

Math Lectures| First Stage

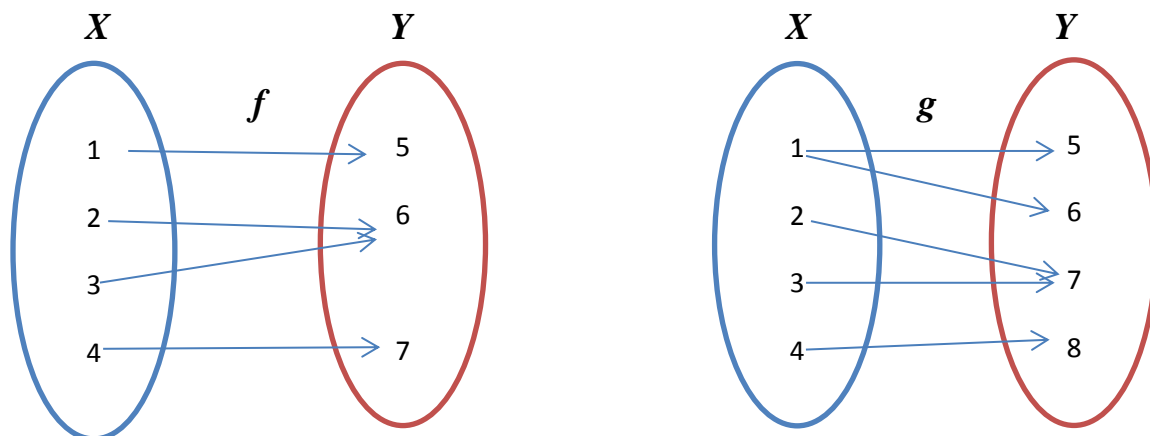
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Lecture One

Functions:

Def : Let each of X and Y be a non-empty set. The relationship f from X to Y is said to be a function from X to Y if for every $x \in X$, there is one element $y \in Y$ satisfy $y = f(x)$. The set x is called a Domain and is denoted by D_f and the set Y is called Range and is denoted by R_f .



We note

- $f : X \rightarrow Y$ is a function. Every element in X has associated with it exactly one element of Y .
- But $g : X \rightarrow Y$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y .
- the domain of f is the set X .
- the range of f is the set of y -values such that $y = f(x)$ for some x in X .

Arithmetic Operations on functions:

- **Sum:** $(f + g)(x) = f(x) + g(x)$.
- **Difference :** $(f - g)(x) = f(x) - g(x)$.
- **Product:** $(f * g)(x) = f(x) * g(x)$.
- **Quotient:** $(f/g)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.

The Limit of function at point:

If the value of a function $f(x)$ approach the value L as x approach b , we say f has limit L as x approach b and we write $\lim_{x \rightarrow b} f(x) = L$.

properties of limits

1. let $f(x) = c$, c is constant then $\lim_{x \rightarrow b} f(x) = \lim_{x \rightarrow b} c = c$.

2. $\lim_{x \rightarrow b} c \cdot f(x) = c \cdot \lim_{x \rightarrow b} f(x)$.

3. $\lim_{x \rightarrow b} (f(x) \pm g(x)) = \lim_{x \rightarrow b} f(x) \pm \lim_{x \rightarrow b} g(x)$.

4. $\lim_{x \rightarrow b} (f(x) \cdot g(x)) = \lim_{x \rightarrow b} f(x) \cdot \lim_{x \rightarrow b} g(x)$.

5. $\lim_{x \rightarrow b} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow b} f(x)}{\lim_{x \rightarrow b} g(x)}$, where $\lim_{x \rightarrow b} g(x) \neq 0$.

6. $\lim_{x \rightarrow b} (\sqrt[n]{f(x)}) = \sqrt[n]{\lim_{x \rightarrow b} f(x)}$, where $\lim_{x \rightarrow b} f(x) > 0$.

7. $\lim_{x \rightarrow b} x = b$.

8. $\lim_{x \rightarrow b} x^n = b^n$.

Examples

1. Evaluating the Limit of a Constant Function at a Point:

Example 1 Let $f(x) = 5$ Find $\lim_{x \rightarrow 1} f(x)$?

Solution: $\lim_{x \rightarrow 1} 5 = 5$. since there is no x in $f(x)$ anyway.

2. Evaluating the Limit of a Polynomial Function at a Point:

Example 2 $f(x) = x^2 + x - 2$. Calculate $\lim_{x \rightarrow 2} f(x)$?

Solution $\lim_{x \rightarrow 2} f(x) = 2^2 + 2 - 2 = 4$.

3. Evaluating the Limit of a Rational Function at a Point:

Example 3 $f(x) = \frac{3x+2}{x-1}$. Calculate $\lim_{x \rightarrow 2} f(x)$?

Note that f is a rational function with implied domain $\text{Dom}(f) = \{x \in \mathbb{R}, x \neq 1\}$.

Solution $\lim_{x \rightarrow 2} f(x) = \frac{3 \cdot 2 + 2}{2 - 1} = 8$.

Example 4 $f(x) = \frac{x^2 - 4}{x - 2}$, Calculate $\lim_{x \rightarrow 2} f(x)$?

Notice that the function $f(x) = \frac{x^2 - 4}{x - 2}$ is not defined when $x = 2$, because the denominator is zero we use the simplification first and then we substitute the value of x .

Solution $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 4$

Example 5: Let $f(x) = \frac{1}{x^2}$ Find $\lim_{x \rightarrow 0} f(x)$ if it exists?

Solution: The value of $f(x)$ do not approach 0 number, so $\lim_{x \rightarrow 0} \frac{1}{x^2}$ does not exist.

4. Two sided limits:

Let $f(x)$ be a function then the right-limit define as $\lim_{x \rightarrow b^+} f(x)$ (the limit of $f(x)$ as approaches a form the right). And the left-limit define as $\lim_{x \rightarrow b^-} f(x)$ (the limit of $f(x)$ as approaches a form the left).

Remark

$\lim_{x \rightarrow b} f(x) = L$ if and only if $\lim_{x \rightarrow b^+} f(x) = L = \lim_{x \rightarrow b^-} f(x)$.

Example 6 Let $f(x) = \begin{cases} 2x - 1 & , x \geq -1 \\ 3x & , x < -1 \end{cases}$,

Find the left and the right $\lim_{x \rightarrow -1} f(x)$? and explain if $\lim_{x \rightarrow -1} f(x)$ if it exists?

Solution :

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2x - 1) = -3 = L_1$$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} 3x = -3 = L_2$$

$\therefore L_1 = L_2 \rightarrow \lim_{x \rightarrow -1} f(x)$ is exists and equal -3.

Example 7 Let $f(x) = \begin{cases} -3x + 1 & , x < 1 \\ x^2 - 2 & , x \geq 1 \end{cases}$, Find the left and the right

$\lim_{x \rightarrow 1} f(x)$? and explain if $\lim_{x \rightarrow 1} f(x)$ if it exists?

Solution :

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 - 2) = -1 = L_1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (-3x + 1) = -2 = L_2$$

We notice $L_1 \neq L_2 \rightarrow \lim_{x \rightarrow 1} f(x)$ does not exist.

Example 8 Calculate the following :

a) $\lim_{x \rightarrow b} (x^3 + 4x^2 - 3)$

b) $\lim_{x \rightarrow b} \frac{x^2 + 3x + 2}{x + 1}$

c) $\lim_{x \rightarrow 2} \sqrt{2x^2 + 1}$

$$d) \lim_{x \rightarrow 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

$$f) \lim_{x \rightarrow 9} \frac{\frac{1}{\sqrt{x}} - \frac{1}{3}}{x - 9}$$

Solution

$$\begin{aligned} a) \lim_{x \rightarrow b} (x^3 + 4x^2 - 3) &= \lim_{x \rightarrow b} x^3 + \lim_{x \rightarrow b} 4x^2 - \lim_{x \rightarrow b} 3 \\ &= b^3 + 4b^2 - 3 \end{aligned}$$

$$b) \lim_{x \rightarrow b} \frac{x^2 + 3x + 2}{x + 1} = \frac{\lim_{x \rightarrow b} (x^2 + 3x + 2)}{\lim_{x \rightarrow b} (x + 1)} = \frac{b^2 + 3b + 2}{b + 1}$$

$$c) \lim_{x \rightarrow 2} \sqrt{2x^2 + 1} = \sqrt{\lim_{x \rightarrow 2} (2x^2 + 1)} = \sqrt{2 \cdot 2^2 + 1} = 3$$

$$d) \lim_{x \rightarrow 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

We can Multiply both numerator and denominator by the conjugate radical expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root).

$$\begin{aligned} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\ &= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} = \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} = \frac{1}{(\sqrt{x^2 + 100} + 10)} \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x \rightarrow 0} \frac{1}{(\sqrt{x^2 + 100} + 10)} = \frac{1}{20}$$

Example 9:
$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 3)(x - 2)} = \frac{4}{-1} = -4$$

$$f) \lim_{x \rightarrow 9} \frac{\frac{1}{\sqrt{x}} - \frac{1}{3}}{x - 9} = \lim_{x \rightarrow 9} \frac{\frac{3 - \sqrt{x}}{3\sqrt{x}}}{(\sqrt{x} - 3)(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{3\sqrt{x}} * \frac{1}{(\sqrt{x} - 3)(\sqrt{x} + 3)}$$

$$\lim_{x \rightarrow 9} \frac{-(\sqrt{x} - 3)}{3\sqrt{x}} * \frac{1}{(\sqrt{x} - 3)(\sqrt{x} + 3)} = \frac{-1}{54}$$

Differentiation:

Derivative definition:

The derivative of a function f is the function f' whose value at x is defined by the

equation: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Notes

- ❖ A function that has derivatives at a point x is said to be differentiable at x .
- ❖ A function that is differentiable at every point of its' domain is called differentiable.
- ❖ The differentiation operation is often denoted by $\frac{df}{dx}$, which read (the derivative of $f(x)$ with respect x .

Example1: Find the derivative of $f(x) = \sqrt{x}$ by definition ?

$$\begin{aligned} \text{Sol. : } f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} * \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Example2: Find the derivative of $f(x) = x^2 + 3x + 2$ by definition ?

$$\begin{aligned} \text{Sol: } f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h) + 2 - (x^2 + 3x + 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 + 3x + 3h + 2 - x^2 - 3x - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2hx + h^2 + 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h + 3)}{h} \end{aligned}$$

$$= 2x + 0 + 3 = 2x + 3.$$

Example 3: Find the derivative of $f(x) = \frac{1}{x}$ by definition ?

$$\begin{aligned} \text{Sol: } f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x - (x+h)}{x(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{x(x+h)} \right) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}. \end{aligned}$$

Example 4: Find the derivative of $f(x) = x^2 + 3$ by definition ?

$$\begin{aligned} \text{Sol: } f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3 - (x^2 + 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 + 3 - x^2 - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = 2x + 0 = 2x. \end{aligned}$$

Slopes and Tangent Lines:

When the value $f'(x)$ exists is called slope of the curve $y = f(x)$ at x . The line through the point $(x, f(x))$ with slope $f'(x) = m$ is the tangent to the curve at x . Now, steps to find the equation of the tangent :-

1. Find a contact point (x_1, y_1) .
2. Find the slope of the curve $m = f'(x)$.
3. Apply following relation $y - y_1 = m(x - x_1)$.

Example 1: Find the equation for the tangent to the curve $f(x) = (2 - x^2)^2$ at $x = 2$?

Sol. :- from steps a above:

1. Find a contact point $y_1 = f(2) = (2 - 2^2)^2 = (-2)^2 = 4$.

$(x_1, y_1) = (2, 4)$

2. Find the slope $m = f'(x)$

$\rightarrow m = f'(x) = 2(2 - x^2) * (-2x) = -4x(2 - x^2)$

$\rightarrow m = f'(2) = -4(2)(2 - 2^2) = -8 * (-2) = 16$

3. Apply the relation $y - y_1 = m(x - x_1)$

$y - 4 = 16(x - 2) = 16x - 32$

$16x - y - 32 + 4 = 0$

$16x - y - 28 = 0$

$y = 16x - 28$

\rightarrow The equation for tangent is $y = 16x - 28$

Example 2 :- Find equation for the tangent to the curve $y = \sqrt{x}$ at $x = 4$

Sol. :

1. firstly find (x_1, y_1) by

$$y_1 = \sqrt{x_1} = \sqrt{4} = 2 \rightarrow (x_1, y_1) = (4, 2)$$

2. Then, find slope $m = f'(x)$

$$\rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

$$\rightarrow m = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

3. Now, Apply the equation :-

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x - 1 + 2$$

\rightarrow The equation for tangent is $y = \frac{1}{4}x + 1$

Example 3:- find equation for the tangent to the curve $y = \sqrt[3]{3x + 5}$ at $x = 1$?

Sol. :

1. find (x_1, y_1) by $y_1 = f(x_1) = f(1) = \sqrt[3]{3(1) + 5} = \sqrt[3]{8} = 2$

$$\rightarrow (x_1, y_1) = (1, 2)$$

2. find slope $m = f'(x)$

$$\rightarrow f'(x) = \frac{1}{3} (3x + 5)^{-2/3} * 3 = \frac{1}{\sqrt[3]{(3x+5)^2}}$$

$$m = f'(1) = \frac{1}{\sqrt[3]{(3(1)+5)^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$$

3. Apply equation

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 1) \rightarrow 4y - 8 = x - 1$$

$$x - 4y + 8 - 1 = 0$$

$$x - 4y + 7 = 0$$

So, equation of tangent is $y = \frac{1}{4}(x + 7)$.

Example 4: find equation for the tangent to the curve $y = x^2$ at the point $(\frac{-1}{2}, \frac{1}{4})$

Sol. : firstly, find slope of curve:

$$f'(x) = 2x \rightarrow m = f'(x) = 2 \left(\frac{-1}{2}\right) = -1$$

Apply equation $y - y_1 = m(x - x_1)$

$$y - \frac{1}{4} = -1 \left(x - \left(\frac{-1}{2}\right)\right)$$

$$y - \frac{1}{4} = -x - \frac{1}{2}$$

$$y = -x - \frac{1}{4}$$

\rightarrow equation of tangent is $y = -x - \frac{1}{4}$