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Electric Circuits Analysis

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Chapter Two
Part 2

Transient Circuits

- ***Second-Order Circuits***
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Chapter Two

Transient Circuits

2.7 *Second-Order Circuits*

In the previous part, we considered circuits with a single storage element (a capacitor or an inductor). Such circuits are first-order because the differential equations describing them are first-order. In this part, we will consider circuits containing two storage elements. These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.

Typical examples of second-order circuits are RLC circuits, in which the three kinds of passive elements are present. Examples of such circuits are shown in Fig. 2.27(a) and (b). Other examples are RL and RC circuits, as shown in Fig. 2.27(c) and (d). It is apparent from Fig. 2.27 that a second-order circuit may have two storage elements of different type or the same type (provided elements of the same type cannot be represented by an equivalent single element).

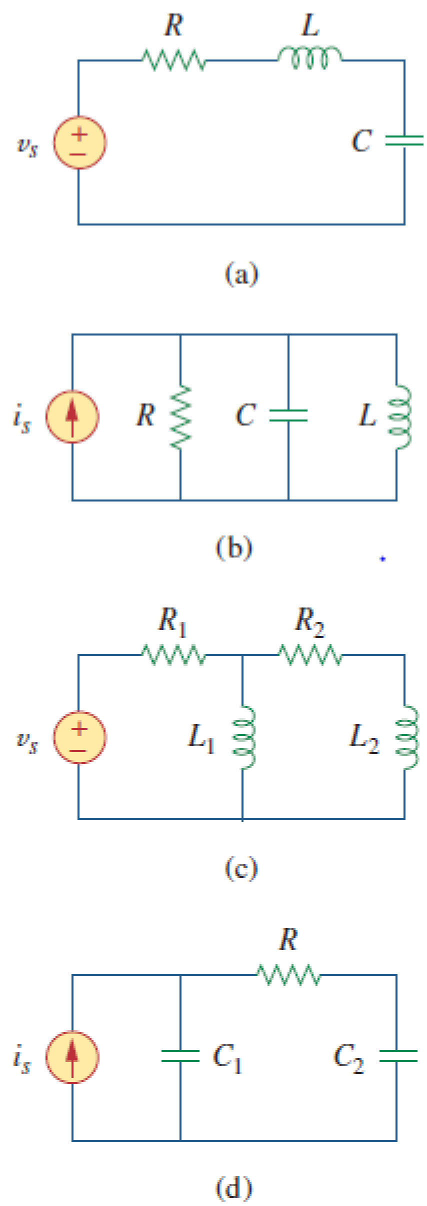

Fig 2.27 Typical examples of second-order circuits.

Our analysis of second-order circuits will be similar to that used for first-order. We will first consider circuits that are excited by the initial conditions of the storage elements. Although these circuits may contain dependent sources, they are free of independent sources. These source-free circuits will give natural responses as expected. Later we will consider circuits that are excited by independent sources.

2.8 The Source Free RLC Series Circuit

Consider the series RLC circuit shown in Fig. 2.28. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0 . Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \qquad \ldots(2.41)$$

$$i(0) = I_0$$

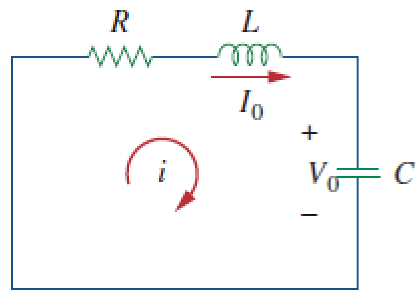

Fig 2.28 A source-free series RLC circuit.

Applying KVL around the loop in Fig. 2.28,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0 \qquad \ldots(2.42)$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \qquad \ldots(2.43)$$

Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let $i = Ae^{st}$ is the solution of Eq (2.43) so, the above equation will be

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

Or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \qquad \ldots(2.44)$$

Since $\boldsymbol{i = Ae^{st}}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad \ldots(2.45)$$

The two roots of above Eq. are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \ldots(2.46)$$

A more compact way of expressing the roots is

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \qquad \ldots(2.47)$$

Where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \ldots(2.48)$$

The roots s_1 and s_2 are called ***natural frequencies***, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; ω_0 is known as the resonant frequency or strictly as the ***undamped natural frequency***, expressed in radians per second (rad/s); and α is the neper frequency or ***the damping factor***, expressed in nepers per second. In terms of ω_0 and $\propto$, Eq. (2.45) can be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad \ldots(2.49)$$

The two values of s in Eq. (2.46 & 2.247) indicate that there are two possible solutions for i, that is, $i_1 = A_1 e^{s_1 t}, \quad i_2 = A_2 e^{s_2 t}$

Since Eq. (2.43) is a linear equation, so the natural response of the series RLC circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \ldots(2.50)$$

Where A_1 & A_2 are the constants and are determined from the initial values of $i(0)$ & $\frac{di(0)}{dt}$.

From Eqs. (2.41 & 2.42),

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

From Eq. (2.47), we can infer that there are three types of solutions:

1- If $\alpha > \omega_0$ we have the overdamped case.
2- If $\alpha < \omega_0$ we have the critically damped case.
3- If $\alpha = \omega_0$ we have the underdamped case.

We will consider each of these cases separately.

A- Overdamped Case ($\alpha > \omega_0$)

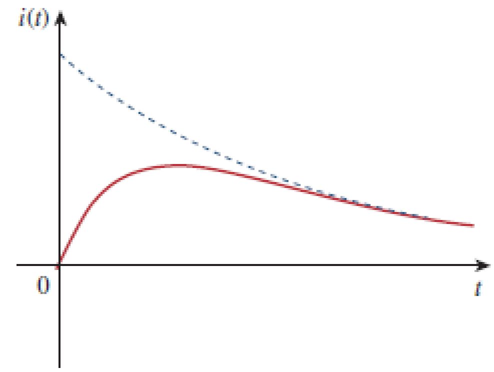

Fig 2.29 Overdamped response.

From Eqs. (2.47) and (2.48), $\alpha > \omega_0$ implies $C > \frac{4L}{R^2}$.

When this happens, both roots s_1 and s_2 are negative and real. The response is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \ldots(2.51)$$

which decays and approaches zero as t increases. Figure 2.29 illustrates a typical overdamped response.

B- Critically Damped Case ($\alpha = \omega_0$)

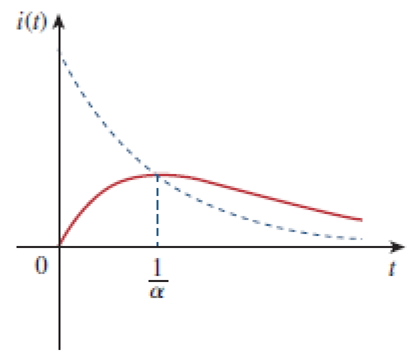

Fig 2.30 critically damped response.

When $\alpha = \omega_0$ implies $C = \frac{4L}{R^2}$, and $s_1 = s_2 = -\alpha = -\frac{R}{2L}$, for this case the response is

$$i(t) = (A_1 t + A_2) e^{-\alpha t} \qquad \ldots(2.52)$$

A critically damped response is shown in Fig. 2.30.

C- Underdamped Case ($\alpha < \omega_0$)

As $\alpha < \omega_0$ implies $C < \frac{4L}{R^2}$, the roots can be written

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

Where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the ***damping frequency.***

The response in this case will be

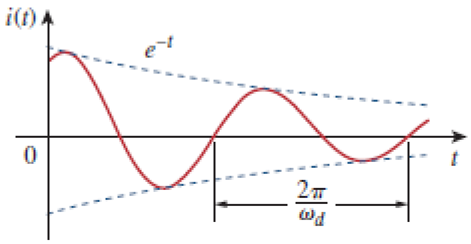

Fig 2.31 underdamped response.

$$i(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t) e^{-\alpha t} \qquad \ldots(2.53)$$

Where $B_1 = A_1 + A_2$ & $B_2 = j(A_1 - A_2)$

The response of this case is shown in Fig. 2. 31.

Example 2.10:- In Fig. 2.28, $R = 40\Omega$, $\mathrm{L} = 4$ H and $\mathrm{C} = 1/4$ F Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

Example 2.11:- Find $i(t)$ in the circuit of Fig. 2.32. Assume that the circuit has reached steady state at $t = 0^-$.

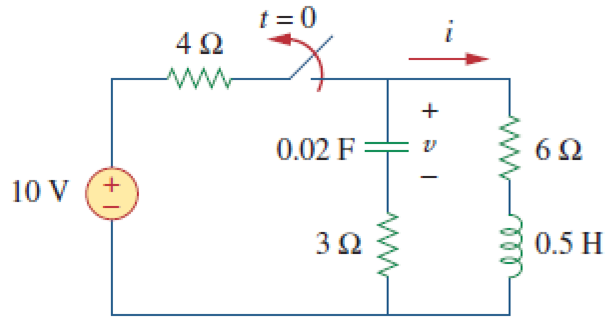

Fig 2.32 For Example 2.11

2.9 ***The Source Free RLC Parallel Circuit***

Parallel RLC circuits find many practical applications, notably in communications networks and filter designs. Consider the parallel RLC circuit shown in Fig. 2.33. Assume initial capacitor voltage V_0 and initial inductor current I_0 . Thus, at $t = 0$,

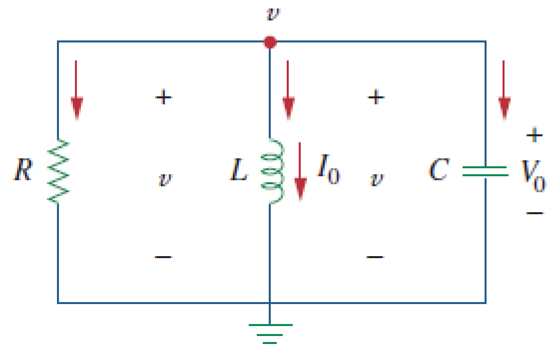

Fig 2.33 A source-free parallel RLC circuit.

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \qquad \ldots(2.54)$$

$$v(0) = V_0$$

By applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0 \qquad \ldots(2.55)$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad \ldots(2.56)$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2 . By following the same reasoning used in establishing in the previous section, the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad \ldots(2.57)$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \qquad \ldots(2.58)$$

Where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \qquad \ldots(2.59)$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions depending on the values of α & ω_0, which are:-

A- Overdamped Case ($\alpha > \omega_0$)

From Eq. (2.59), $\alpha > \omega_0$ implies $L > 4R^2C$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \ldots(2.60)$$

B- Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$ implies $L = 4R^2C$, and $s_1 = s_2 = -\alpha = -\frac{1}{RC}$, for this case the response is

$$v(t) = (A_1 t + A_2)e^{-\alpha t} \qquad \ldots(2.61)$$

C- Underdamped Case ($\alpha < \omega_0$)

As $\alpha < \omega_0$ implies $L < 4R^2C$, In this case the roots are complex and may be expressed as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

Where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, & the response is

$$v(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \ldots(2.62)$$

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$ .The first term is known from Eq. (2.54). We find the second term by combining Eqs. (2.54) and (2.55), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

Or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \qquad \ldots(2.63)$$

Example 2.12:- In the parallel circuit in the figure 2.33, find $v(t)$, for $v(0) = 5V, i(0) = 0, L = 1\ H$ & $C = 10\ mF$. Consider three case $R = 1.923\ \Omega, R = 5\ \Omega$ & $R = 6.25\ \Omega$.

Example 2.13:- Find $v(t)$ for $t > 0$ in the RLC circuit of Fig. 2.34.

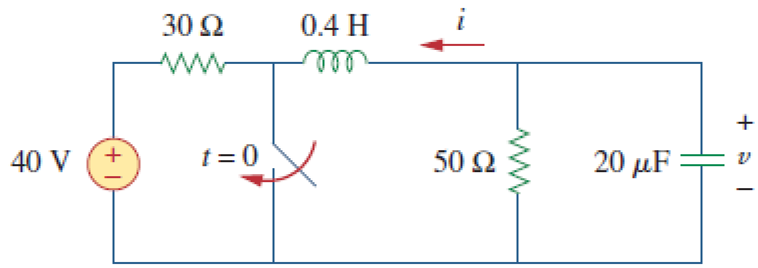

Fig 2.34 For Example 2.13

2.10 Step Response of a Series RLC Circuit

As we learned in the preceding chapter, the step response is obtained by the sudden application of a dc source. Consider the series RLC circuit shown in Fig. 2.35. Applying KVL around the loop for $t > 0$,

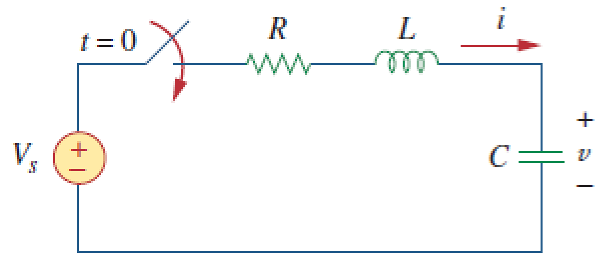

Fig 2.35 Step voltage applied to a series RLC circuit.

$$L\frac{di}{dt} + Ri + v = V_s \qquad \ldots(2.63)$$

But $i = C\frac{dv}{dt}$, so by substituting i in Eq. (2.63) and rearranging terms , we will have

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \qquad \ldots(2.64)$$

The solution to above Eq. has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$ that is,

$$v(t) = v_t(t) + v_{ss}(t)$$

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

$$v_{ss}(t) = v(\infty) = V_s$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

$\ldots(2.65)$

The constants A_1 and A_2 in each case can be determined from the initial conditions $v(0)$ and $dv(0)/dt$.

Example 2.14:- Find $v(t)$ & $i(t)$ for $t > 0$ in circuit in the Fig. 2.36. Consider three case $R = 5\ \Omega, R = 4\ \Omega$ & $R = 1\ \Omega$.

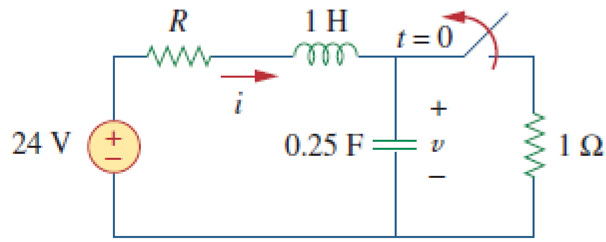

Fig 2.36 For Example 2.14

2.11 Step Response of a Parallel RLC Circuit

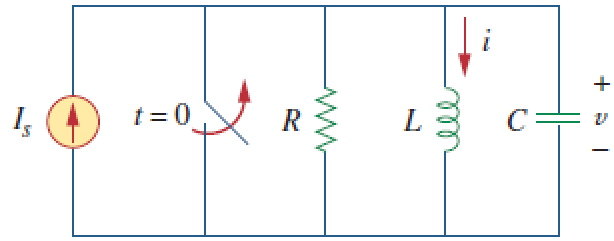

Fig 2.37 Parallel RLC circuit with an applied

Consider the parallel RLC circuit shown in Fig. 2.37. We want to find i due to a sudden application of a dc current. Applying KCL at the top node for $t > 0$

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \qquad \ldots(2.66)$$

But $v = L\frac{di}{dt}$, so by substituting v in Eq. (2.63) and rearranging terms , we get

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \qquad \ldots(2.67)$$

The solution to above Eq. has two components: the transient response $i_t(t)$ and the steady-state response $i_{ss}(t)$ that is,

$$i(t) = i_t(t) + i_{ss}(t) \qquad \ldots(2.68)$$

As we proceed in the last section, the response will be

$$\begin{aligned} i(t) &= I_s + A_1e^{s_1t} + A_2e^{s_2t} && \text{(Overdamped)} \\ i(t) &= I_s + (A_1 + A_2t)e^{-\alpha t} && \text{(Critically damped)} \\ i(t) &= I_s + (A_1\cos\omega_d t + A_2\sin\omega_d t)e^{-\alpha t} && \text{(Underdamped)} \end{aligned} \qquad \ldots(2.69)$$

The constants A_1 and A_2 in each case can be determined from the initial conditions $i(0)$ and $di(0)/dt$.

Example 2.15:- Find $i(t)$ & $i_R(t)$ for $t > 0$ in circuit in the Fig. 2.38.

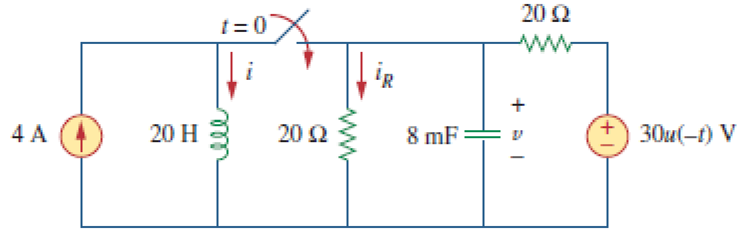

Fig 2.36 For Example 2.15