4. Solution of Simultaneous Differential Equations using Laplace Transform

Steps:

1. Take Laplace transform for all coupled equations
2. Substitute the initial conditions
3. Solve the resulting equations using any suitable method such as matrix inversion, elimination method or Cramer determinant method.
4. Take inverse transforms for all of the solutions.

Example: Solve the following coupled equations using Laplace transform method, given that $x(0) = 0$, $y(0) = 1$

$$\dot{x} + y = e^{-t}$$

$$\dot{y} - x = 3e^{-t}$$

By taking Laplace transform for both equations:

$$sX(s) - x(0) + Y(s) = \frac{1}{s+1}$$

$$sY(s) - y(0) - X(s) = \frac{3}{s+1}$$

By substituting the initial conditions and re-arranging:

$$sX(s) + Y(s) = \frac{1}{s+1}$$

$$-X(s) + sY(s) = \frac{s+4}{s+1}$$

Using Cramer’s method (straightforward):

$$X(s)=\frac{\begin{vmatrix}\frac{1}{s+1} & 1\\ \frac{s+4}{s+1} & s\end{vmatrix}}{\begin{vmatrix}s & 1\\ -1 & s\end{vmatrix}},\quad and\ Y(s)=\frac{\begin{vmatrix}s & \frac{1}{s+1}\\ -1 & \frac{s+4}{s+1}\end{vmatrix}}{\begin{vmatrix}s & 1\\ -1 & s\end{vmatrix}}$$

$$X(s)=\frac{\frac{s}{s+1}-\frac{s+4}{s+1}}{s^2+1}=\frac{-4}{(s^2+1)(s+1)},\quad and$$

$$Y(s)=\frac{\frac{s(s+4)}{s+1}-\frac{-1}{s+1}}{s^2+1}=\frac{s^2+4s+1}{(s^2+1)(s+1)}$$

Processing $X(s)$ to find $x(t)$:

$$X(s)=\frac{As+B}{s^2+1}+\frac{C}{(s+1)}$$

For the complex conjugate pole: $s=\alpha\pm i\beta,\ where\ \alpha=0,\ and\ \beta=1$

$$R_a(s)\big|_{s=\alpha+i\beta}=\frac{-4}{s+1}=\frac{-4}{i+1}=\frac{-4}{i+1}\times\frac{1-i}{1-i}=\frac{1}{2}\left(-4+i4\right)=-2+i2$$

$$C_a=2,\ D_a=-2$$

$$C=\frac{-4}{s^2+1}\bigg|_{s=-1}=-2$$

Therefore:

$$x(t)=\frac{e^0}{1}\left(2\cos t-2\sin t\right)-2e^{-t}=2\cos t-2\sin t-2e^{-t}$$

Processing $Y(s)$ to find $y(t)$:

$$Y(s)=\frac{As+B}{s^2+1}+\frac{C}{(s+1)}$$

$$R_a(s)\big|_{s=i} = \frac{s^2+4s+1}{s+1} = \frac{i4}{i+1} = \frac{i4}{i+1}\times\frac{1-i}{1-i} = \frac{1}{2}\left(i4+4\right) = 2+i2$$

$$C_a = 2,\ D_a = 2$$

$$C = \frac{s^2+4s+1}{s^2+1}\bigg|_{s=-1} = -1$$

Therefore:

$$y(t) = \frac{e^0}{1}\left(2\cos t + 2\sin t\right) - e^{-t} = 2\cos t + 2\sin t - e^{-t}$$

Example: Solve the following system of simultaneous equations, representing two-degree of freedom system, using Laplace transform method, subjected to the following initial conditions $x(0) = 0$, $y(0) = 1$ and $\dot{x}(0) = \dot{y}(0) = 0$

$$2\ddot{x} + 12x - 4y = 0$$
$$4\ddot{y} - 4x + 4y = 0$$

Solution:

By taking Laplace transform for both equations:

$$2\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 12X(s) - 4Y(s) = 0$$
$$4\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] - 4X(s) + 4Y(s) = 0$$

By substituting the initial conditions and re-arranging:

$$(2s^2+12)X(s) - 4Y(s) = 0$$
$$-4X(s) + \ (4s^2+4)Y(s) = 4s$$

Using Cramer's method:

$$X(s) = \frac{\begin{vmatrix} 0 & -4 \\ 4s & 4s^2+4 \end{vmatrix}}{\begin{vmatrix} 2s^2+12 & -4 \\ -4 & 4s^2+4 \end{vmatrix}},\quad and\ Y(s) = \frac{\begin{vmatrix} 2s^2+12 & 0 \\ -4 & 4s \end{vmatrix}}{\begin{vmatrix} 2s^2+12 & -4 \\ -4 & 4s^2+4 \end{vmatrix}}$$

$$X(s)=\frac{16s}{8s^4+56s^2+32}=\frac{2s}{s^4+7s^2+4},\quad and$$
$$Y(s)=\frac{8s^3+48s}{8s^4+56s^2+32}=\frac{s^3+6s}{s^4+7s^2+4}$$

The denominator of bot X(s) and Y(s) can be solved by casting it to second degree polynomial using $s^2=\varphi$:

Processing $X(s)$:

$$X(s)=\frac{2s}{s^4+7s^2+4}=\frac{2s}{(s^2+0.6277)(s^2+6.3723)}$$

There are two complex conjugate poles:

$$p_1=\alpha_1\pm i\beta_1=0\pm i\sqrt{0.6277}=\pm i0.7923,$$
$$p_2=\alpha_2\pm i\beta_2=0\pm i\sqrt{6.3723}=\pm i2.5243$$

$$R_a(s)\big|_{s=i0.7923}=\left[\frac{2s}{(s^2+6.3723)}\right]_{s=i0.7923}=\frac{i1.5846}{-0.6277+6.3723}=i0.2758$$
$$C_a=0.2758,\ D_a=0$$
$$R_b(s)\big|_{s=i2.5243}=\left[\frac{2s}{(s^2+0.6277)}\right]_{s=i2.5243}=\frac{i5.0486}{-6.3723+0.6277}=-i0.8788$$
$$C_b=-0.8788,\ D_b=0$$

Therefore:

$$x(t)=\frac{e^0}{0.7923}\left(0.2758\cos 0.7923t\right)+\frac{e^0}{2.5243}\left(-0.8788\cos 2.5243t\right)$$
$$=0.3481(\cos 0.7923t-\cos 2.5243t)$$

Processing $Y(s)$:

$$Y(s)=\frac{s^3+6s}{s^4+7s^2+4}=\frac{s^3+6s}{(s^2+0.6277)(s^2+6.3723)}$$

There are two complex conjugate poles:

$$p_1 = \alpha_1 \pm i\beta_1 = 0 \pm i\sqrt{0.6277} = \pm i0.7923,$$
$$p_2 = \alpha_2 \pm i\beta_2 = 0 \pm i\sqrt{6.3723} = \pm i2.5243$$

$$R_a(s)\Big|_{s=i0.7923} = \left[\frac{s^3+6s}{(s^2+6.3723)}\right]_{s=i0.7923} = \frac{i4.2564}{-0.6277+6.3723} = i0.7409$$

$$C_a = 0.7409,\ D_a = 0$$

$$R_b(s)\Big|_{s=i2.5243} = \left[\frac{s^3+6s}{(s^2+0.6277)}\right]_{s=i2.5243} = \frac{-i0.9374}{-6.3723+0.6277} = i0.1632$$

$$C_b = 0.1632,\ D_b = 0$$

Therefore:

$$y(t) = \frac{e^0}{0.7923}\left(0.7409\cos 0.7923t\right) + \frac{e^0}{2.5243}\left(0.1632\cos 2.5243t\right)$$
$$= 0.9351\cos 0.7923t + 0.06464\cos 2.5243t$$