Chapter Two

The Transient Circuits

- **RL Circuits**
- **RC Circuits**
- **RLC Circuits**

The analysis of circuits containing inductors and/or capacitors is dependent upon the formulation and solution of the Integra differential equations that characterize the circuits. The solution of the differential equation represents a response of the circuit, and it is known by many names:

- *The source-free response may be called the **natural response**, the **transient response**, the **free response**, or the **complementary function**, but because of its more descriptive nature, we will most often call it the **natural response**.*
- *When we consider independent sources acting on a circuit, part of the response will resemble the nature of the particular source (or forcing function) used; this part of the response, called the particular solution, the **steady-state response**, or **the forced response**.*
- *In other words, the **complete response** is the sum of the natural response and the forced response.*

We will consider several different methods of solving these differential equations. The mathematical manipulation, however, is not circuit analysis.

2.1 RL Circuit:

We begin our study of transient analysis by considering the simple series RL circuit shown in Fig. 2.1. Let us designate the time-varying current as i (t); we will represent the value of i (t) at t = 0 as I_0; in other words, i (0) = I_0.We therefore have

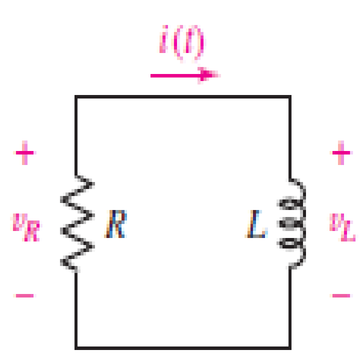


Fig. 2.1 A series RL circuit for which i(t) is to be determined, subject to the initial condition that i(0) = I_0.

$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \qquad [1]$$

Our goal is an expression for i(t) which satisfies this equation and also has the value I_0 at t = 0. The solution may be obtained by several different methods.

One very direct method of solving a differential equation consists of writing the equation in such a way that the variables are separated, and then integrating each side of the equation. The variables in Eq. [1] are i and t, and it is apparent that the equation may be multiplied by dt, divided by i, and arranged with the variables separated:

$$\frac{di}{i} = -\frac{R}{L}dt \qquad [2]$$

After a little manipulation, we find that the current i(t) is given by

$$i(t) = I_0 e^{-Rt/L} \qquad [3]$$

We check our solution by first showing that substitution of Eq. [3] in Eq. [1] yields the identity 0 = 0, and then showing that substitution of t = 0 in Eq. [3] produces i (0) = I_0. Both steps are necessary; the solution must satisfy the differential equation which characterizes the circuit, and it must also satisfy the initial condition.

Let us now consider the nature of the response in the series RL circuit. We have found that the inductor current is represented by

$$i(t) = I_0 e^{-\frac{R}{L}t}$$

At t = 0, the current has value I_0, but as time increases, the current decreases and approaches zero. The shape of this decaying exponential is seen by the plot of $i(t)/I_0$ versus t shown in Fig. 2.2.

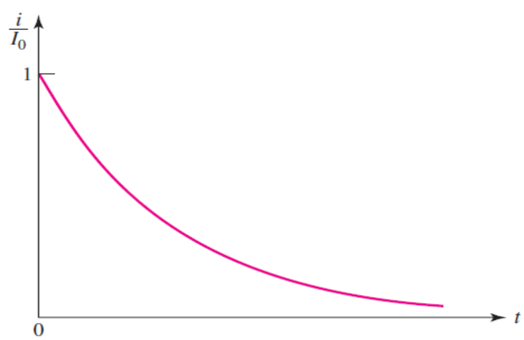


Fig. 2.2: The plot of $i(t)/I_0$ versus t.

Where $\boldsymbol{\tau = \frac{L}{R}}$

The last equation become

$$i(t) = I_0 e^{-t/\tau}$$

Example 2.1: Determine both i_1 and i_L in the circuit shown in Fig. 2.6a for t > 0.

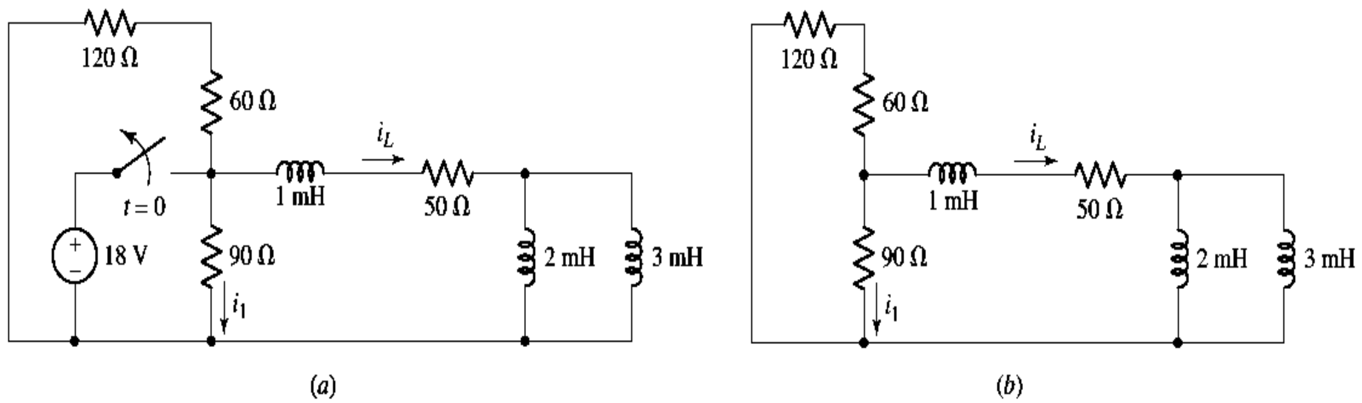


Fig. 2.3

Solution:

After t = 0, when the voltage source is disconnected as shown in Fig. 2.3b, we easily calculate an equivalent inductance,

$L_{eq} = 2 \times 3/(2 + 3) + 1 = 2.2$ mH

an equivalent resistance, in series with the equivalent inductance,

$R_{eq} = 90(60 + 120)/(90 + 180) + 50 = 110\ \Omega$

and the time constant,

$\boldsymbol{\tau = L_{eq}/R_{eq}} = 2.2 \times 10^{-3}/110 = 20\ \mu s$

Thus, the form of the natural response is $Ke^{-50,000t}$, where K is an unknown constant. Considering the circuit just prior to the switch opening ($t = 0^-$), $i_L = 18/50$ A. Since $i_L(0^+) = i_L(0^-)$, we know that $i_L = 18/50$ A or 360 mA at $t = 0^+$ and so

$$i_L = \begin{cases} 360\ mA & t < 0 \\ 360e^{-50,000t}\ mA & t \geq 0 \end{cases}$$

There is no restriction on i_1 changing instantaneously at t = 0, so its value at $t = 0^-$ (18/90 A or 200 mA) is not relevant to finding i_1 for t > 0. Instead, we must find $i_1(0^+)$ through our knowledge of $i_L(0^+)$.

Using current division,

$$i_1(0^+) = -i_L(0^+)(120 + 60)/(120 + 60 + 90) = -240 \text{ mA}$$

Hence,

$$i_1 = \begin{cases} 200\ mA & t < 0 \\ -240e^{-50{,}000t}\ mA & t \geq 0 \end{cases}$$

H.W.: At t = 0.15 s in the circuit of Fig. 2.4, find the value of (a) i_L; (b) i_1; (c) i_2.

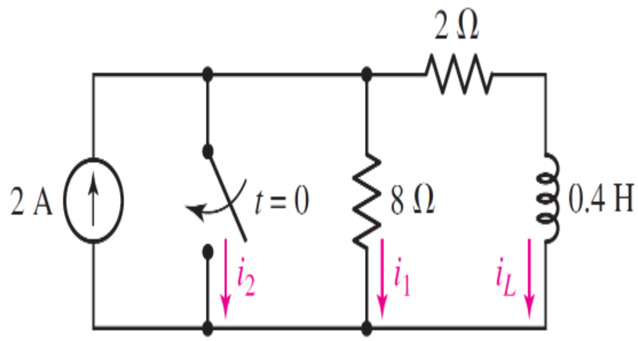


Fig. 2.4.

2.2 RC Circuit

A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 2.5. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be the voltage v(t) across the capacitor. Since the capacitor is initially charged, we can assume that at time t=0 the initial voltage is

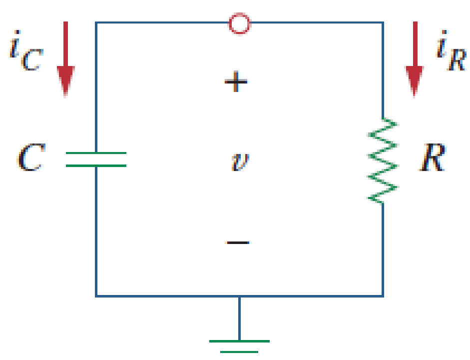


Fig. 2.5 RC circuit.

$$v(0) = V_0$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2$$

Applying KCL at the top node of the circuit in Fig. 2.5 yields

$$i_C + i_R = 0$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

This is a *first-order differential equation*, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where ln A is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC}$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0e^{-t/RC}$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the natural response of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in Fig. 2.6. Note that at t=0 we have the correct initial condition as in Equations above As t increases, the voltage decreases toward zero. The rapidity with whichthe voltage decreases is expressed in terms of the time constant, denoted by **τ** , the lowercase Greek letter tau.

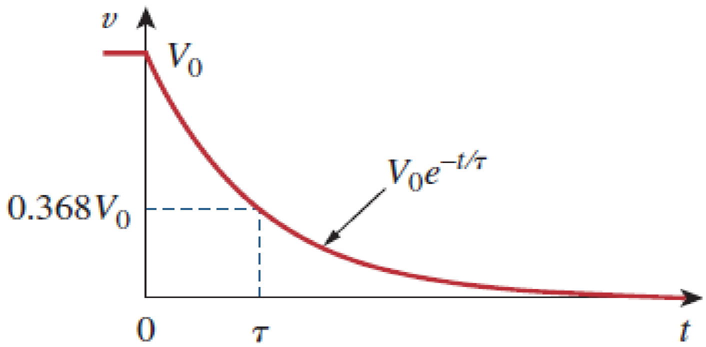


Fig. 2.6 The voltage response of the RC circuit.

Where $\tau = RC$

The last equation become

$$v(t) = V_0e^{-t/\tau}$$

Example 2.2 :- In figure 2.7a find $\boldsymbol{V_c}$ **,** $\boldsymbol{V_x}$ **,** $\boldsymbol{i_x}$ where $\mathbf{V_c(0)}$=15v

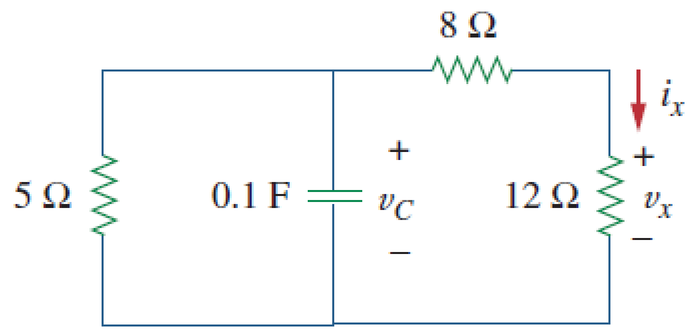


Fig. 2.7a

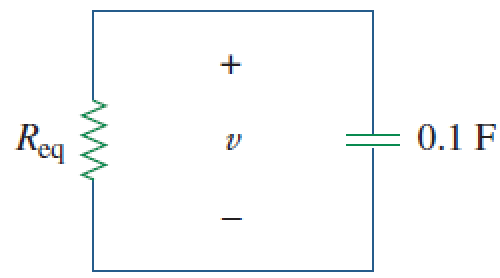


Fig. 2.5b

2.3 The Unit-Step Function:

The unit step function u(t) is 0 for negative values of t and 1 for positive values of t.

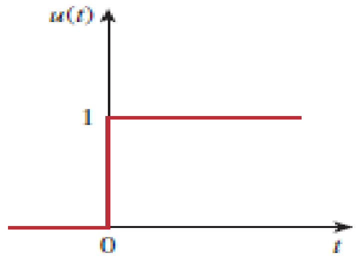


Fig. 2.6 unit step

In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

The unit step function is undefined at **t=0** where it changes abruptly from 0 to 1. It is dimensionless, like other mathematical functions such as sine and cosine. Figure 2.7 depicts the unit step function. If the abrupt change occurs at **t=to** (where) instead of **t=0** the unit step function becomes

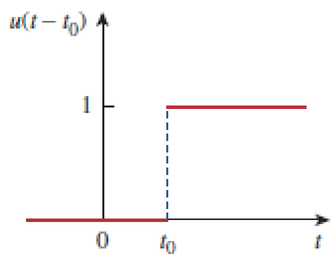


Fig. 2.7

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0)$$

2.4 Step Response of an RC Circuit

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a step response.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig. 2.8(a) which can be replaced by the circuit in Fig. 2.8(b), where **Vs** is a constant dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage **Vo** on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0$$

Fig. 2.8 An RC circuit with voltage step input.

Where $\mathbf{V(0^-)}$ is the voltage across the capacitor just before switching and $\mathbf{V(0^+)}$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

where v is the voltage across the capacitor. For **t** > **0** Equation above becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

Thus,

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

This is known as the complete response (or total response) of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that **$V_s > V_0$** a plot of is shown in Fig. 2.9.

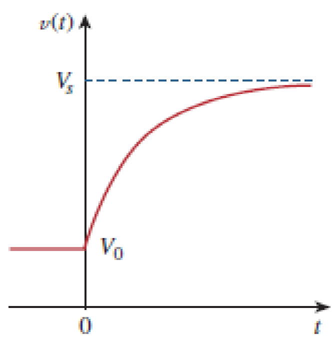


Fig. 2.9

If we assume that the capacitor is uncharged initially, we set $\mathbf{V_0 = 0}$ in last Equation so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from First Equation in this page using We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \qquad t > 0$$

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$

Fig. 2.10 Step response of an RC circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Figure 2.10 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$

The complete response is composed of two parts, the natural response and the forced response.

The natural response is a characteristic of the circuit and not of the sources. Its form may be found by considering the source-free circuit, and it has an amplitude that depends on both the initial amplitude of the source and the initial energy storage.

The forced response has the characteristics of the forcing function; it is found by pretending that all switches were thrown a long time ago. Since we are presently concerned only with switches and dc sources, the forced response is merely the solution of a simple dc circuit problem.

$$\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$$

$$v = v_n + v_f$$

where

$$v_n = V_o e^{-t/\tau}$$

and

$$v_f = V_s(1 - e^{-t/\tau})$$

Another way of looking at the complete response is to break into two components—one temporary and the other permanent, i.e.,

$$\text{Complete response} = \underset{\text{temporary part}}{\text{transient response}} + \underset{\text{permanent part}}{\text{steady-state response}}$$

or

$$v = v_t + v_{ss}$$

where

$$v_t = (V_o - V_s)e^{-t/\tau}$$

and

$$v_{ss} = V_s$$

The transient response $\mathbf{V_t}$ is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus,

The **transient response** is the circuit's temporary response that will die out with time.

The steady-state response $\mathbf{V_{ss}}$ is the portion of the complete response that remains after the transient response has died out. Thus,

The **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

Whichever way we look at it, the complete response may be written as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

where **V(0)** is the initial voltage at $\mathbf{t=0^+}$ and **V(∞)** is the final or SteadyState value. Thus, to find the step response of an RC circuit requires three things

Note that if the switch changes position at time instead of at there is a time delay in the response so that Equation becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau}$$

where $\mathbf{V(t_0)}$ is the initial value at $\mathbf{t=t_0^+}$ Keep in mind that Equations applies only to step responses, that is, when the input excitation is constant.

Example 2.3 The switch in Fig. 2.11 has been in position A for a long time. At the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value $at\ t = 1\ s\ and\ t = 4\ s$.

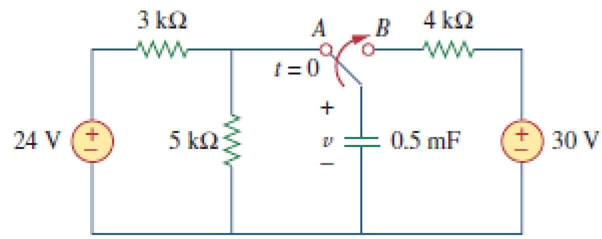


Fig. 2.11

Example 2.4 In Fig. 2.12, the switch has been closed for a long time and is opened at t= 0. Find i and v for all time.

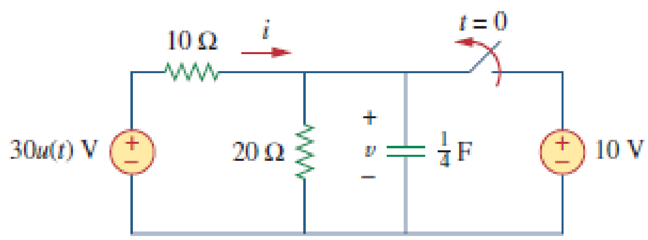


Fig. 2.12

Solution of Example 2.13: (a) for t<0 (b) for t > 0.

2.5 Step Response of an RL Circuit

Consider the RL circuit in Fig. 2.14(a), which may be replaced by the circuit in Fig. 2.14(b). Again, our goal is to find the inductor current I as the circuit response. Rather than apply Kirchhoff's laws, we will use the simple technique in last equations. Let the response be the sum of the transient response and the steady-state response,

Fig 2.14 An RL circuit with a step input voltage.

$$i = i_t + i_{ss}$$

We know that the transient response is always a decaying exponential, that is

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R}$$

where A is a constant to be determined.

The steady-state response is the value of the current a long time after the switch in Fig. 2.14(a) is closed. We know that the transient response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage **Vs** appears across R. Thus, the steady-state response is

$$i_{ss} = \frac{V_s}{R}$$

$$i = Ae^{-t/\tau} + \frac{V_s}{R}$$

We now determine the constant **A** from the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than **Vs** Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0$$

Thus, at $t = 0$ Equation becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for A in Equation we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$$

This is the complete response of the RL circuit. It is illustrated in Fig. 2.15. The response may be written as

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

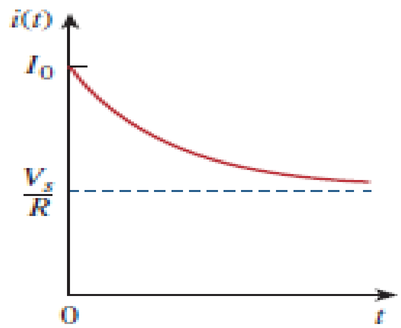


Fig. 2.15 Total response of the RL circuit with initial inductor current **I_0**.

where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively. Thus, to find the step response of an RL circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$ last equation becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

If **$I_0=0$** then

$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases}$	$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$

This is the step response of the RL circuit with no initial inductor current. The voltage across the inductor is obtained from Equation $v = L di/dt$ using We get

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$

or

$$v(t) = V_s e^{-t/\tau}u(t)$$

Figure 2.16 shows the step responses in Equations

Fig. 2.16 Step responses of an RL circuit with no initial inductor current: (a) current response, (b) voltage response.

Example 2.5 Find $i(t)$ in the circuit of Fig. 2.17 for Assume that the switch has been closed for $t > 0$ a long time.

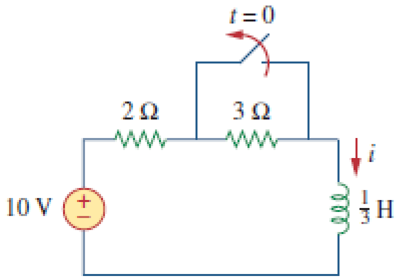


Fig. 2.17

Example 2.6 At $t = 0$ switch 1 in Fig. 2.18 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for t = 2 s and t = 5 s.

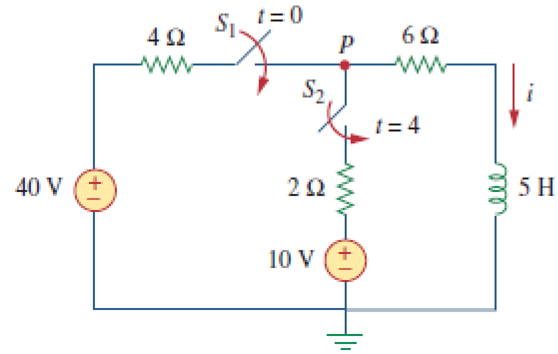


Fig. 2.18

H.W

1- The switch in Fig. 2.19 has been closed for a long time. It opens at $t = 0$ Find for

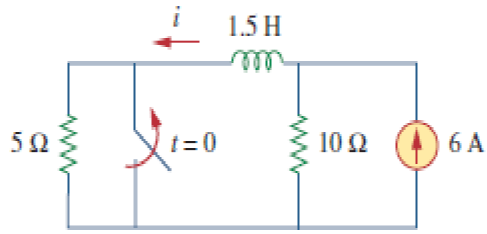


Fig. 2.19

2- Switch $\mathbf{S_1}$ in Fig. 2.20 is closed at $t = 0$ and switch $\mathbf{S_2}$ is closed at $t = 2\ s$ Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$

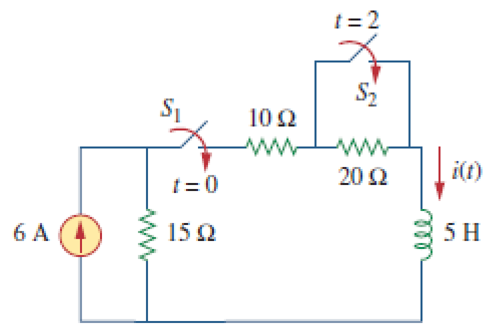


Fig. 2.20

3- Find $v(t)$ for $t > 0$ in the circuit of Fig. 2.21. Assume the switch has been open for a long time and is closed at $t = 0$ Calculate $v(t)$ at $t = 0.5$

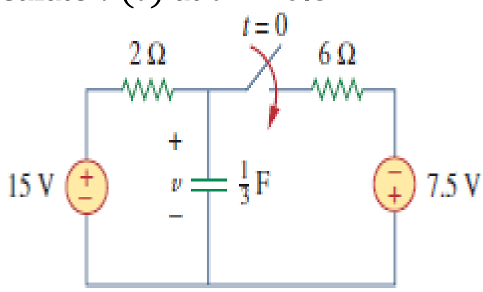


Fig. 2.21

4- The switch in Fig. 2.22 is closed at $t = 0$ Find $i(t)$and $v(t)$for all time.
Note that $u(-t) = 1\ for\ t < 0\ and\ 0\ for\ t > 0$. Also, $u(-t) = 1 - u(t)$.

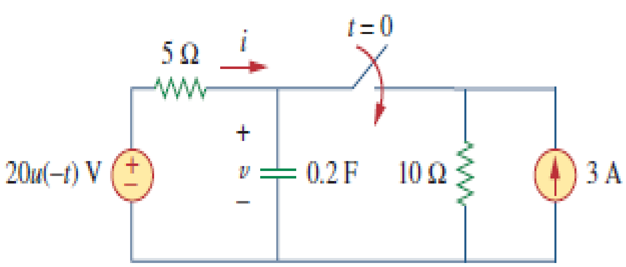


Fig. 2.22

2.6 RLC circuit:

In this section, we consider more complex circuits, which contain both an inductor and a capacitor. The result is a second-order differential equation for any voltage or current of interest. Now we need two initial conditions to solve each differential equation.

Such circuits occur routinely in a wide variety of applications, including oscillators and frequency filters. They are also very useful in modelling a number of practical situations, such as automobile suspension systems, temperature controllers, and even the response of an airplane to changes in elevator and aileron positions.

Typical examples of second-order circuits are RLC circuits, in which the three kinds of passive elements are present. Examples of such circuits are shown in Fig. 2.23(a) and (b).

Fig 2.23 RLC circuits

2.6.1 Free Series RLC Circuit

An understanding of the natural response of the series RLC circuit is a necessary background for future studies in filter design and communications networks. Consider the series RLC circuit shown in Fig. 2.24. The circuit is being excited by the energy initially stored in the capacitor and inductor.

The energy is represented by the initial capacitor voltage $\mathbf{V_0}$ and initial inductor current $\mathbf{I_0}$. Thus, at $t = 0$,

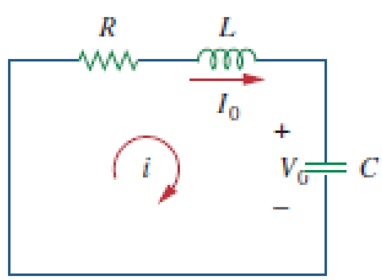


Fig. 2.24 series RLC circuit

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \qquad \text{(a1)}$$

$$i(0) = I_0 \qquad \text{(b1)}$$

Applying KVL around the loop in Fig. 2.24,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0 \qquad (2)$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \qquad (3)$$

This is a second-order differential equation and is the reason for calling the RLC circuits in this chapter second-order circuits. Our goal is to solve Eq. (3). To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial values of some i and The initial value of i is given in Eq. (1b). We get the initial value of the derivative of i from Eqs. (1a) and (2); that is,

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \qquad (4)$$

With the two initial conditions in Eqs. (1b) and (4), we can now solve Eq. (3). Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st} \qquad (5)$$

where A and s are constants to be determined. Substituting Eq. (5) into Eq. (3) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \qquad (6)$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad (7)$$

This quadratic equation is known as the characteristic equation of the differential Eq. (3), since the roots of the equation dictate the character of i. The two roots of Eq. (7) are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad (8a)$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad (8b)$$

A more compact way of expressing the roots is

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \qquad (9)$$

Where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad (10)$$

The roots $\mathbf{S_1}$ and $\mathbf{S_2}$ are called natural frequencies, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit $\boldsymbol{\omega_0}$; is known as the resonant frequency or strictly as the undamped natural frequency, expressed in radians per second (rad/s); and $\boldsymbol{\alpha}$ is the neper frequency or the damping factor, expressed in nepers per second. In terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\omega_0}$, Eq. (7) can be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad (11)$$

The variables **s** and $\boldsymbol{\omega_0}$ are important quantities we will be discussing throughout the rest of the text. The two values of s in Eq. (9) indicate that there are two possible solutions for i, each of which is of the form of the assumed solution in Eq. (5); that is,

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t} \qquad (12)$$

Since Eq. (3) is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of Eq. (3). A complete or total solution of Eq. (3) would therefore require a linear combination of i_1 and i_2 . Thus, the natural response of the series RLC circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (13)$$

where the constants $\mathbf{A_1}$ and $\mathbf{A_2}$ are determined from the initial values $i(0)$ and $di(0)/dt$ in Eqs. (1) and (2).

From Eq. (9), we can infer that there are three types of solutions:

1. If $\alpha > \omega_0$ we have the overdamped case.

2. If $\alpha = \omega_0$ we have the critically damped case.

3. If $\alpha < \omega_0$ we have the underdamped case.

The response is overdamped when the roots of the circuit's characteristic equation are unequal and real, critically damped when the roots are equal and real, and underdamped when the roots are complex.

We will consider each of these cases separately.

Overdamped Case $\alpha > \omega_0$

From Eqs. (8) and (9), $\alpha > \omega_0$ implies $C > 4L/R^2$ When this happens, both roots **S_1** and **S_2** are negative and real. The response is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{14}$$

which decays and approaches zero as t increases. Figure 2.25 illustrates a typical overdamped response.

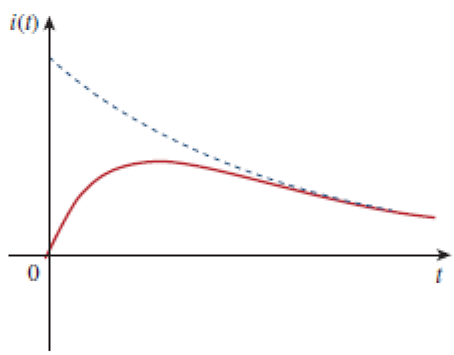


Fig. 2.25 overdamped

Critically Damped Case $\alpha = \omega_0$

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{15}$$

For this case, Eq. (13) yields

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

where . $\mathbf{A_3} = \mathbf{A_1} + \mathbf{A_2}$ This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant $\mathbf{A_3}$. What then could be wrong? Our assumption of an exponential solution is incorrect for the special case of critical damping. Let us go back to Eq. (3). When $\alpha = \omega_0 = R/2L$, Eq. (3) becomes

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

or

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0 \qquad (16)$$

If we let

$$f = \frac{di}{dt} + \alpha i$$

then Eq. (16) becomes

$$\frac{df}{dt} + \alpha f = 0 \qquad (17)$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$ where is a constant. Equation (17) then becomes

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

or

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \qquad (18)$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \qquad (19)$$

Integrating both sides yields

$$e^{\alpha t} i = A_1 t + A_2$$

or

$$i = (A_1 t + A_2)e^{-\alpha t} \qquad (20)$$

where $\mathbf{A_2}$ is another constant. Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}} \qquad (21)$$

A typical critically damped response is shown in Fig. 2.26 In fact, Fig. 2.26 is a sketch of $i(t) = t\ e^{-\alpha t}$ which reaches a maximum value of $e^{-1}/\alpha\ at\ t = 1/\alpha$, one time constant, and then decays all the way to zero.

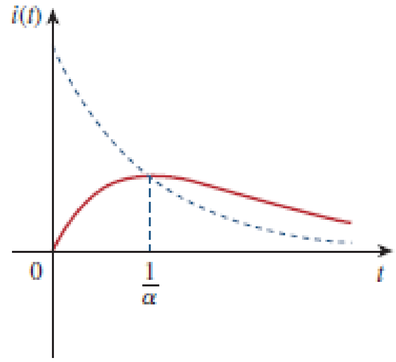


Fig. 2.26 critically damped

<u>Underdamped Case $\alpha < \omega_0$</u>

When $\alpha < \omega_0$, $C < 4L/R^2$ The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \quad (22a)$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \quad (22b)$$

where $j = \sqrt{-1}$and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$which is called the damping frequency. Both and are natural frequencies because they help determine the natural response; while ω_0 is often called the undamped natural frequency , ω_d is called the damped natural frequency. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned} \quad (23)$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \quad (24)$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned} \quad (25)$$

Replacing constants (**A_1** + **A_2**) and j (**A_1** - **A_2**) with constants **B_1** and **B_2** we write

$$\boxed{i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)} \quad (26)$$

With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature. The response has a time constant of $1/\alpha$ and a period $T = 2\pi/\omega_d$ of Figure 2.27 depicts a typical underdamped response. Figure 8.9 assumes for each case that $i(0) = 0$

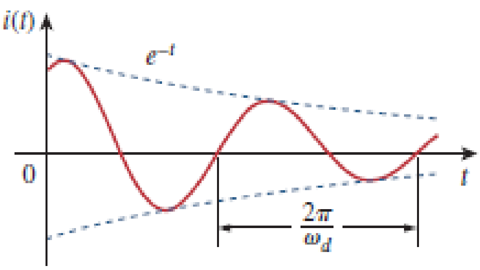


Fig. 2.27 underdamped

<u>Example 2.7</u> In Fig. 2.24 , R=40Ω , L= 4 H and C= 0.25 F Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

<u>Example 2.8</u> Find $i(t)$ in the circuit of Fig. 2.28. Assume that the circuit has reached steady state at $t = 0^{-}$.

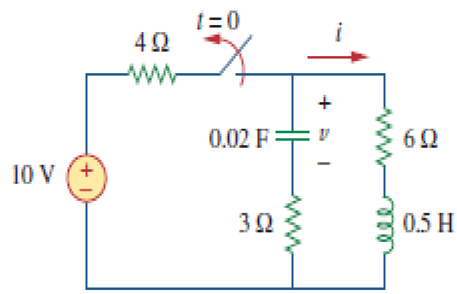
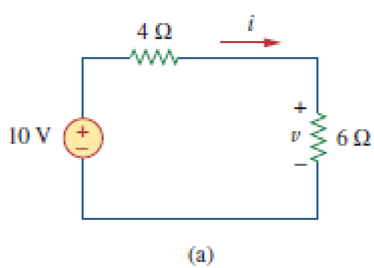


Fig. 2.28



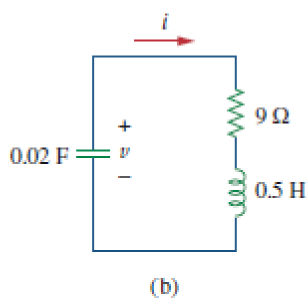


Fig. 2.29 : The circuit in Fig. 2.28 : (a) for t < 0, (b) for t > 0.

2.6.2 Free Parallel RLC Circuit

Parallel RLC circuits find many practical applications, notably in communications networks and filter designs.

Consider the parallel RLC circuit shown in Fig. 2.30. Assume initial inductor current $\mathbf{I_0}$ and initial capacitor voltage $\mathbf{V_0}$,

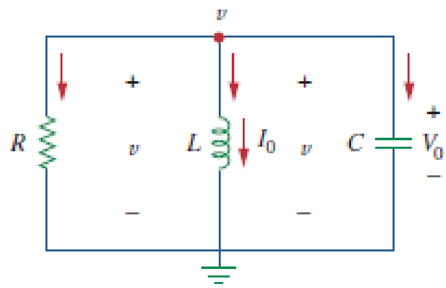


Fig. 2.30 parallel RLC circuit

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \qquad (27a)$$

$$v(0) = V_0 \qquad (27b)$$

Since the three elements are in parallel, they have the same voltage v across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0 \qquad (28)$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad (29)$$

We obtain the characteristic equation by replacing the first derivative by **s** and the second derivative by $\mathbf{s^2}$. By following the same reasoning used in establishing Eqs. (3) through (7), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad (30)$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \qquad (31)$$

where

$$\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{32}$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$ or $\alpha < \omega_0$

Let us consider these cases separately.

Overdamped Case

From Eq. (32), $\alpha > \omega_0$ when $L > 4R^2C$ The roots of the characteristic equation are real and negative. The response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{33}$$

Critically Damped Case

For $\alpha = \omega_0$, $L = 4R^2C$ The roots are real and equal so that the response is

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} \tag{34}$$

Underdamped Case

When $\alpha < \omega_0$, $L < 4R^2C$ In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{35}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{36}$$

The response is

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \tag{37}$$

The constants and in each case can be determined from the initial conditions. We need and The first term is known from Eq. (27b). We find the second term by combining Eqs. (27) and (28), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \quad (38)$$

The voltage waveforms are similar to those shown in Figure 2.25 , 2.26 2.27 and will depend on whether the circuit is overdamped, underdamped, or critically damped.

Example 2.9 In the parallel circuit of Fig. 2.30, find $v(t)$ for $t > 0$ assuming $v(0) = 5v$, $i(0) = 0A$, $L = 1\ H$, $C = 10mF$ Consider these cases: $R = 1.923\ \Omega, R = 5\ \Omega, R = 6.25\ \Omega$.

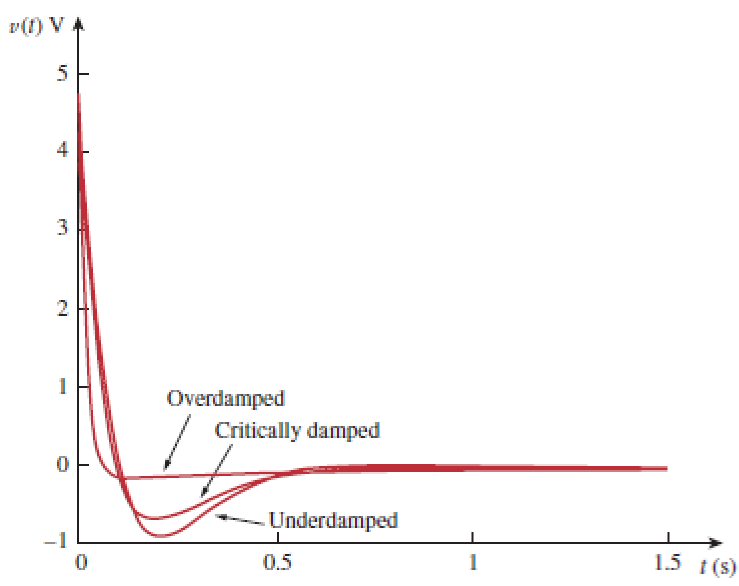


Fig. 2.31 responses for three degrees of damping.

Example 2.10 Find v(t) for t > 0 in the RLC circuit of Fig. 2.32

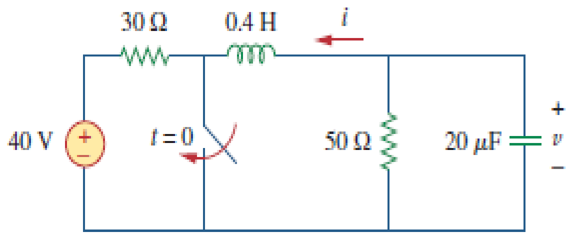


Fig. 2.32

Type	Condition	Criteria	α	ω_0	Response
Parallel Series	Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$A_1e^{s_1t} + A_2e^{s_2t}$, where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$
Parallel Series	Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(A_1t + A_2)$
Parallel Series	Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

H.W.

1- Series RLC circuit has R= 10Ω , L=5 H and C=2 mF find α , ω_0 , S_1 and S_2 and what type of natural response will the circuit have?

2- The circuit in Fig. 2.33 has reached steady state at $t = 0^-$ If the make before- break switch moves to position b at $t = 0$ calculate $i(t)$ for $t > 0$

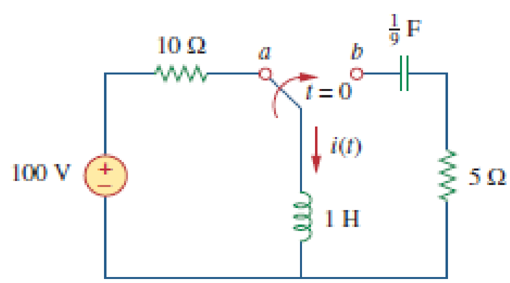


Fig. 2.33

3- Parallel RLC circuit has $R = 2\Omega$, $L = 0.4\ H$ and $C = 25\ mF$, $v(0) = 0$ and $i(0) = 50mA$ Find $v(t)$ for $t > 0$

4- Refer to the circuit in Fig. 2.34. Find $v(t)$ for $t > 0$.

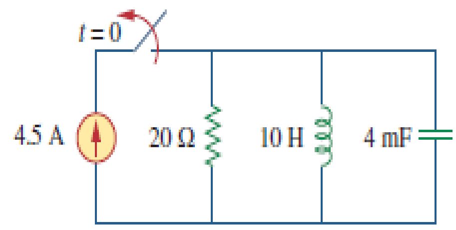


Fig. 2.34

2.7 Step Response of a Series RLC Circuit

As we learned in the preceding chapter, the step response is obtained by the sudden application of a dc source. Consider the series RLC circuit shown in Fig. 2.35. Applying KVL around the loop for $t > 0$,

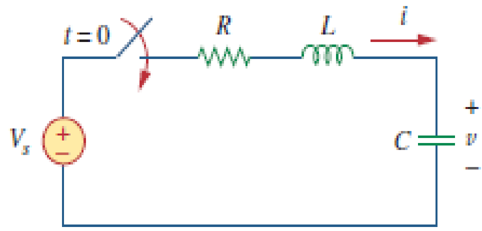


Fig. 2.35 Step voltage applied to a series RLC circuit.

$$L\frac{di}{dt} + Ri + v = V_s \qquad (39)$$

But

$$i = C\frac{dv}{dt}$$

Substituting for i in Eq. (39) and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \qquad (40)$$

which has the same form as Eq. (3). More specifically, the coefficients are the same (and that is important in determining the frequency parameters) but the variable is different. (Likewise, see Eq. (47).) Hence, the characteristic equation for the series RLC circuit is not affected by the presence of the dc source.

The solution to Eq. (40) has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$ that is,

$$v(t) = v_t(t) + v_{ss}(t) \qquad (41)$$

The transient response $v_t(t)$ is the component of the total response that dies out with time. The form of the transient response is the same as the form of the solution obtained in Section 2.3 for the source-free circuit, given by Eqs. (14), (21), and (26). Therefore, the transient response $v_t(t)$ for the overdamped, underdamped, and critically damped cases are:

$$v_t(t) = A_1e^{s_1t} + A_2e^{s_2t} \qquad \text{(Overdamped)} \qquad (42a)$$

$$v_t(t) = (A_1 + A_2t)e^{-\alpha t} \qquad \text{(Critically damped)} \qquad (42b)$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \qquad (42c)$$

The steady-state response is the final value of $v(t)$. In the circuit in Fig. 2.35 , the final value of the capacitor voltage is the same as the source voltage **Vs** . Hence

$$v_{ss}(t) = v(\infty) = V_s \quad (43)$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1e^{s_1t} + A_2e^{s_2t} \quad \text{(Overdamped)}$$
$$v(t) = V_s + (A_1 + A_2t)e^{-\alpha t} \quad \text{(Critically damped)}$$
$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)}$$
(44)

Example 2.11 For the circuit in Fig. 2.36, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: R=5Ω , R=4Ω and R=1Ω .

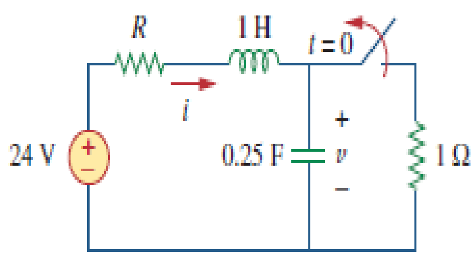


Fig. 2.36

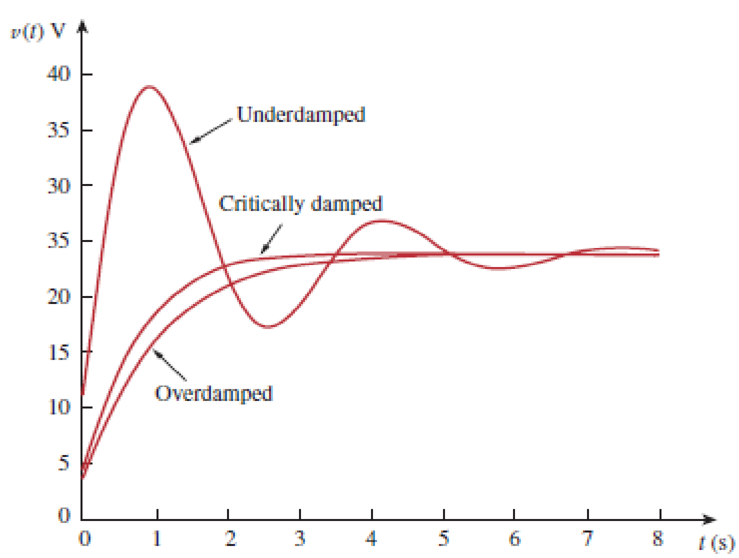


Fig. 2.36 response for three degrees of damping.

2.8 Step Response of a Parallel RLC Circuit

Consider the parallel RLC circuit shown in Fig. 2.37. We want to find i due to a sudden application of a dc current. Applying KCL at the top node for $t > 0$

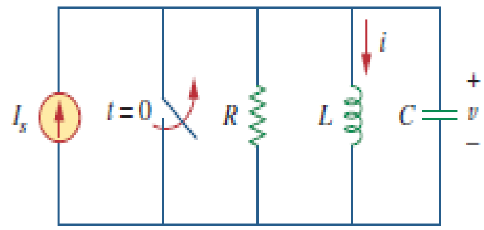


Fig. 2.37 Parallel RLC circuit with an applied current.

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \tag{46}$$

But

$$v = L\frac{di}{dt}$$

Substituting for v in Eq. (46) and dividing by LC, we get

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \tag{47}$$

which has the same characteristic equation as Eq. (29).

The complete solution to Eq. (47) consists of the transient response $i_t(t)$ and the steady-state response $i_{ss}(t)$that is,

$$i(t) = i_t(t) + i_{ss}(t) \tag{48}$$

The transient response is the same as what we had in Section 2.4. The steady-state response is the final value of i. In the circuit in Fig. 2.37, the final value of the current through the inductor is the same as the source current Thus,

$$\begin{aligned} i(t) &= I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)} \\ i(t) &= I_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)} \\ i(t) &= I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)} \end{aligned} \tag{49}$$

Example 2.12 In the circuit of Fig. 2.38, find $i(t)$ and $i_R(t)$ for $t > 0$.

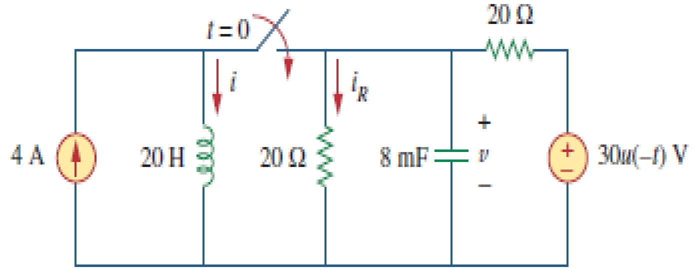


Fig. 2.38

H.W

1- Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 2.39.

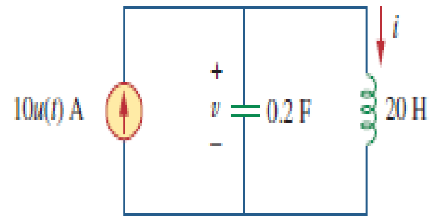


Fig. 2.39

2- Having been in position a for a long time, the switch in Fig. 2.40 is moved to position b at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.

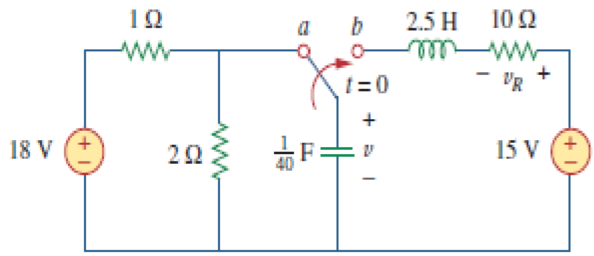


Fig. 2.40

3- The switch in the circuit of Fig. 2.41 has been in the left position for a long time; it is moved to the right at t = 0. Find (a) dv/dt at t = 0+; (b) v at t = 1 ms; (c) t0, the first value of t greater than zero at which v = 0.

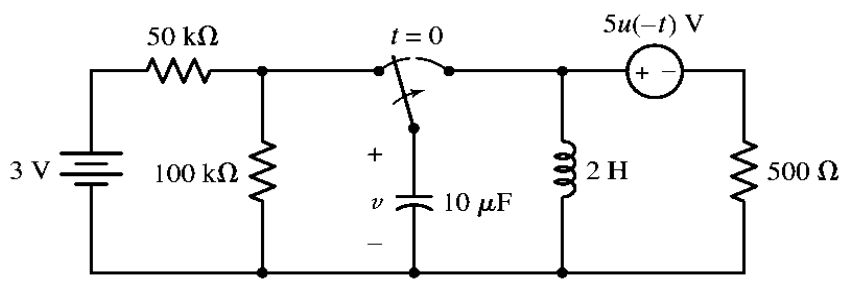


Fig. 2.41

4- Determine **i_L(t)** for the circuit of Fig. 2.42, and plot the waveform.

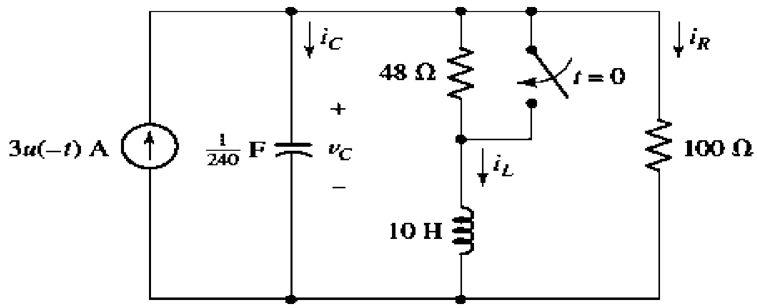


Fig. 2.42