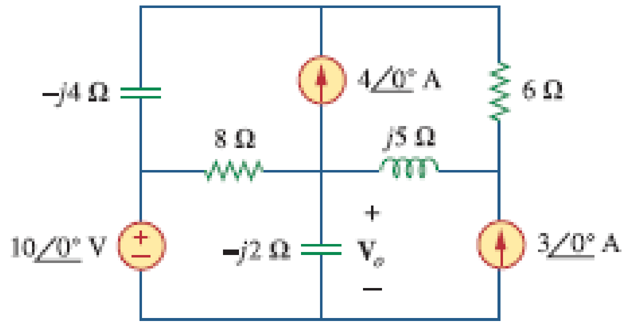
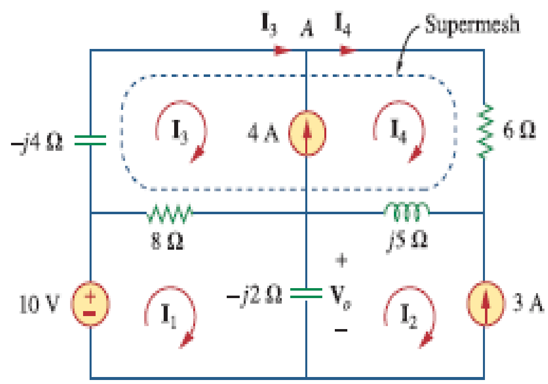


Figure 3.16



H.W. 1. Determine current $\mathbf{I_0}$ in the circuit shown in Figure 3.17

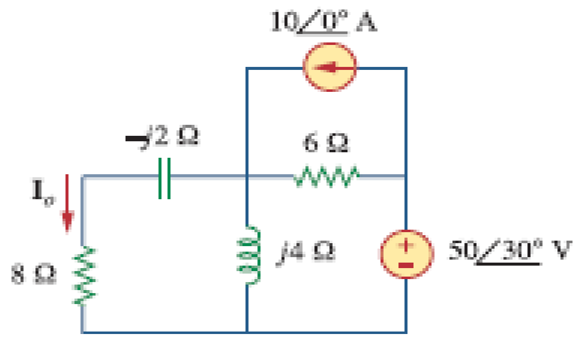


Figure 3.17

2. Determine current $\mathbf{I_0}$ in the circuit shown in Figure 3.18

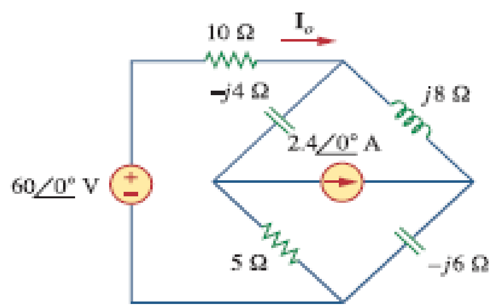


Figure 3.18

3. Obtain expressions for the time-domain currents i1 and i2 in the circuit given as Fig. 3.19.

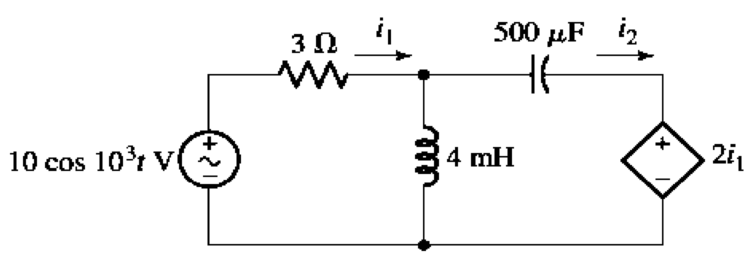


Figure 3.19

4. Use mesh analysis on the circuit of Fig. 3.20 to find I_1 and I_2.

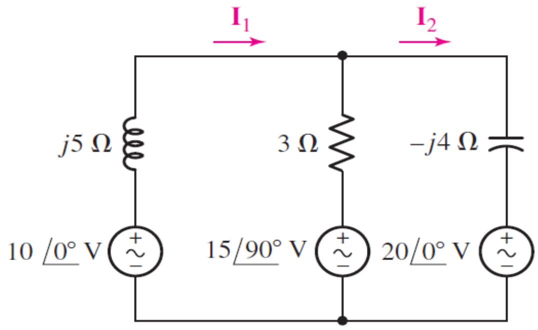


Fig. 3.20

3.6 Superposition AC Analysis

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques covered in Chapter 1.

2. Repeat step 1 for each of the other independent sources.

3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Example 3.7

Use superposition theorem to find $\mathbf{I_0}$ for figure 3.21

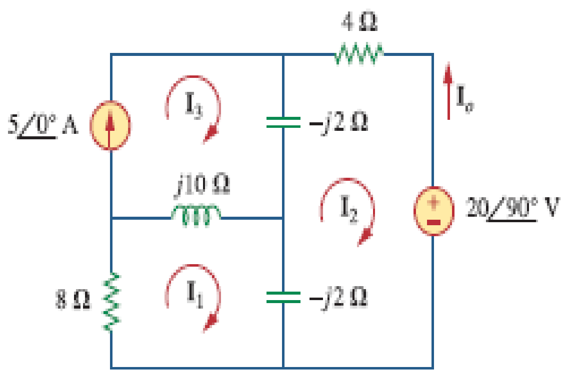
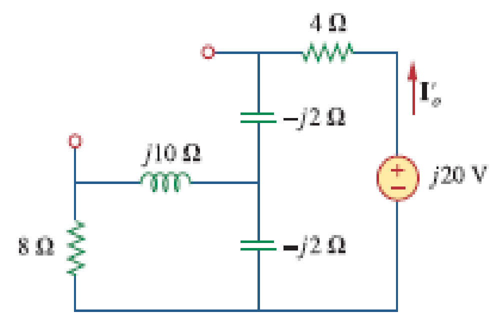
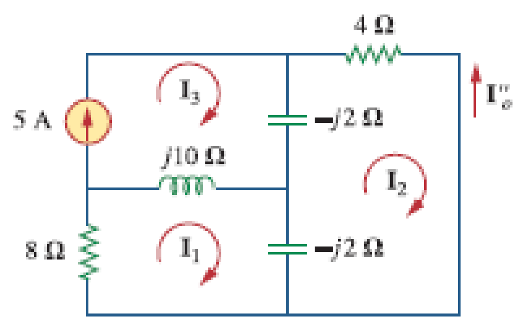


Figure 3.21



(a)



(b)

Example 3.8

Use superposition theorem to find $\mathbf{V_0}$ for figure 3.22

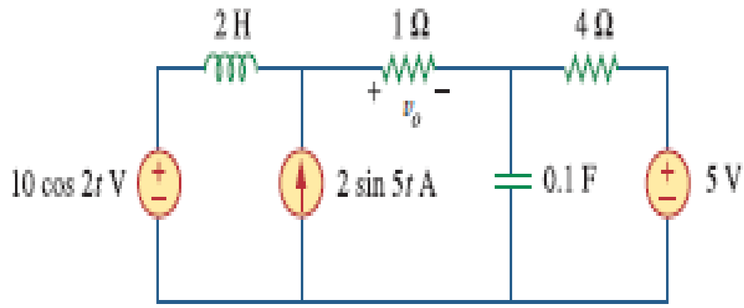
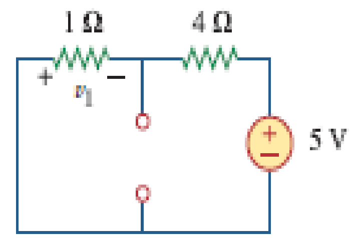
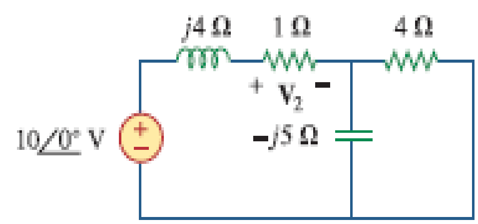


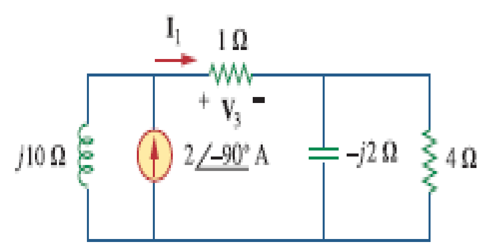
figure 3.22



(a)



(b)



(c)

3.7 Thevenin and Norton AC analysis

Example 3.9

Find Thevenin equivalent at terminal a-b for figure 3.23

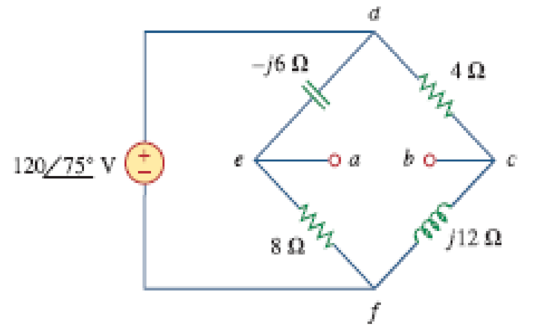
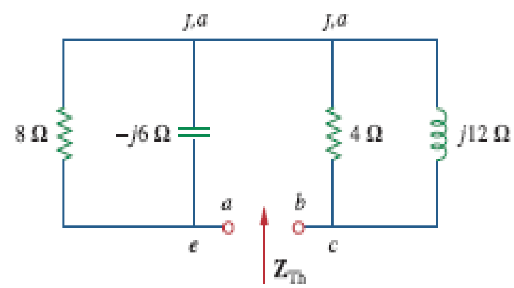
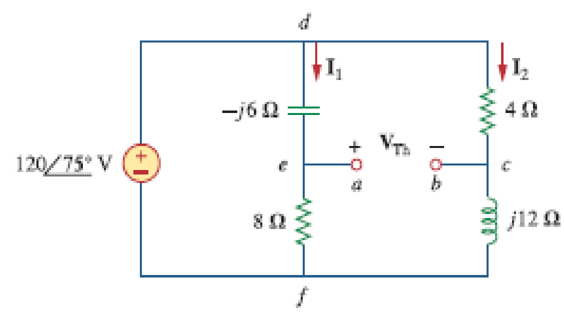


Figure 3.23



(a)



(b)

Example 3.10

Find Thevenin equivalent at terminal a-b for figure 3.24

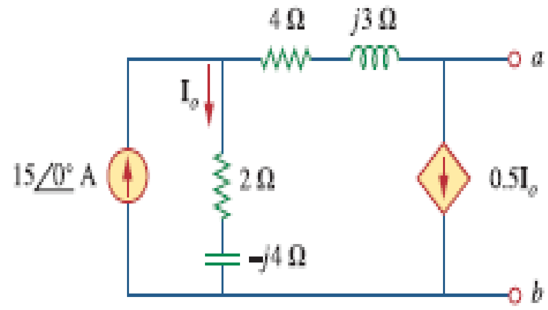
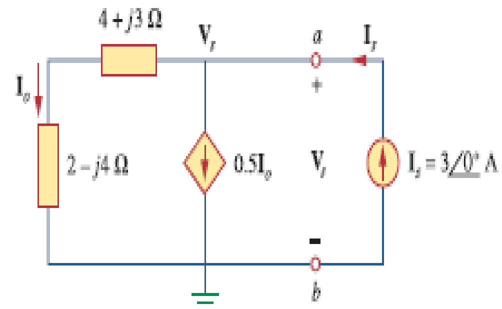
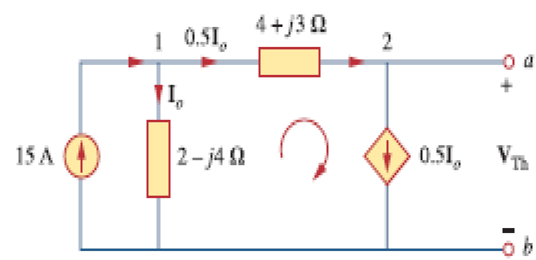


figure 3.24





Example 3.11

Find Norton equivalent at terminal a-b for figure 3.25

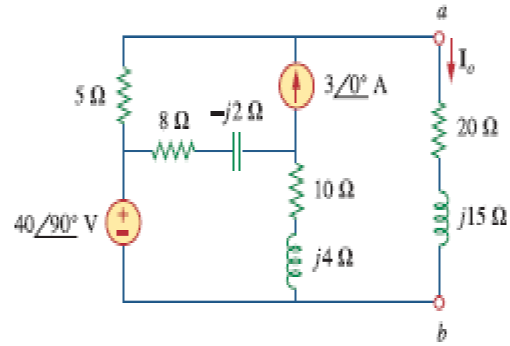
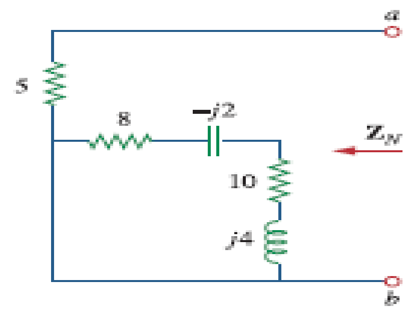
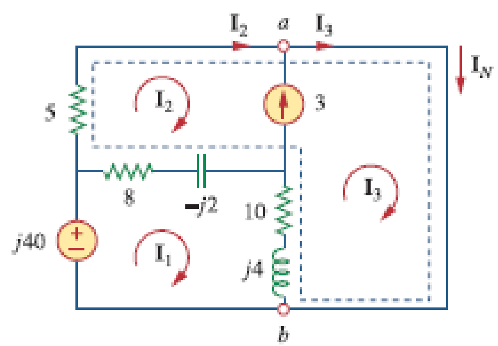
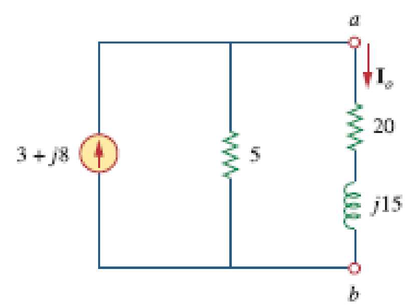


figure 3.25

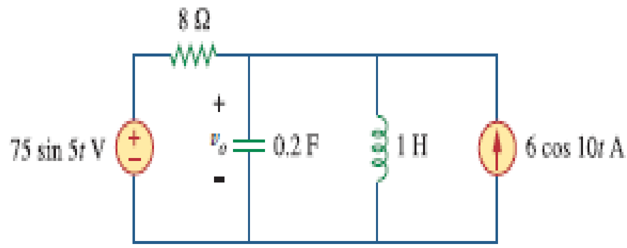




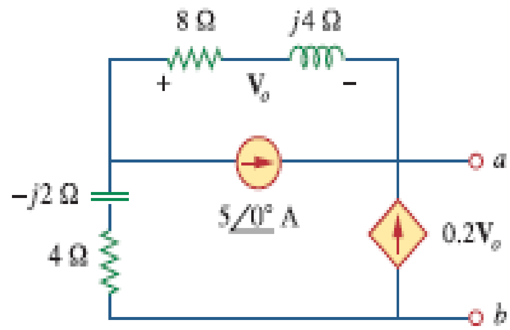


H.W

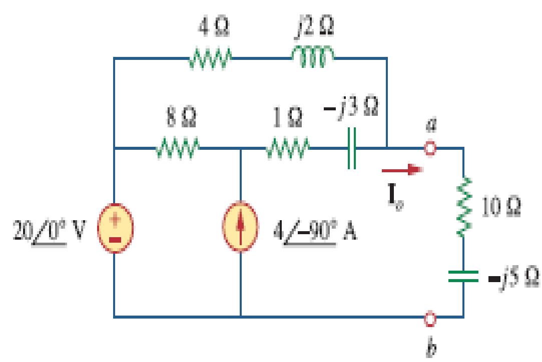
1. Use superposition theorem to find V0 for figure below



2. Find Thevenin equivalent at terminal a-b for figure below



3. Find Norton equivalent at terminal a-b for figure below and find $\mathbf{I_0}$



3.8 AC Power Analysis

Power analysis is of paramount importance. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device - every fan, motor, lamp, pressing iron, TV, personal computer-has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance. The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

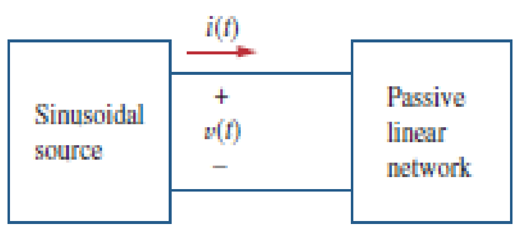
3.8.1 Instantaneous and Average Power

the instantaneous power p(t) absorbed by an element is the product of the instantaneous voltage v(t) across the element and the instantaneous current i(t) through it. Assuming the passive sign convention,

$$\boxed{p(t) = v(t)i(t)} \qquad (3.22)$$

The instantaneous power (in watts) is the power at any instant of time.

It is the rate at which an element absorbs energy. Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 3.26 . Let the voltage and current at the terminals of the circuit be



$$\boldsymbol{p(t) = v(t)i(t) = Vm\,Im\,cos(\omega t + \phi)cos\,\omega t} \qquad (3.23)$$

which we will find convenient to rewrite in a form obtained by using the trigonometric identity for the product of two cosine functions. Thus,

$$\boldsymbol{p(t) = \frac{V_m I_m}{2}[\cos(2\omega t + \phi) + cos\phi] = \frac{V_m I_m}{2} cos\phi + \frac{V_m I_m}{2}\cos(2\omega t + \phi)} \qquad (3.24)$$

The last equation possesses several characteristics that are true in general for circuits in the sinusoidal steady state. One term, the first is not a function of time; and a second term is included which has a cyclic variation at twice the applied frequency. Since this term is a cosine wave, and since sine waves and cosine waves have average values which are zero (when averaged over an integral number of periods), this example suggests that the average power is $1/2\, Vm\, Im$ cos φ; as we will see shortly, this is indeed the case.

3.8.2 AVERAGE POWER

Now let us obtain the general result for the sinusoidal steady state. We assume the general sinusoidal voltage

$$\boldsymbol{v(t) = Vm\, cos(\omega t + \theta)} \qquad (3.25)$$

and current

$$\boldsymbol{i(t) = Im\, cos(\omega t + \phi)} \qquad (3.26)$$

associated with the device in question. The instantaneous power is

$$\boldsymbol{p(t) = Vm\, Im\, cos(\omega t + \theta) cos(\omega t + \phi)} \qquad (3.27)$$

Again expressing the product of two cosine functions as one-half the sum of the cosine of the difference angle and the cosine of the sum angle,

$$\boldsymbol{p(t) = \frac{V_m I_m}{2} \cos(\theta - \phi) + \frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi)} \qquad (3.28)$$

we may save ourselves some integration by an inspection of the result.

The first term is a constant, independent of t. The remaining term is a cosine function; p(t) is therefore periodic, and its period is ½T. Note that the period T is associated with the given current and voltage, and not with the power; the power function has a period ½T. However, we may integrate over an interval of T to determine the average value if we wish; it is necessary only that we also divide by T. Our familiarity with cosine and sine waves, however, shows that the average value of either over a period is zero. There is thus no need to integrate Eq. [3.29] formally; by inspection, the average value of the second term is zero over a period T (or ½T), and the average value of the first term, a constant, must be that constant itself. Thus,

$$\boldsymbol{P = \frac{V_m I_m}{2} \cos(\theta - \phi)} \qquad (3.29)$$

Example 3.12

Given the time-domain voltage $v = 4\cos(\pi t/6)$ V, find both the average power and an expression for the instantaneous power that result when the corresponding phasor voltage $V = 4\angle 0^\circ$ V is applied across an impedance $Z = 2\angle 60^\circ$ Ω.

Solution:

The phasor current is $V\angle Z = 2\angle -60^\circ$ A, and so the average power is

$$P = \tfrac{1}{2}(4)(2)\cos 60^\circ = 2 \quad \text{W}$$

We can write the time-domain voltage,

$$v(t) = 4\cos(\pi t/6) \qquad \text{V}$$

and the time-domain current,

$$i(t) = 2\cos(\pi t/6 - 60^\circ) \quad \text{A}$$

The instantaneous power, therefore, is given by their product:

$$\begin{aligned} p(t) &= 8\cos(\pi t/6)\cos(\pi t/6 - 60^\circ) \\ &= 2 + 4\cos(\pi t/3 - 60^\circ) \qquad \text{W} \end{aligned}$$

Example 3.13

Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 3.26

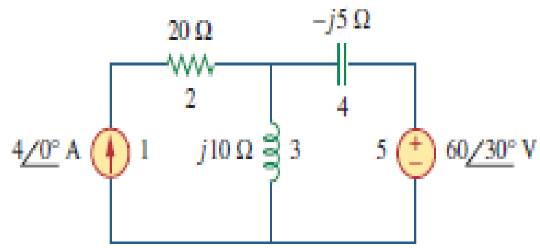
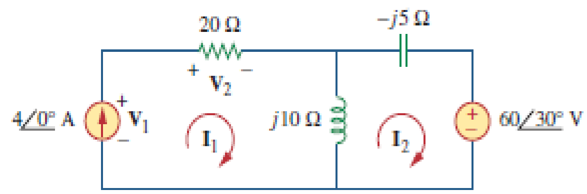


Fig. 3.26



3.8.3 Maximum Average Power Transfer

In Chapter 1 we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load $\mathbf{R_L}$ Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $\mathbf{R_L} = \mathbf{R_{Th}}$ We now extend that result to ac circuits.

Consider the circuit in Fig. 3.27, where an ac circuit is connected to a load $\mathbf{Z_L}$ and is represented by its Thevenin equivalent. The load is usually represented by an impedance, which may model an electric motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z_{Th}}$ and the load impedance $\mathbf{Z_L}$ are

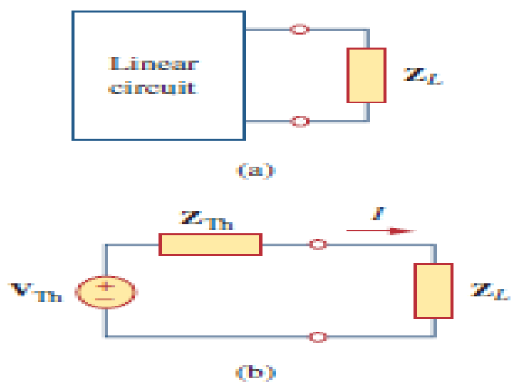


Fig. 3.27 Finding the maximum average power transfer:

(a) circuit with a load, (b) the Thevenin equivalent.

$$\mathbf{Z}_{Th} = R_{Th} + jX_{Th}$$
$$\mathbf{Z}_L = R_L + jX_L \qquad (3.30)$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + \mathbf{Z}_L} = \frac{\mathbf{V}_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)} \qquad (3.31)$$

the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L/2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2} \qquad (3.32)$$

Our objective is to adjust the load parameters and so that P is maximum. To do this we set $\frac{\partial P}{\partial R_L}$and $\frac{\partial P}{\partial X_L}$ equal to zero. From Eq. (3.32), we obtain

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 R_L(X_{\text{Th}} + X_L)}{[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{3.33}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\text{Th}}|^2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2 - 2R_L(R_{\text{Th}} + R_L)]}{2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{3.34}$$

Setting $\frac{\partial P}{\partial X_L}$ to zero gives

$$X_L = -X_{\text{Th}} \tag{3.35}$$

and setting $\frac{\partial P}{\partial R_L}$ to zero results in

$$R_L = \sqrt{R_{\text{Th}}^2 + (X_{\text{Th}} + X_L)^2} \tag{3.36}$$

Combining Eqs. (3.35) and (3.36) leads to the conclusion that for maximum average power transfer, $\mathbf{Z_L}$ must be selected so that $\mathbf{X_L = -X_{Th}}$ and i.e.,

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*} \tag{3.37}$$

For maximum average power transfer, the load impedance $\mathbf{Z_L}$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z_{Th}}$.

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}} \tag{3.38}$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

Example 3.14

Determine the load impedance $\mathbf{Z_L}$ that maximizes the average power drawn from the circuit of Fig. 3.28. What is the maximum average power?

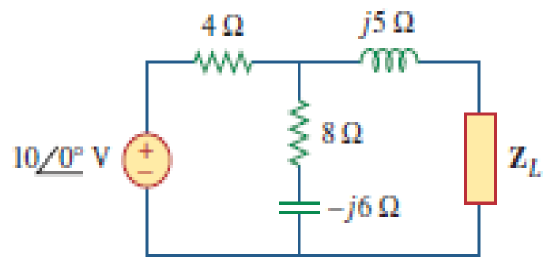
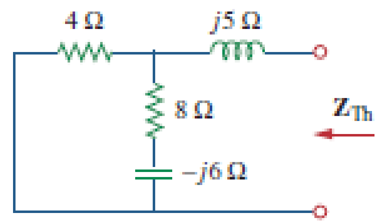
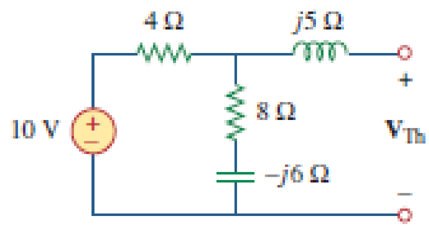


Fig. 3.28.





Example 3.15

In the circuit in Fig. 3.29, find the value of that will absorb the maximum average power. Calculate that power.

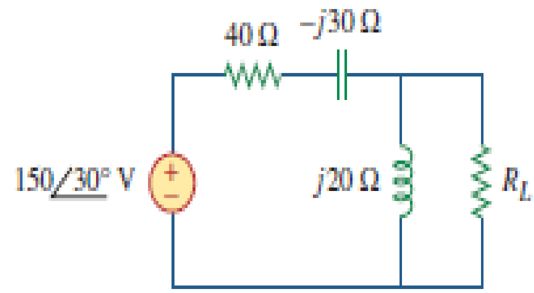


Fig. 3.29

3.8.4 Effective or RMS Value

Use of RMS Values to Compute Average Power

The average power delivered to an R ohm resistor by a sinusoidal current is

$$\mathbf{P = \tfrac{1}{2} I^2_m R}$$

Since $I_{eff} = Im/\sqrt{2}$, the average power may be written as

$$\mathbf{P = I_{eff}^2 R}$$

The other power expressions may also be written in terms of effective values:

$$\mathbf{P = V_{eff} I_{eff} \cos(\theta - \phi)}$$

$$\mathbf{P = V_{eff}^2 / R}$$

3.8.5 APPARENT POWER AND POWER FACTOR

The product of the effective values of the voltage and current is not the average power; we define it as the apparent power.

$$\mathbf{P = V_{eff} I_{eff}}$$

Dimensionally, apparent power must be measured in the same units as real power, since $\mathbf{\cos(\theta - \varphi)}$ is dimensionless; but in order to avoid confusion, the term volt-amperes, or VA, is applied to the apparent power.

Since $\mathbf{\cos(\theta - \varphi)}$ cannot have a magnitude greater than unity, the magnitude of the real power can never be greater than the magnitude of the apparent power.

The ratio of the real or average power to the apparent power is called the power factor, symbolized by PF. Hence,

$$\mathbf{PF = average\ power/apparent\ power = V_{eff} I_{eff} \cos(\theta - \phi)/V_{eff} I_{eff} = \cos(\theta - \phi)}$$

In the sinusoidal case, the power factor is simply $\mathbf{\cos(\theta - \phi)}$**,** where $\mathbf{(\theta - \phi)}$ is the angle by which the voltage leads the current. This relationship is the reason why the angle $\mathbf{(\theta - \phi)}$ is often referred to as the PF angle.

For a purely resistive load, the voltage and current are in phase, $\mathbf{(\theta - \phi)}$ is zero, and the PF is unity.

Example 3.16

Calculate values for the average power delivered to each of the two loads shown in Fig. 3.30, the apparent power supplied by the source, and the power factor of the combined loads.

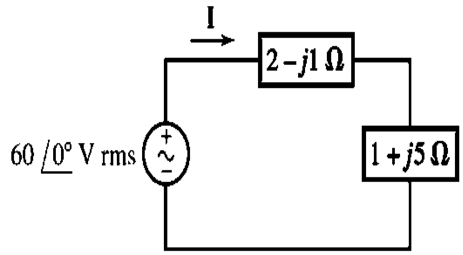


Fig. 3.30

Solution:

We require I_{eff}:

$$I = 60\angle 0^o/(3 + j4) = 12\angle{-53.13^\circ} \quad \text{A rms}$$

so $I_{eff} = 12$ A rms, and ang $I = -53.13^\circ$.

The average power delivered to the top load is given by

$$P_{upper} = I^2_{eff}R_{top} = (12)^2(2) = 288\,\text{W}$$

and the average power delivered to the right load is given by

$$P_{lower} = I^2_{eff}R_{right} = (12)^2(1) = 144 \quad \text{W}$$

The source itself supplies an apparent power of $V_{eff}\, I_{eff} = (60)(12) = 720$ VA.

Finally, the power factor of the combined loads is found by considering the voltage and current associated with the combined loads.

$$PF = P/V_{eff}\, I_{eff} = (P_{upper} + P_{lower})/V_{eff}\, I_{eff} = 432/(60*12) = 0.6 \text{ lagging}$$

H.W.: For the circuit of Fig. 3.31, determine the power factor of the combined loads if $Z_L = 10$ Ω.

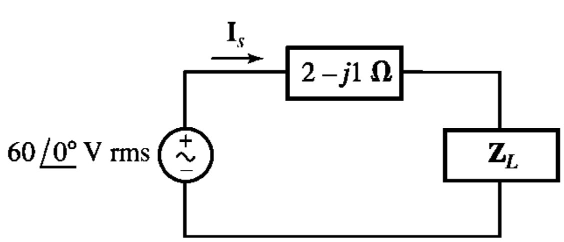


Fig. 3.31

3.8.6 COMPLEX POWER

In this section, we define complex power to allow us to calculate the various contributions to the total power in a clean, efficient fashion. The magnitude of the complex power is simply the apparent power. The real part is the **average power** and—as we are about to see—the imaginary part is a new quantity, termed the **reactive power**, which describes the rate of energy transfer into and out of reactive load components (e.g., inductors and capacitors).

If we first inspect the polar or exponential form of the complex power,

$$\mathbf{S} = \mathbf{V_{eff}\, I_{eff}\, e^{\,j\,(\theta-\phi)}}$$

we see that the magnitude of S, $V_{eff}I_{eff}$, is the apparent power. The angle of S, $(\theta - \phi)$, is the PF angle (i.e., the angle by which the voltage leads the current).

In rectangular form, we have

$$\mathbf{S = P + jQ}$$

$$\mathbf{P = V_{eff}\, I_{eff}\cos(\theta-\phi)} \qquad \text{average power}$$

$$\mathbf{Q = V_{eff}\, I_{eff}\sin(\theta-\phi)} \qquad \text{reactive power}$$

Quantity	Symbol	Formula	Units
Average power	P	$V_{eff}I_{eff}\cos(\theta-\phi)$	watt (W)
Reactive power	Q	$V_{eff}I_{eff}\sin(\theta-\phi)$	volt-ampere-reactive (VAR)
Complex power	$\mathbf{S}$	$P + jQ$	
		$V_{eff}I_{eff}\underline{/\theta-\phi}$	volt-ampere (VA)
		$\mathbf{V}_{eff}\mathbf{I}^{*}_{eff}$	
Apparent power	$\lvert\mathbf{S}\rvert$	$V_{eff}I_{eff}$	volt-ampere (VA)

Example 3.17

An industrial consumer is operating a 50 kW (67.1 hp) induction motor at a lagging PF of 0.8. The source voltage is 230 V rms. In order to obtain lower electrical rates, the customer wishes to raise the PF to 0.95 lagging. Specify a suitable solution.

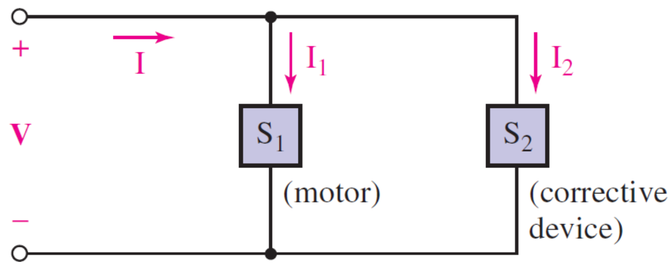


Fig. 3.32

Solution:

The complex power supplied to the induction motor must have a real part of 50 kW and an angle of $\cos^{-1}(0.8)$, or 36.9°. Hence,

$$S_1 = 50\angle 36.9^\circ/0.8 = 50 + j37.5 \quad \text{kVA}$$

In order to achieve a PF of 0.95, the total complex power must become

$$S = S_1 + S_2 = (50/0.95)\angle\cos^{-1}(0.95) = 50 + j16.43 \qquad \text{kVA}$$

Thus, the complex power drawn by the corrective load is

$$S_2 = -j21.07 \qquad \text{kVA}$$

The necessary load impedance Z_2 may be found in several simple steps. We select a phase angle of 0° for the voltage source, and therefore the current drawn by Z_2 is

$$I_2^* = S_2/V = -j21{,}070/230 = -j91.6 \qquad \text{A}$$

or

$$I_2 = j91.6 \qquad \text{A}$$

Therefore,

$$Z_2 = V/I_2 = 230/j91.6 = -j2.51\ \Omega$$

If the operating frequency is 60 Hz, this load can be provided by a 1056 μF capacitor connected in parallel with the motor. However, its initial cost, maintenance, and depreciation must be covered by the reduction in the electric bill.

H.W.:

1- For the circuit shown in Fig. 3.33, find the complex power absorbed by the (a) 1 Ω resistor; (b) −j10 Ω capacitor; (c) 5 + j10 Ω impedance; (d) source.

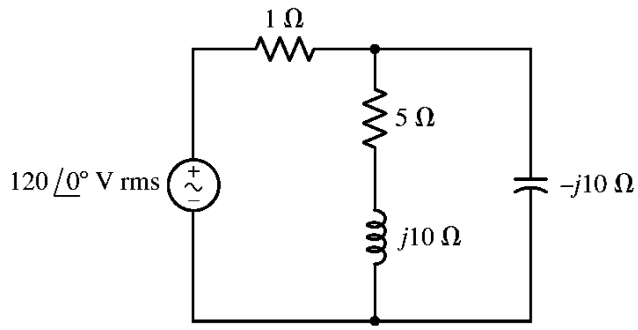


Fig. 3.33

2- A 440 V rms source supplies power to a load Z_L = 10 + j2 Ω through a transmission line having a total resistance of 1.5 Ω. Find (a) the average and apparent power supplied to the load; (b) the average and apparent power lost in the transmission line; (c) the average and apparent power supplied by the source; (d) the power factor at which the source operates.