3. Laplace Transform

$$F(s)=\mathcal{L}\{f(t)\}=\int_0^{\infty} e^{-st} f(t)\,dt$$ **"Laplace transform"**

$$f(t)=\mathcal{L}^{-1}F(s)=\frac{1}{i\,2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds$$ **"Complex Inversion formula"**

Example: Find Laplace transform for the function f(t) = 1

$$F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{1\}=\int_0^{\infty} e^{-st}.1dt=\left.\frac{e^{-st}}{-s}\right|_0^{\infty}=\frac{-1}{s}[0-1]=\frac{1}{s}$$

Table (1) Elementary Laplace transforms

f(t)	F(s)	f(t)	F(s)
a	$\frac{a}{s}$	e^{at}	$\frac{1}{s-a}$
t^n	$\begin{cases}\frac{\Gamma(n+1)}{s^{n+1}} & n>-1 \\ \frac{n!}{s^{n+1}} & n \text{ a positive integer}\end{cases}$		
$\sin at$	$\frac{a}{s^2+a^2}$	$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$f_1(t)\mp f_2(t)$	$F_1(s)\mp F_2(s)$	$\int_a^t f(t)dt$	$\frac{1}{s}F(s)+\frac{1}{s}\int_a^0 f(t)dt$
$e^{at}f(t)$	$F(s-a)$	$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$

****See the new Laplace Transform Table**

Laplace Transform of Derivative

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}.f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}dt$$

$$0 - f(0) + s\int_0^\infty f(t)e^{-st}dt = s\mathcal{L}\{f(t)\} - f(0)$$

Therefore:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Generally:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots\cdots f^{(n-1)}(0)$$

- **Laplace Transform for integrals**

$$\mathcal{L}\left\{\int_a^t f(t)dt\right\} = \int_0^\infty\left(\int_a^t f(t)dt\right)e^{-st}\,dt = \left[\int_a^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty f(t)\left(\frac{e^{-st}}{-s}\right)dt$$

$$= \underbrace{\int_a^\infty f(t)dt\left(\frac{e^{-\infty}}{-s}\right)}_{=0} - \int_a^0 f(t)dt\left(\frac{e^0}{-s}\right) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}\int_a^0 f(t)dt + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$So\ \ far: \mathcal{L}\left\{\int_a^t f(t)dt\right\} = \frac{1}{s}F(s) - \frac{1}{s}\int_0^a f(t)dt$$

This can be extended to double, triple and higher integration.

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s)$$

Example: Find the Laplace transform for the functions

$f(t) = \sin^2 t$

$f(t) = \sin^2 t \quad , f'(t) = 2\sin t\cos t = \sin 2t$

$$\mathcal{L}\{f'(t)\} = \frac{2}{s^2+4}$$

but :

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

since : $f(0) = \sin 0 = 0$

$$sF(s) = \frac{2}{s^2+4}$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{2}{s(s^2+4)}$$

$$Or \qquad \sin^2 t = \frac{1}{2}(1-\cos 2t) = f(t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}$$

Example: Find the Laplace transform for $\int_0^t \cos 4t \, dt$

$$\because \ \mathcal{L}\{\cos 4t\} = \frac{s}{s^2+16}$$

$$\therefore \mathcal{L}\left\{\int_0^t \cos 4t \, dt\right\} = \frac{1}{s}\cdot\frac{s}{s^2+16} = \frac{1}{s^2+16}$$

Or

$$\int_0^t \cos 4t \, dt = \frac{\sin 4t}{4} \quad \rightarrow \quad \mathcal{L}\left\{\frac{\sin 4t}{4}\right\} = \frac{1}{4}\cdot\frac{4}{s^2+16} = \frac{1}{s^2+16}$$

Shifting in s-domain

If f(t) has the transform F(s) (where s >k), then $e^{at}f(t)$ has the transform F(s - a) (where s - a >k).

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Proof: $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a)$

We can show that

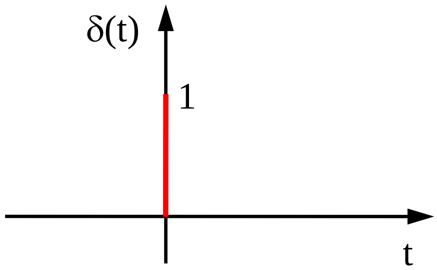
$$\mathcal{L}\left(e^{-at}\cos bt\right) = \frac{s+a}{(s+a)^2+b^2} \quad , \mathcal{L}\left(e^{-at}\sin bt\right) = \frac{b}{(s+a)^2+b^2}$$

$$\mathcal{L}\left(e^{-at}t^n\right) = \frac{n!}{(s+a)^{n+1}}$$

Unit Delta Function (Dirac Function)

Unit impulse (delta) function

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

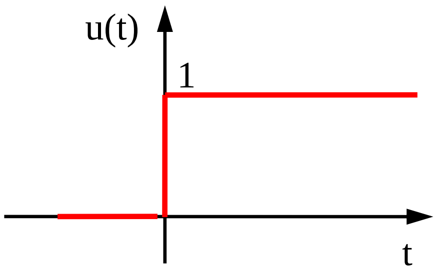


$$\mathcal{L}(\delta(t)) = 1$$

Unit Step Function (Heaviside Function)

Unit step function, u(t)

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



Step Function (Heaviside function)

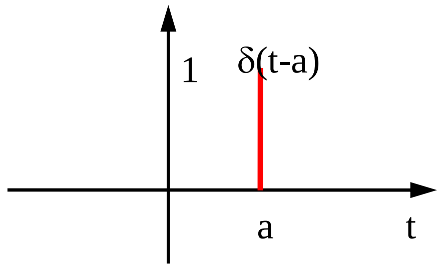
$$\mathcal{L}(u(t)) = \int_0^\infty 1.e^{-st}dt = \frac{1}{s}$$

Notes:

(1) $\delta(t) = \dfrac{du(t)}{dt}$

(2) $\int_0^{\infty} \delta(t)dt = u(t)$

Shifting (Translated) functions



$$\delta(t-a) = \begin{cases} 1 & t = a \\ 0 & t \neq a \end{cases}$$

$$\mathcal{L}\left(\delta(t-a)\right) = \int_0^{\infty} \delta(t-a)e^{-st}dt = e^{-sa}$$

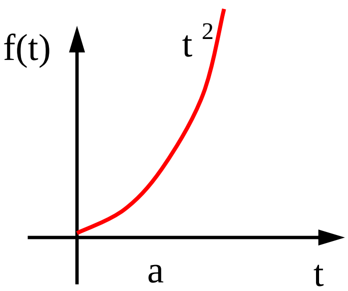
$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

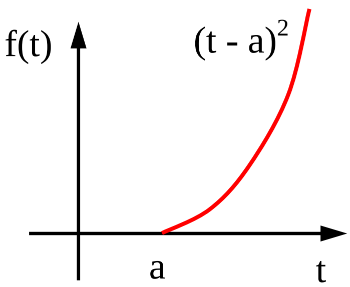
$$\mathcal{L}\left(u(t-a)\right) = \int_0^{\infty} u(t-a)e^{-st}dt = \int_0^{a} 0.dt + \int_a^{\infty} 1.e^{-st}dt$$

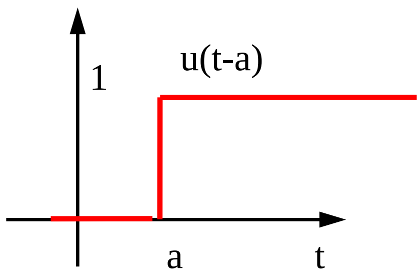
$$= \frac{-1}{s}\left[e^{-st}\right]_a^{\infty} = \frac{-1}{s}\left[0 - e^{-as}\right] = \frac{e^{-as}}{s}$$

Shifting in Time-domain

For example;







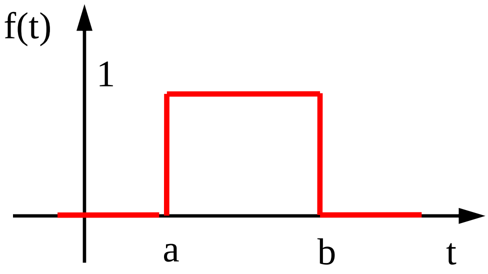
If we take the expression:

$$(t-a)^2 u(t-a) = \begin{cases} 0 & t < a \\ (t-a)^2 & t \geq a \end{cases}$$

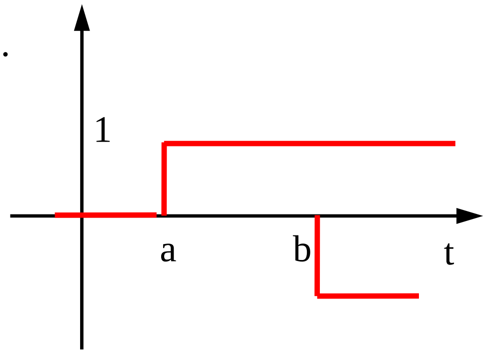
More generally:

$$f\left(t-a\right)u(t-a)$$

Example: What is the equation of the function whose graphs in shown below;

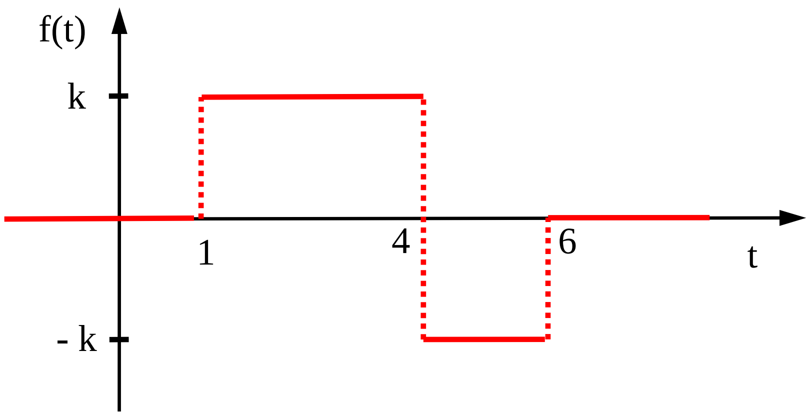


Sol.



$$f(t)=u(t-a)-u(t-b)$$

Example:



$$f(t)=k\left[u(t-1)-2u(t-4)+u(t-6)\right]$$

$$\mathcal{L}\left\{f(t-a)u(t-a)\right\}=e^{-as}F\left(s\right) \qquad \textbf{(t-Shifting)}$$

we can also write: $\mathcal{L}\left\{f(t)u(t-a)\right\}=e^{-as}\mathcal{L}\left\{f\left(t+a\right)\right\}$

Or, we can write

$$f(t-a)u(t-a)=\mathcal{L}^{-1}\left\{e^{-as}F\left(s\right)\right\}$$

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\}=\int_0^\infty f(t-a)u(t-a)e^{-st}dt=\int_a^\infty f(t-a)e^{-st}dt$$

Now let $t-a=\tau,\quad dt=d\tau$, then

$$\int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau=e^{-as}\int_0^\infty f(\tau)e^{-s\tau}d\tau=e^{-as}F(s)$$

Example: Find $\mathcal{L}\left\{\cos(t-1)u(t-1)\right\}$

$$\mathcal{L}\left\{\cos(t-1)u(t-1)\right\}=e^{-s}\frac{s}{s^2+1}\qquad\text{where } f(t)=\cos t\ \rightarrow F(s)=\frac{s}{s^2+1}$$

Example: Find $\mathcal{L}\left\{\sin(2t-2)u(t-1)\right\}$

$$F(t)=\sin(2t-2)u(t-1)=\sin 2(t-1)u(t-1)$$

$$\therefore \mathcal{L}\left\{F(t)\right\}=e^{-s}\frac{2}{s^2+4}\qquad \textit{where}\ \mathcal{L}\left\{\sin 2t\right\}=\frac{2}{s^2+4}$$

Example: Find $F(t)=\cos(t-1)u(t-2)$

$$\begin{aligned}F(t)&=\cos(t-2+1)u(t-2)\\&=\left[\cos(t-2)\cos 1-\sin(t-2)\sin 1\right]u(t-2)\end{aligned}$$

$$\therefore \mathcal{L}\left\{F(t)\right\}=\left[\frac{s\cos 1}{s^2+1}-\frac{\sin 1}{s^2+1}\right]e^{-2s}$$

Example: Write the following using unit step functions and find its transform

$$f(t)=\begin{cases} 2 & if\, 0<t<1 \\ \frac{1}{2}t^2 & if\, 1<t<\frac{1}{2}\pi \\ \cos t & if \quad t>\frac{1}{2}\pi \end{cases}$$

Solution: Step (1) In terms of unit step functions

$$f(t)=2\left[u(t)-u(t-1)\right]+\frac{1}{2}t^2\left[u(t-1)-u(t-\frac{1}{2}\pi)\right]+\left(\cos t\right)u(t-\frac{1}{2}\pi)$$

Step (2) Find Laplace transform:

$$\mathcal{L}\left\{2\left[u(t)-u(t-1)\right]\right\}=2\left(1-e^{-s}\right)/s$$

Using the formula: $\mathcal{L}\{f(t)u(t-a)\}=e^{-as}\mathcal{L}\{f(t+a)\}$

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\}=e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2+t+\frac{1}{2}\right\}=e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)$$

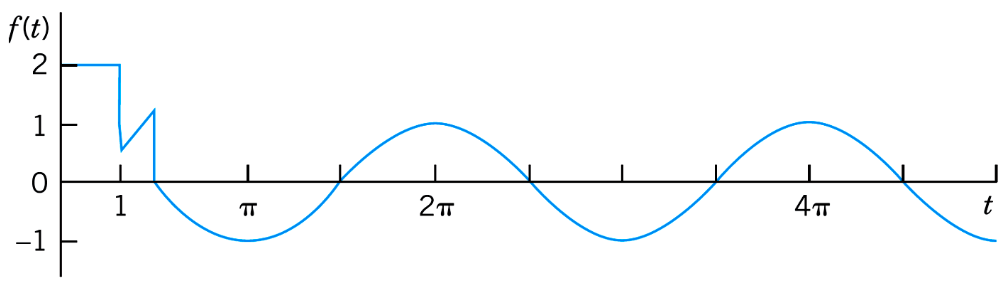
Similarly;

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-\frac{\pi}{2})\right\}=e^{-\pi s/2}\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)$$

$$\mathcal{L}\left\{\cos t u(t-\frac{\pi}{2})\right\}=e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}=e^{-\pi s/2}\mathcal{L}\{-\sin(t)\}=-e^{-\pi s/2}\left(\frac{1}{s^2+1}\right)$$

Finally;

$$\therefore \mathcal{L}\{F(t)\}=\frac{2}{s}-\frac{2}{s}e^{-s}+e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)+e^{-\pi s/2}\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)-e^{-\pi s/2}\left(\frac{1}{s^2+1}\right)$$



Differentiation of Transforms $(t^n f(t))$

$$\mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n F(s)}{ds^n}$$

Proof: $F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt \quad \Rightarrow \quad F'(s) = -\int_0^\infty f(t)te^{-st}dt$

Consequently:

$\mathcal{L}\left(tf(t)\right) = -F'(s)$ hence $\mathcal{L}^{-1}\left(F'(s)\right) = -tf(t)$

Example: Find the inverse transform of $\ln\left(1+\frac{b^2}{s^2}\right)$

Sol. $F(s) = \ln\left(1+\frac{b^2}{s^2}\right) = \ln\left(\frac{s^2+b^2}{s^2}\right)$

$$F'(s) = \frac{d}{ds}\left[\ln\left(s^2+b^2\right) - \ln s^2\right] = \frac{2s}{s^2+b^2} - \frac{2s}{s^2}$$

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2}{s}\right\} = 2\cos bt - 2 = -tf(t)$$

$$f(t) = \frac{2}{t}(1-\cos bt)$$

Integration of Transforms

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds \qquad hence \qquad \mathcal{L}^{-1}\left\{\int_s^\infty F(s)ds\right\} = \frac{f(t)}{t}$$

Proof: $\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right]ds$

We may reverse the order of integration, that is,

$$\int_s^{\infty} F(s)ds = \int_0^{\infty}\left[\int_s^{\infty} e^{-st} f(t)ds\right]dt = \int_0^{\infty}\left[f(t)\int_s^{\infty} e^{-st}ds\right]dt$$

$$= \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt = \int_0^{\infty} e^{-st}\left[\frac{f(t)}{t}\right]dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Example: What is $\mathcal{L}\left\{\frac{\sin kt}{t}\right\}$?

Sol. $\because f(t) = \sin kt$

$$\mathcal{L}\left\{\frac{\sin kt}{t}\right\} = \int_s^{\infty} \mathcal{L}\{\sin kt\}ds = \int_s^{\infty} \frac{k}{s^2+k^2}ds = \tan^{-1}\frac{k}{s}\bigg|_s^{\infty}$$

$$= 0 - \tan^{-1}\frac{k}{s} = -\tan^{-1}\frac{k}{s}$$

Example: What is $f(t)\ if\ \mathcal{L}\{f(t)\} = \frac{s}{\left(s^2-1\right)^2}$?

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left\{\int_s^{\infty}\frac{s}{\left(s^2-1\right)^2}ds\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{2\left(s^2-1\right)}\bigg|_s^{\infty}\right\}$$

$$\Rightarrow f(t) = t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\left(0+\frac{1}{s^2-1}\right)\right\} = t\,\mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\right\}$$

$$= \frac{t}{4}\left(e^{t}-e^{-t}\right) = \frac{t\sinh t}{2}$$

Partial Fraction Method

$$F(s)=\frac{Y(s)}{Q(s)}$$ **In condition Y(s) has a degree less then Q(s)**

Notes:

(1) $\mathcal{L}\{A\}=A\delta(t)$

(2) $\mathcal{L}\{k\,s\}=k\dfrac{d\delta(t)}{dt}$

1. Unrepeated or simple root (s - a)

$$F(s)=\frac{Y(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots$$

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=A_1e^{a_1t}+A_2e^{a_2t}+\cdots$$

$$A_1=\lim_{s\to a_1}\left\{(s-a_1)\frac{Y(s)}{Q(s)}\right\}=\left[(s-a_1)\frac{Y(s)}{Q(s)}\right]_{s=a_1}$$

$$A_2=\lim_{s\to a_2}\left\{(s-a_2)\frac{Y(s)}{Q(s)}\right\}=\left[(s-a_2)\frac{Y(s)}{Q(s)}\right]_{s=a_2}$$

Example: Find $\mathcal{L}^{-1}$ *for* $F(s)=\dfrac{s-2}{s^2+5s+6}$

$$F(s)=\frac{s-2}{s^2+5s+6}=\frac{s-2}{(s+2)(s+3)}=\frac{A}{s+2}+\frac{B}{s+3}$$

$$A=\lim_{s\to-2}\left\{\cancel{(s+2)}\,\frac{s-2}{\cancel{(s+2)}\,(s+3)}\right\}=\left[\frac{s-2}{s+3}\right]_{s=-2}=-4$$

$$B=\lim_{s\to-3}\left\{\cancel{(s+3)}\,\frac{s-2}{(s+2)\,\cancel{(s+3)}}\right\}=5$$

$$\therefore\ f(t)=-4e^{-2t}+5e^{-3t}$$

2. Repeated roots (s − a)^m:

where *m* is a positive integer

$$F(s)=\frac{Y(s)}{Q(s)}=\frac{A_m}{(s-a)^m}+\frac{A_{m-1}}{(s-a)^{m-1}}+\frac{A_{m-2}}{(s-a)^{m-2}}+\cdots+\frac{A_1}{(s-a)}$$

$$f(t)=e^{at}\left[A_m\frac{t^{m-1}}{(m-1)!}+A_{m-1}\frac{t^{m-2}}{(m-2)!}+A_{m-2}\frac{t^{m-3}}{(m-3)!}+\cdots+A_1\right]$$

$$R_a(s)=(s-a)^m\frac{Y(s)}{Q(s)}$$

$$A_m=\lim_{s\to a}\left\{(s-a)^m\frac{Y(s)}{Q(s)}\right\}=R_a(s)\big|_{s=a}$$

$$A_{m-1}=\frac{1}{1!}\lim_{s\to a}\left\{\frac{d}{ds}R_a(s)\right\}$$

Generally:

$$A_{m-k}=\frac{1}{k!}\lim_{s\to a}\left\{\frac{d^k}{ds^k}R_a(s)\right\}\quad where\ \ k=1,\,2,\,3,\cdots(m-1)$$

Example: Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2(s-2)}\right\}$

$$F(s)=\frac{s+2}{(s+1)^2(s-2)}=\frac{A_2}{(s+1)^2}+\frac{A_1}{(s+1)}+\frac{B}{s-2}$$

$$R_a(s) = \cancel{(s+1)^2}\,\frac{s+2}{\cancel{(s+1)^2}\,(s-2)} = \frac{s+2}{s-2}$$

$$A_2 = \lim_{s\to -1}\{R_a(s)\} = \left[\frac{s+2}{s-2}\right]_{s=-1} = \frac{-1}{3}$$

$$A_1 = \frac{1}{1!}\lim_{s\to -1}\left\{\frac{d}{ds}R_a(s)\right\} = \left[\frac{d}{ds}\left(\frac{s+2}{s-2}\right)\right]_{s=-1} = \left[\frac{(s-2)-s-2}{(s-2)^2}\right]_{s=-1} = \frac{-4}{9}$$

$$B = \lim_{s\to 2}\left\{\cancel{(s-2)}\,\frac{s+2}{(s+1)^2\,\cancel{(s-2)}}\right\} = \left[\frac{s+2}{(s+1)^2}\right]_{s=2} = \frac{4}{9}$$

$$\therefore F(s) = \frac{-1/3}{(s+1)^2} - \frac{4/9}{(s+1)} + \frac{4/9}{s-2}$$

$$f(t) = e^{-t}\left(\frac{-1}{3}\frac{t}{1!} - \frac{4}{9}\right) + \frac{4}{9}e^{2t}$$

3**. Unrepeated complex root** $(s-p)(s-\bar{p})$

where *p* is a complex number $p = \alpha + i\beta$, $\bar{p} = \alpha - i\beta$

$$F(s) = \frac{Y(s)}{Q(s)} = \frac{As+B}{(s-p)(s-\bar{p})}$$

$$f(t) = \frac{e^{\alpha t}}{\beta}\left[C_a \cos\beta t + D_a \sin\beta t\right]$$

where

C_a *is imaginary part of* $R_a(s)\big|_{s=a}$

D_a *is real part of* $R_a(s)\big|_{s=a}$

$$\because R_a(s) = \left[(s-a)(s-\bar{a})\frac{Y(s)}{Q(s)}\right]$$

Example: Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+2s+5)(s+1)}\right\}$

$s^2 + 2s + 5$

$s = \frac{-2 \mp \sqrt{4 - 4*5}}{2}$

$s = -1 \mp i2$

$\therefore \alpha = -1$

$\beta = 2$

$$F(s)=\frac{As+B}{s^2+2s+5}+\frac{C}{s+1}$$

$$F(s)=\frac{As+B}{(s+1-i2)\left(s+1+i2\right)}+\frac{E}{s+1}$$

$$R_a(s)=\cancel{(s^2+2s+5)}\,\frac{s+2}{\cancel{\left(s^2+2s+5\right)}\left(s+1\right)}=\frac{s+2}{s+1}$$

$$R_a(s)\Big|_{s=-1+i2}=\left[\frac{s+2}{s+1}\right]_{s=-1+i2}=\frac{1+i2}{i2}=1-i\frac{1}{2}$$

$$C_a=\frac{-1}{2}\quad,\quad D_a=1$$

$$E=\lim_{s\to-1}\left\{\frac{s+2}{s^2+2s+5}\right\}=R_c\left(s\right)\Big|_{s=-1}=\frac{1}{4}$$

$$f(t)=\frac{e^{-t}}{2}\left(-\frac{1}{2}\cos 2t+\sin 2t\right)+\frac{1}{4}e^{-t}$$